The Higgs Troika

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Abstract

There are three composite electroweak Higgs bosons stemming from top quark, tau neutrino, and tau lepton condensations. Each of the three condensations gives rise to masses of four different fermions. The fermion mass hierarchies within these three cohorts are dictated by four-fermion condensations, which break two global chiral symmetries. The four-fermion condensations induce axion-like pseudo-Nambu-Goldstone bosons and can be dark matter candidates. In addition to the 125 GeV Higgs boson observed at the LHC, we anticipate detection of tau neutrino composite Higgs boson via the charm quark decay channel. Tau neutrino condensation may contribute substantially to the muon anomalous magnetic moment. On the other hand, a feeble antisymmetric condensation might be gravitationally relevant and reflected as large-scale CMB anisotropies.

Keywords Clifford algebra; chiral symmetry breaking; axion; strong CP problem; flavor mixing; muon anomalous magnetic moment; CMB anisotropies; suppressed proton decay

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1   Introduction

Dimensionless ratios between parameters appearing in a physical theory can not be accidentally small. This naturalness principle is elegantly defined by 't Hooft [1]: a quantity should be small only if the underlying theory becomes more symmetric as that quantity tends to zero. Weakly broken symmetry ensures that the smallness of a parameter is preserved against possible perturbative disturbances. The standard model Higgs sector is unnatural since even if one takes the massless Higgs boson limit, the symmetry of standard model is not enhanced. Perturbative quantum corrections tend to draw the smaller electroweak scale towards Planck scale.

One way of addressing the naturalness problem is to replace the fundamental Higgs boson with a fermion-antifermion condensation, such as in technicolor [2–4] and top condensation models [5–14]. The Higgs sector is an effective description of the low energy physics represented by composite boson field. The condensation is induced via dynamical symmetry breaking mechanism, which is a profound concept in physics. It is introduced into relativistic quantum field theory by Nambu and Jona-Lasinio (NJL) [15], inspired by earlier Bardeen-Cooper-Schriefer (BCS) theory of superconductivity [16].

A challenge facing the composite Higgs model is to account for the vast range of fermion masses which span five orders of magnitude. The current paper is an effort towards explaining the fermion mass hierarchies in the context of composite electroweak Higgs bosons. We propose two global chiral symmetries $U(1)_\alpha$ and $U(1)_\beta$, in addition to the local gauge symmetries. There are four-fermion as well as two-fermion condensations. In accordance with naturalness principle, the chiral symmetries play a pivotal role in determining the relative magnitudes of four-fermion condensations, and consequently giving rise to fermion mass hierarchies.

Our approach is based on the framework of six-dimensional Clifford algebra $C\ell_{0,6}$ [17, 18]. Clifford algebra, also known as geometric algebra or spacetime algebra (for the specific case of $C\ell_{1,3}$), is a powerful mathematical tool with various applications in physics [19–23]. Including right-handed neutrinos, there are 16 Weyl fermions with $16 \times 2 = 32$ complex components (64 real components) within each of the three fermion families. One generation of fermions can be represented by an algebraic spinor, which is a linear combination of all $2^6 = 64$ basis elements of six-dimensional Clifford algebra $C\ell_{0,6}$. The $SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_{B-L}$ local gauge symmetries, which encompasses standard model symmetries, are naturally embedded in the algebraic structure. Besides the binary Clifford algebra, ternary Clifford algebra [24, 25] is also leveraged in constructing flavor projection operators. They serve the purpose of determining allowable flavor-mixing interactions.

This paper is structured as follows: Section 2 introduces algebraic spinors and local gauge symmetries. In section 3, we study flavor-mixing interactions, chiral symmetry breaking by fermion condensations, and fermion mass hierarchies. In the last section we draw our conclusions.

2 Clifford Algebra and Gauge Symmetries

Standard model fermions (plus right-handed neutrinos) can be represented by algebraic spinors of six-dimensional Clifford algebra $C\ell_{0,6}$. The Lagrangian of the algebraic spinors accommodates local gauge symmetries $SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_{B-L}$. This section is a brief review of the algebraic spinor of $C\ell_{0,6}$ and the related symmetries. More details, including specifics about the mappings between conventional matrix representation and $C\ell_{0,6}$ formulation, can be found in Refs. [17, 18].

2.1 Fermions as Algebraic Spinors

The six-dimensional Clifford algebra $C\ell_{0,6}$ is defined by vector basis $\{\Gamma_j; j = 1, 2, \ldots, 6\}$ satisfying

$$\Gamma_j \Gamma_k + \Gamma_k \Gamma_j = -2\delta_{jk}. \tag{1}$$

The four-dimensional Dirac spacetime algebra $C\ell_{1,3}$ is a sub-algebra of $C\ell_{0,6}$ with basis

$$\gamma_0 = \Gamma_1 \Gamma_2 \Gamma_3, \tag{2}$$
$$\gamma_1 = \Gamma_4, \tag{3}$$
$$\gamma_2 = \Gamma_5, \tag{4}$$
$$\gamma_3 = \Gamma_6. \tag{5}$$

Rather than being a vector, γ_0 here is a composite trivector, which departs from the other Clifford algebra based approaches [19–23].

Color projection operators are given by

$$P_{rd} = \frac{1}{4}(1 + i\gamma_1\Gamma_1 - i\gamma_2\Gamma_2 - i\gamma_3\Gamma_3), \tag{6}$$

$$P_{gr} = \frac{1}{4}(1 - i\gamma_1\Gamma_1 + i\gamma_2\Gamma_2 - i\gamma_3\Gamma_3), \tag{7}$$

$$P_{bl} = \frac{1}{4}(1 - i\gamma_1\Gamma_1 - i\gamma_2\Gamma_2 + i\gamma_3\Gamma_3), \tag{8}$$

$$P_{l} = \frac{1}{4}(1 + i\gamma_1\Gamma_1 + i\gamma_2\Gamma_2 + i\gamma_3\Gamma_3), \tag{9}$$

where i is the unit pseudoscalar

$$i = \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5 \Gamma_6 = \gamma_0 \gamma_1 \gamma_2 \gamma_3, \tag{10}$$

which squares to -1, anticommutes with Clifford-odd elements, and commutes with Clifford-even elements.

The lepton projection operator P_l can be regarded as projection to the fourth color. The quark projection operator P_q is the sum of red, green, and blue projections

$$P_q = P_{rd} + P_{gr} + P_{bl}. \tag{11}$$

Additionally, we introduce another set of projection operators

$$P_\pm = \frac{1}{2}(1 \pm i\Gamma_1\Gamma_2), \tag{12}$$

for the purpose of differentiating electroweak up- and down-type fermions.

One generation of fermions can be represented by an algebraic spinor, which is a linear combination of all $2^6 = 64$ basis elements of Clifford algebra $C\ell_{0,6}$. Due to the Fermion nature, the linear combination coefficients are Grassmann-odd. Note that the algebraic spinor is a $C\ell_{0,6}$-valued, Grassmann-odd, and real (rather than complex) function of *four-dimensional* spacetime, albeit $C\ell_{0,6}$ is six-dimensional.

Spinors with left/right chirality correspond to Clifford-odd/even multivectors

$$\psi_L^a = \frac{1}{2}(\psi^a + i\psi^a i), \tag{13}$$

$$\psi_R^a = \frac{1}{2}(\psi^a - i\psi^a i), \tag{14}$$

where three generations of spinors are denoted as ψ^a, with $a = 1, 2, 3$. We identify projections of spinors

$$\psi^a = (P_+ + P_-)(\psi_L^a + \psi_R^a)(P_q + P_l) \tag{15}$$

with quarks and leptons as shown in table 1.

First Generation	Second Generation	Third Generation
$u_L = P_+\psi_L^1 P_q$	$c_L = P_+\psi_L^2 P_q$	$t_L = P_+\psi_L^3 P_q$
$d_L = P_-\psi_L^1 P_q$	$s_L = P_-\psi_L^2 P_q$	$b_L = P_-\psi_L^3 P_q$
$\nu_L = P_+\psi_L^1 P_l$	$\nu_{\mu L} = P_+\psi_L^2 P_l$	$\nu_{\tau L} = P_+\psi_L^3 P_l$
$e_L = P_-\psi_L^1 P_l$	$\mu_L = P_-\psi_L^2 P_l$	$\tau_L = P_-\psi_L^3 P_l$
$u_R = P_-\psi_R^1 P_q$	$c_R = P_-\psi_R^2 P_q$	$t_R = P_-\psi_R^3 P_q$
$d_R = P_+\psi_R^1 P_q$	$s_R = P_+\psi_R^2 P_q$	$b_R = P_+\psi_R^3 P_q$
$\nu_R = P_-\psi_R^1 P_l$	$\nu_{\mu R} = P_-\psi_R^2 P_l$	$\nu_{\tau R} = P_-\psi_R^3 P_l$
$e_R = P_+\psi_R^1 P_l$	$\mu_R = P_+\psi_R^2 P_l$	$\tau_R = P_+\psi_R^3 P_l$

Table 1: Three generations of fermions as projections of algebraic spinors. Quarks stand for sum of red, green, and blue colors.

2.2 Gauge Fields and Covariant Derivatives

The $SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_{B-L}$ gauge-covariant derivatives of fermion fields are

$$D_{L\mu}\psi_L^a = (\partial_\mu + W_{L\mu})\psi_L^a + \psi_L^a(W_{BL\mu} + G_\mu), \tag{16}$$
$$D_{R\mu}\psi_R^a = (\partial_\mu + W_{R\mu})\psi_R^a + \psi_R^a(W_{BL\mu} + G_\mu), \tag{17}$$

where $\mu = 0, 1, 2, 3$. Here the gauge fields are defined to absorb gauge coupling constants.

The $SU(3)_c$ strong interaction G_μ is expressed as (summation convention for repeated indices is adopted)

$$G_\mu = G_\mu^k T_k, \tag{18}$$

where

$$\left(T_1, \ldots, T_8 \right) = \begin{pmatrix} \frac{1}{4}(\gamma_1\Gamma_2 + \gamma_2\Gamma_1), \frac{1}{4}(\Gamma_1\Gamma_2 + \gamma_1\gamma_2), \frac{1}{4}(\Gamma_1\gamma_1 - \Gamma_2\gamma_2), \\ \frac{1}{4}(\gamma_1\Gamma_3 + \gamma_3\Gamma_1), \frac{1}{4}(\Gamma_1\Gamma_3 + \gamma_1\gamma_3), \\ \frac{1}{4}(\gamma_2\Gamma_3 + \gamma_3\Gamma_2), \frac{1}{4}(\Gamma_2\Gamma_3 + \gamma_2\gamma_3), \\ \frac{1}{4\sqrt{3}}(\Gamma_1\gamma_1 + \Gamma_2\gamma_2 - 2\Gamma_3\gamma_3). \end{pmatrix} \tag{19}$$

The $SU(2)_L$ left-handed weak interaction $W_{L\mu}$, $U(1)_R$ right-handed weak interaction $W_{R\mu}$, and $U(1)_{B-L}$ interaction $W_{BL\mu}$ are of the form

$$W_{L\mu} = \frac{1}{2}(W_{L\mu}^1\Gamma_2\Gamma_3 + W_{L\mu}^2\Gamma_1\Gamma_3 + W_{L\mu}^3\Gamma_1\Gamma_2), \tag{20}$$

$$W_{R\mu} = \frac{1}{2}W_{R\mu}^3\Gamma_1\Gamma_2, \tag{21}$$

$$W_{BL\mu} = \frac{1}{2}W_{BL\mu}^J J, \tag{22}$$

where

$$J = \frac{1}{3}(\gamma_1\Gamma_1 + \gamma_2\Gamma_2 + \gamma_3\Gamma_3). \tag{23}$$

Thanks to the properties

$$P_q J = \frac{1}{3}P_q i, \tag{24}$$

$$P_l J = -P_l i, \tag{25}$$

$W_{BL\mu}$ is equivalent to

$$W_{BL\mu} = \frac{1}{2}W_{BL\mu}^J(B - L)i, \tag{26}$$

where B and L are baryon and lepton numbers, respectively. Along with another property,

$$\Gamma_1 \Gamma_2 P_\pm = \mp i P_\pm, \tag{27}$$

the standard model electromagnetic charges for all individual fermions can thus be correctly derived [17, 18].

It can be verified that the product of lepton projector P_l with any generator in color algebra (19) is zero

$$P_l T_k = 0. \tag{28}$$

As a result, leptons are $SU(3)_c$ singlets. They do not interact with gluons.

After dynamic symmetry breaking of $SU(2)_L \times U(1)_R \times U(1)_{B-L}$, which will be discussed in later subsections, the remaining massless interactions are G_μ and electromagnetic field A_μ. The G_μ and A_μ part of gauge-covariant derivative is cast into the form

$$D_\mu \psi^a = (\partial_\mu + \frac{1}{2} A_\mu \Gamma_1 \Gamma_2) \psi^a + \psi^a (\frac{1}{2} A_\mu J + G_\mu). \tag{29}$$

Note that electromagnetic field is non-chiral and makes no distinction between left- and right-handed spinors.

2.3 Gauge-Invariant Lagrangian

The gauge-invariant Lagrangian reads

$$\mathcal{L}_{World} = \mathcal{L}_{Fermion} + \mathcal{L}_{Yang-Mills} + \mathcal{L}_{Gravity} + \mathcal{L}_{Multi-Fermion}. \tag{30}$$

The fermion kinetic Lagrangian can be written as

$$\mathcal{L}_{Fermion} = \hat{i} \left\langle \bar{\psi}_L^a \gamma^\mu D_{L\mu} \psi_L^a + \bar{\psi}_R^a \gamma^\mu D_{R\mu} \psi_R^a \right\rangle, \tag{31}$$

where $\langle \ldots \rangle$ stands for Clifford-scalar part of enclosed expression, and $\bar{\psi}_{L/R}^a$ are defined as

$$\bar{\psi}_{L/R}^a = (\psi_{L/R}^a)^\dagger \gamma_0 = (\mp \tilde{\psi}_{L/R}^a) \gamma_0. \tag{32}$$

Here reversion of $\psi_{L/R}^a$, denoted $\tilde{\psi}_{L/R}^a$, reverses the order in any product of Clifford vectors. Note that $\hat{i}$ is the mathematical imaginary number. It is different from Clifford algebra $C\ell_{0,6}$ pseudoscalar i. Imaginary number $\hat{i}$ commutes with all Clifford algebra elements.

The local gauge symmetries can be extended to $SO(1,3)_{Lorentz} \times SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_{B-L}$, in the unified theory of gravity and Yang-Mills interactions [18]. Gravity is treated as gauge theory of local Lorentz symmetry. We will not go into the details of gravity and Yang-Mills Lagrangians in this paper, whereas multi-fermion interactions will be specified in the following section.

Note that spacetime itself can be replaced by Clifford space (C-space) in a more ambitious approach [21].

2.4 Is Quantum Theory Real?

The mathematical imaginary number $\hat{i}$ is ubiquitous in physics theories. The original objective of Clifford algebra (or geometric algebra) approach to physics is to abandon the imaginary number and replace it with certain even element in Clifford algebra. This initiative, pioneered by Hestenes [19], has been fairly successful in a wide variety of physics domains [19–23], such as rotational symmetries, Dirac equation, gauge field theories, and quantum theory in the first quantization form. It's why we write $Cl_{0,6}$ pseudoscalar as i in the first place.

When it comes to quantum field theory (QFT) beyond tree approximation (with loop corrections), the jury is still out with regard to the status of imaginary number. As we have witnessed in the last subsection, imaginary number $\hat{i}$ appears in the spinor Lagrangian. Can we replace it with $Cl_{0,6}$ pseudoscalar i? The answer is no. Given the Grassmann-odd nature of the spinor field, it turns out that an action in the form of

$$S_{Fermion} = \int \left\langle i\bar{\psi}_L^a \gamma^\mu D_{L\mu}\psi_L^a + i\bar{\psi}_R^a \gamma^\mu D_{R\mu}\psi_R^a \right\rangle d^4x \tag{33}$$

can be proved to be equivalent to zero, after neglecting surface integral terms.

One might reckon that what really matters in the path integral formalism of QFT is

$$e^{\hat{i}\int \mathcal{L}d^4x}, \tag{34}$$

with additional source terms added in the Lagrangian. Since we know that

$$\hat{i}\mathcal{L}_{Fermion} = -\left\langle \bar{\psi}_L^a \gamma^\mu D_{L\mu}\psi_L^a + \bar{\psi}_R^a \gamma^\mu D_{R\mu}\psi_R^a \right\rangle \tag{35}$$

is 'real', shall we claim that we can do away with imaginary number after all? The answer is still no.

We know that QFT propagators have poles. They are not properly defined without a prescription on integral in the vicinity of the poles. The beautiful Lorentz-invariant Feynman propagator hinges on the contour integral on the complex plane. Feynman's $\hat{i}\epsilon$ trick introduces the imaginary number through the back door. Equating the imaginary number with a Clifford algebra element in this context would seem rather unnatural[1].

The interaction part of the Lagrangian is 'real' though, be it Yang-Mills or multi-fermion interaction. Nevertheless, QFT loop integral would pick up an extra $\hat{i}$, via proper contour integral on the complex plane (or Wick rotation of time axis). Therefore, a self-energy loop diagram yields an 'imaginary' correction to $\mathcal{L}_{Fermion}$. Unless we come up with some other innovative ways of performing integral around the propagator poles, we have to live with the imaginary number $\hat{i}$.

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[1]Of course, one can introduce a separate Clifford bivector $\hat{i} = \Gamma_7\Gamma_8$ to the original $Cl_{0,6}$, so that it commutes with all $Cl_{0,6}$ elements. However, this approach is not conducive to any additional physics insight.

3 Chiral Symmetries and Fermion Mass Hierarchies

Dynamical symmetry breaking (DSB) is introduced into relativistic quantum field theory by Nambu and Jona-Lasinio [15]. The NJL model is based on a four-fermion interaction, which is strong enough to induce fermion-antifermion condensation via DSB mechanism. Four-fermion interactions are not renormalizable in the conventional sense. They can be regarded as effective representations of underlying renormalizable theory.

We subscribe to the general notion that multi-fermion interactions are instrumental in driving DSB and giving rise to fermion masses. The Higgs sector is just an effective Ginzburg-Landau-type description of the low energy physics represented by composite boson fields. Four-fermion as well as two-fermion condensations are introduced in this section. Two global chiral symmetries play the crucial role of dictating the relative magnitudes of four-fermion condensations, and consequently shaping fermion mass hierarchies.

3.1 Flavor Projection Operators

For the purpose of investigating allowable flavor-mixing multi-fermion interactions, we resort to another kind of Clifford algebra involving ternary communication relationships [24, 25] rather than the usual binary ones. Let's consider a ternary Clifford algebra with a single vector ζ satisfying

$$\zeta^3 = 1, \tag{36}$$

with ζ commuting with $C\ell_{0,6}$. We introduce three projection operators which involve both binary and ternary Clifford algebra elements

$$\zeta^0 = \frac{1}{3}(1 + \zeta + \zeta^2), \tag{37}$$

$$\zeta^+ = \frac{1}{3}(1 + e^{\frac{2\pi}{3}i}\zeta + e^{-\frac{2\pi}{3}i}\zeta^2), \tag{38}$$

$$\zeta^- = \frac{1}{3}(1 + e^{-\frac{2\pi}{3}i}\zeta + e^{\frac{2\pi}{3}i}\zeta^2). \tag{39}$$

Flavor projection operators for the three families of fermions are defined by [18]

$$P^1 = P_q\zeta^- + P_l\zeta^0, \tag{40}$$
$$P^2 = P_q\zeta^+ + P_l\zeta^-, \tag{41}$$
$$P^3 = P_q\zeta^0 + P_l\zeta^+, \tag{42}$$

where P_q and P_l are quark and lepton projection operators, respectively.

Note that ζ operators are assigned to quarks and leptons in disparate patterns. As will be demonstrated in later subsections, this particular layout is structured to facilitate

mixing between first and second generation quarks as well as second and third generation leptons. The mixing stems from properties

$$\zeta^0 A = A\zeta^0, \tag{43}$$
$$\zeta^+ A = A\zeta^-, \tag{44}$$
$$\zeta^- A = A\zeta^+, \tag{45}$$

for any $Cl_{0,6}$-odd element A, since pseudoscalar i in the definition of ζ operators anticommutes with $Cl_{0,6}$-odd A.

We know that flavor mixing is observed between all generations, giving rise to distinct configurations of CKM and PMNS matrices. The above flavor projection assignment captures the most significant mixing effects. To allow for further mixing possibilities, we can potentially adopt other flavor projection assignments. The rule of thumb is that only one ansatz should be used for a given Lagrangian term, while separate terms are permitted to adopt different ansatzes. For the current paper, we will focus on the schema defined in (40, 41, 42). The interaction terms subjected to other flavor projection assignments are presumably associated with suppressed coupling constants. We leave the study of these subdued interactions to future research.

If we take a step back and think twice about it, we would realize that the traditional way of generation assignment is rather arbitrary. For instance, there is no compelling rational for categorizing (u, d) and (ν, e) into the same generation. It is done so purely out of convenience, since (u, d, e) are the lightest bunch and constitute the bulk of building blocks of our universe.

It makes more sense to call a spade a spade by adopting the following naming convention,

$$Generation \quad \zeta^0 : \quad t, b, \nu, e, \tag{46}$$
$$Generation \quad \zeta^+ : \quad c, s, \nu_\tau, \tau, \tag{47}$$
$$Generation \quad \zeta^- : \quad u, d, \nu_\mu, \mu, \tag{48}$$

the flavor projection operators are thus simply $(\zeta^0, \zeta^\pm)$, without expressly referencing quark and lepton projection operators.

It's not merely about playing nomenclature music chairs. We know that the algebraic spinor of $Cl_{0,6}$ can actually accommodate $SU(4)$ gauge symmetry [17, 18], which encompasses $SU(3)_c \times U(1)_{B-L}$. If nature allows for such gauge interaction, the coset $SU(4)/(U(3)_c \times U(1)_{B-L})$ related gauge fields would transform down/up quarks into muons/muon neutrinos of the same ζ^- generation, instead of into electrons/neutrinos of ζ^0 generation as usually assumed for proton decay. Because of the heavy muon mass, the tree-level amplitude for proton decay into meson and positive muon would therefore be suppressed.

Going forward, we will strictly follow the flavor projection regime which stipulates

that fermions and flavor projection operators should always stick together in pairs as

$$\psi^1 P^1, \psi^2 P^2, \psi^3 P^3, \tag{49}$$

$$P^1 \bar{\psi}^1, P^2 \bar{\psi}^2, P^3 \bar{\psi}^3. \tag{50}$$

Let's test the flavor projection rule with a generalized fermion kinetic Lagrangian

$$\mathcal{L}_{Fermion} = \hat{i} \left\langle P^a \bar{\psi}_L^a \gamma^\mu D_{L\mu} \psi_L^b P^b + P^a \bar{\psi}_R^a \gamma^\mu D_{R\mu} \psi_R^b P^b \right\rangle. \tag{51}$$

Given the orthogonal properties

$$\zeta^0 \zeta^- = \zeta^0 \zeta^+ = \zeta^- \zeta^+ = 0, \tag{52}$$

one can easily verify that there is no flavor-mixing cross term in the Lagrangian. Flavor mixing is permitted for multi-fermion interactions, which will be studied in later subsections.

For sake of simplifying notations, from now on we will not explicitly write down flavor projection operators. Each fermion field should implicitly assume an accompanying flavor projection operator.

3.2 Right-Handed-Only Four-Fermion Interactions

The right-handed-only interactions are responsible for dynamically generating neutrino Majorana masses. The $SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_{B-L}$ gauge-invariant interaction Lagrangian contains

$$\mathcal{L}_{Majorana} = G_{11} \left\langle \bar{\nu}_R \Gamma_2 \Gamma_3 \nu_R \bar{\nu}_R \Gamma_2 \Gamma_3 \nu_R \right\rangle \tag{53}$$

$$+ G_{23} \left\langle \bar{\nu}_{\mu R} \Gamma_2 \Gamma_3 \nu_{\tau R} \bar{\nu}_{\tau R} \Gamma_2 \Gamma_3 \nu_{\mu R} \right\rangle + h.c., \tag{54}$$

where G_{11} and G_{23} are coupling constants. The bivector $\Gamma_2 \Gamma_3$ can be replaced by arbitrary combination of $\Gamma_2 \Gamma_3$ and $\Gamma_1 \Gamma_3$. But it does not change the overall picture.

As stated earlier, flavor projection operators are implicitly attached to fermions. The flavor-mixing terms (54) between μ and τ neutrinos are controlled by the properties of flavor projection operators. Other right-handed-only four-fermion interactions are also allowable. Some of them are examined elsewhere [18]. Since they don't contribute to Majorana type two-fermion condensations, these additional terms are not enumerated here.

3.3 Right-Left-Mixing Four-Fermion Interactions

Right-left-mixing interactions are employed by top condensation model [5–9] for dynamically breaking electroweak symmetry. The simplest version of top condensation model assumes top quark-antiquark condensation only. For the purpose of saturating electroweak

scale, the scenario has been extended, among others [9], to neutrino condensations [10–14] as well.

Based on the flavor projection operator properties, three or more condensations are required to generate Dirac masses for all three generations of fermions. We adopt the minimalist approach by allowing for top quark, tau neutrino, and tau lepton condensations only. Our premise is that any other fermion condensations are either nonexistent or negligible. As will be shown later, this unique set of condensations is consistent with fermion mass hierarchies controlled by two chiral symmetries.

The $SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_{B-L}$ gauge-invariant right-left-mixing interactions pertaining to these three condensations are

$$\mathcal{L}_t = \frac{1}{4} g_t \left\langle \bar{q}_L^3 \gamma^\mu q_L^3 \bar{t}_R \gamma_\mu t_R \right\rangle \tag{55}$$

$$+ \frac{1}{4} g_{t\nu} \left\langle \bar{l}_L^1 \gamma^\mu q_L^3 \bar{t}_R \gamma_\mu \nu_R \right\rangle + h.c. \tag{56}$$

$$- g_{tb} \left\langle \bar{q}_L^3 q_L^3 \bar{t}_R b_R \right\rangle + h.c. \tag{57}$$

$$- g_{te} \left\langle \bar{l}_L^1 q_L^3 \bar{t}_R e_R \right\rangle + h.c., \tag{58}$$

$$\mathcal{L}_{\nu_\tau} = \frac{1}{4} g_{\nu_\tau} \left\langle \bar{l}_L^3 \gamma^\mu l_L^3 \bar{\nu}_{\tau R} \gamma_\mu \nu_{\tau R} \right\rangle \tag{59}$$

$$+ \frac{1}{4} g_{\nu_\tau c} \left\langle \bar{q}_L^2 \gamma^\mu l_L^3 \bar{\nu}_{\tau R} \gamma_\mu c_R \right\rangle + h.c. \tag{60}$$

$$- g_{\nu_\tau \mu} \left\langle \bar{l}_L^2 l_L^3 \bar{\nu}_{\tau R} \mu_R \right\rangle + h.c. \tag{61}$$

$$- g_{\nu_\tau d} \left\langle \bar{q}_L^1 l_L^3 \bar{\nu}_{\tau R} d_R \right\rangle + h.c., \tag{62}$$

$$\mathcal{L}_\tau = \frac{1}{4} g_\tau \left\langle \bar{l}_L^3 \gamma^\mu l_L^3 \bar{\tau}_R \gamma_\mu \tau_R \right\rangle \tag{63}$$

$$+ \frac{1}{4} g_{\tau s} \left\langle \bar{q}_L^2 \gamma^\mu l_L^3 \bar{\tau}_R \gamma_\mu s_R \right\rangle + h.c. \tag{64}$$

$$- g_{\tau \nu_\mu} \left\langle \bar{l}_L^2 l_L^3 \bar{\tau}_R \nu_{\mu R} \right\rangle + h.c. \tag{65}$$

$$- g_{\tau u} \left\langle \bar{q}_L^1 l_L^3 \bar{\tau}_R u_R \right\rangle + h.c., \tag{66}$$

where $g_{...}$ are coupling constants. The left-handed doublets q_L^a and l_L^a are understood as the sum of up- and down-type fermions. For example, q_L^1 and l_L^1 denote

$$q_L^1 = u_L + d_L, \tag{67}$$

$$l_L^1 = \nu_L + e_L. \tag{68}$$

The permissible flavor-mixing patterns of these four-fermion terms are dictated by the properties of flavor projection operators. In the ζ naming parlance, top cohort $\mathcal{L}_t$ corresponds to ζ^0 generation, while tau neutrino and tau lepton cohorts $\mathcal{L}_{\nu_\tau}$ and $\mathcal{L}_\tau$ represent mixture of $\zeta^\pm$ generations.

The vector interactions contain pairs γ^μ / γ_μ, while the others are scalar interactions. One can potentially add bivector interactions with pairs $\gamma^\mu \gamma^\nu / \gamma_\mu \gamma_\nu$, though they more or

less behave in a similar manner as the scalar counterparts. As such, we will not write down bivector interactions separately.

Other right-left-mixing four-fermion interactions are also allowed. For instance, the following flavor-changing charged interactions

$$\mathcal{L}_{Quark-Mixing} = g_{us} \left\langle \bar{q}_L^2 q_L^1 \bar{u}_R s_R \right\rangle + h.c. \tag{69}$$
$$+ g_{cd} \left\langle \bar{q}_L^1 q_L^2 \bar{c}_R d_R \right\rangle + h.c. \tag{70}$$

mix first and second generation quarks. They flip up-type quarks to down-type quarks, and *vice versa*. They are not directly driving the three electroweak condensations in question.

Note that flavor-changing neutral interactions, such as

$$\left\langle \bar{q}_L^2 \gamma^\mu q_L^1 \bar{d}_R \gamma_\mu s_R \right\rangle, \tag{71}$$

are identically zero, since for Clifford-even $\bar{d}_R \gamma_\mu s_R$ we have

$$P^1 \bar{d}_R \gamma_\mu s_R P^2 = \zeta^- \bar{d}_R \gamma_\mu s_R \zeta^+ = \bar{d}_R \gamma_\mu s_R \zeta^- \zeta^+ = 0. \tag{72}$$

3.4 Four-Fermion Condensations and Chiral Symmetry Breaking

As mentioned earlier, four-fermion interactions are invariant under $SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_{B-L}$ gauge transformations. These gauge symmetries are local and related to gauge interactions. In this subsection, we investigate two additional global chiral symmetries $U(1)_\alpha$ and $U(1)_\beta$ associated with right-handed fermions,

$$U(1)_\alpha: \quad \psi_R^a \quad \Rightarrow \quad \psi_R^a e^{\alpha i}, \tag{73}$$
$$U(1)_\beta: \quad P_- \psi_R^a P_q \quad \Rightarrow \quad P_- \psi_R^a P_q e^{\beta i}, \tag{74}$$
$$P_+ \psi_R^a P_q \quad \Rightarrow \quad P_+ \psi_R^a P_q e^{-\beta i}, \tag{75}$$
$$P_- \psi_R^a P_l \quad \Rightarrow \quad P_- \psi_R^a P_l e^{-\beta i}, \tag{76}$$
$$P_+ \psi_R^a P_l \quad \Rightarrow \quad P_+ \psi_R^a P_l e^{\beta i}. \tag{77}$$

The α-type chiral transformation rotates all right-handed fermions by the same phase $e^{\alpha i}$. The β-type chiral transformation rotates up-type quarks (u_R, c_R, t_R) and down-type leptons (e_R, μ_R, τ_R) by $e^{\beta i}$, while it rotates down-type quarks (d_R, s_R, b_R) and up-type leptons (ν_R, $\nu_{\mu R}$, $\nu_{\tau R}$) by $e^{-\beta i}$. One can replace $U(1)_\alpha$ with $U(1)_A$:

$$\psi_R^a \Rightarrow \psi_R^a e^{\alpha i} = e^{\alpha i} \psi_R^a, \tag{78}$$
$$\psi_L^a \Rightarrow \psi_L^a e^{-\alpha i} = e^{\alpha i} \psi_L^a, \tag{79}$$

without affecting the following discussion.

Note that the definitions of chiral symmetries are not explicitly generation dependent, differing from earlier efforts of linking generation-dependent chiral symmetry [26] with fermion mass hierarchy.

The fermion kinetic Lagrangian and right-handed-only four-fermion interactions are invariant under $U(1)_\alpha$ and $U(1)_\beta$ transformations. However, some of the right-left-mixing four-fermion interactions do not observe the chiral symmetries. The symmetry violation pattern is summarized in table 2. The top row respects both $U(1)_\alpha$ and $U(1)_\beta$ symmetries, while the bottom row violates both symmetries. On the Lagrangian level, $U(1)_\alpha$ invariance is controlled by whether or not two right-handed fermions are both up-type or both down-type (equivalently, vector vs scalar interactions), while $U(1)_\beta$ invariance is determined by whether or not two right-handed fermions are both quarks or both leptons.

t Cohort	ν_τ Cohort	τ Cohort	$U(1)_\alpha$	$U(1)_\beta$
g_t	g_{ν_τ}	g_τ	$\checkmark$	$\checkmark$
$g_{t\nu}e^{-2\beta i}$	$g_{\nu_\tau c}e^{2\beta i}$	$g_{\tau s}e^{-2\beta i}$	$\checkmark$	$\times$
$g_{tb}e^{2\alpha i}$	$g_{\nu_\tau \mu}e^{2\alpha i}$	$g_{\tau\nu_\mu}e^{2\alpha i}$	$\times$	$\checkmark$
$g_{te}e^{2\alpha i+2\beta i}$	$g_{\nu_\tau d}e^{2\alpha i-2\beta i}$	$g_{\tau u}e^{2\alpha i+2\beta i}$	$\times$	$\times$

Table 2: The $U(1)_\alpha$ and $U(1)_\beta$ symmetry violation pattern grouped by top quark, tau neutrino, and tau lepton cohorts. Four-fermion interactions are represented by coupling constants along with the extra phases originated from the chiral transformations of right-handed fermions.

The fundamental theory ought to be $U(1)_\alpha$ and $U(1)_\beta$ invariant. Therefore, the coupling constants in the second, third, and fourth rows should be promoted to composite boson fields representing four-fermion condensations. These composite boson fields are endowed with proper chiral charges to make the Lagrangian $U(1)_\alpha$ and $U(1)_\beta$ invariant by netting out the extra phases in table 2. The notion of four-fermion condensation has been proposed in Ref. [18]. Four-fermion condensations produce various pseudo-Nambu-Goldstone bosons and can be dark matter candidates.

By virtue of DSB mechanism, the four-fermion condensations are induced by the underlying eight-fermion interactions which honor all $SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_{B-L} \times U(1)_\alpha \times U(1)_\beta$ symmetries. For instance, a g_{tb} related eight-fermion interaction takes the form

$$\mathcal{L}_{tb} = G_{tb} \left\langle (\bar{b}_R t_R \bar{q}_L^3 q_L^3)(\bar{q}_L^3 q_L^3 \bar{t}_R b_R) \right\rangle. \tag{80}$$

The coupling constant g_{tb} is effectively the four-fermion condensation of

$$G_{tb}\bar{b}_R t_R \bar{q}_L^3 q_L^3 \quad \rightarrow \quad g_{tb} P_q \zeta^0, \tag{81}$$

which breaks the $U(1)_\alpha/U(1)_A$ symmetry, while leaving the $SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_{B-L}$ gauge symmetries intact. Note that due to the explicit chiral symmetry breaking originated from quantum anomaly and instanton effects, the pseudo-Nambu-Goldstone boson associated with the $U(1)_\alpha/U(1)_A$ symmetry breaking acquires a mass in a similar fashion as the axion [30–32]. Given that the four-fermion condensations are local gauge (especially electroweak) singlets, they are more in line with the invisible axion models [33–36]. It is worthwhile to study further the four-fermion condensations as a possible solution to strong CP problem.

The magnitudes of $U(1)_\alpha$ symmetry breaking four-fermion condensations are presumably smaller than the $U(1)_\beta$ counterparts, thus making $U(1)_\alpha$ the primary agent and $U(1)_\beta$ the secondary agent in establishing the hierarchies of four-fermion coupling constants.

We believe that the constants g_t, g_{ν_τ}, g_τ, G_{11}, and G_{23} are of the same order, since these couplings observe both chiral symmetries[2]. In light of the symmetry breaking pattern of table 2, the coupling constants within the same column can be progressively smaller from the top row to the bottom row, in accordance with 't Hooft's naturalness principle. As stated earlier, we make the assumption that $U(1)_\alpha$ is primary and $U(1)_\beta$ is secondary. As a result, symmetry breaking of $U(1)_\alpha$ matters more, determining the order of the second and third row. It is supported by the fact that charm quark is heavier than muon as shown in next subsection.

3.5 Two-Fermion Condensations and Gauge Symmetry Breaking

Two-fermion condensations are resulted from four-fermion interactions studied in prior subsections. There are five composite Higgs bosons corresponding to two Majorana condensations and three electroweak condensations,

$$H_{11} \quad \sim \quad \Lambda^{-2}\hat{i}i\bar{\nu}_R\Gamma_2\Gamma_3\nu_R \quad \rightarrow \quad \frac{1}{2}\upsilon_{11}P_l\gamma_0\zeta^0, \tag{82}$$

$$H_{23} \quad \sim \quad \Lambda^{-2}\hat{i}i\bar{\nu}_{\mu R}\Gamma_2\Gamma_3\nu_{\tau R} \quad \rightarrow \quad \frac{1}{2}\upsilon_{23}P_l\zeta^-\gamma_0\zeta^+, \tag{83}$$

$$h_t \quad \sim \quad \Lambda^{-2}\hat{i}q_L^3 i\bar{t}_R \quad \rightarrow \quad \upsilon_t P_+\zeta^0, \tag{84}$$

$$h_{\nu_\tau} \quad \sim \quad \Lambda^{-2}\hat{i}l_L^3 i\bar{\nu}_{\tau R} \quad \rightarrow \quad \upsilon_{\nu_\tau} P_+\zeta^-, \tag{85}$$

$$h_\tau \quad \sim \quad \Lambda^{-2}\hat{i}l_L^3 i\bar{\tau}_R \quad \rightarrow \quad \upsilon_\tau P_-\zeta^-, \tag{86}$$

where Λ is a cutoff scale.

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[2]Alternatively, one might assume that there is a hierarchy between G_{11}/G_{23} and the other four-fermion coupling constants. An interaction such as $G_{11}\langle(\bar{\nu}_R + \bar{t}_R)\Gamma_2\Gamma_3(\nu_R + t_R)(\bar{\nu}_R + \bar{t}_R)\Gamma_2\Gamma_3(\nu_R + t_R)\rangle$ has $SU(4)$ symmetry for the combination of ν_R and t_R, which is not shared by the right-left-mixing four-fermion interactions. Based on naturalness principle, $g_{...}$ can thus be much smaller than G_{11}/G_{23}.

Majorana Higgs bosons H_{11} and H_{23} correspond to right-handed neutrino condensations. The Majorana condensations break the symmetries from $SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_{B-L}$ to the standard model symmetries $SU(3)_c \times SU(2)_L \times U(1)_Y$, where $U(1)_Y$ is the hypercharge gauge symmetry. Gauge field Z' acquires a mass as a consequence [18] .

Electroweak Higgs bosons h_t , h_{ν_τ}, and h_τ represent top quark, tau neutrino, and tau lepton condensations, respectively. The electroweak condensations break the electroweak symmetries. As a result, gauge fields $W^\pm$ and Z gain masses.

Collectively, these five composite Higgs bosons break the symmetries from $SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_{B-L}$ to $SU(3)_c \times U(1)_{em}$, where $U(1)_{em}$ is the electromagnetic gauge symmetry.

With fermion pairs in the four-fermion Lagrangians approximated by condensation values, the resultant fermion mass terms are

$$\Lambda^{-2}\mathcal{L}_{Mass} = G_{11}\upsilon_{11}\hat{i}\left\langle i\gamma_0 \bar{\nu}_R \Gamma_2 \Gamma_3 \nu_R \right\rangle \tag{87}$$

$$+ G_{23}\upsilon_{23}\hat{i}\left\langle i\gamma_0 \bar{\nu}_{\mu R}\Gamma_2\Gamma_3\nu_{\tau R}\right\rangle + G_{23}\upsilon_{23}\hat{i}\left\langle i\gamma_0\bar{\nu}_{\tau R}\Gamma_2\Gamma_3\nu_{\mu R}\right\rangle \tag{88}$$

$$+ g_t\upsilon_t\hat{i}\left\langle i\bar{t}t\right\rangle + g_{t\nu}\upsilon_t\hat{i}\left\langle i\bar{\nu}\nu\right\rangle + g_{tb}\upsilon_t\hat{i}\left\langle i\bar{b}b\right\rangle + g_{te}\upsilon_t\hat{i}\left\langle i\bar{e}e\right\rangle \tag{89}$$

$$+ g_{\nu_\tau}\upsilon_{\nu_\tau}\hat{i}\left\langle i\bar{\nu}_\tau\nu_\tau\right\rangle + g_{\nu_\tau c}\upsilon_{\nu_\tau}\hat{i}\left\langle i\bar{c}c\right\rangle + g_{\nu_\tau \mu}\upsilon_{\nu_\tau}\hat{i}\left\langle i\bar{\mu}\mu\right\rangle + g_{\nu_\tau d}\upsilon_{\nu_\tau}\hat{i}\left\langle i\bar{d}d\right\rangle \tag{90}$$

$$+ g_\tau\upsilon_\tau\hat{i}\left\langle i\bar{\tau}\tau\right\rangle + g_{\tau s}\upsilon_\tau\hat{i}\left\langle i\bar{s}s\right\rangle + g_{\tau\nu_\mu}\upsilon_\tau\hat{i}\left\langle i\bar{\nu}_\mu\nu_\mu\right\rangle + g_{\tau u}\upsilon_\tau\hat{i}\left\langle i\bar{u}u\right\rangle . \tag{91}$$

The first two lines are Majorana mass terms of neutrinos. The rest are Dirac mass terms. Note that imaginary number $\hat{i}$ in the mass terms is from fermion condensations which pick up an extra $\hat{i}$ via self-energy contour integral on the complex plane (or equivalently Wick rotation of time axis). The conventional formulation does not distinguish between $\hat{i}\bar{\psi}$ and $i\bar{\psi}$. Thus the combination of $\hat{i}$ and i in the mass terms reduces to -1, yielding real masses. In the context of algebraic spinors, equating $\hat{i}$ with i is not viable. Given the Grassmann-odd nature of fermion fields, a mass term such as

$$\left\langle \bar{t}t \right\rangle \tag{92}$$

can be verified to be equivalent to zero.

The Majorana mass terms (88) mix second and third generation neutrinos $\nu_{\mu R}$ and $\nu_{\tau R}$[3], as evidenced in the observation of neutrino oscillations [27–29]. Majorana scale (also called seesaw scale) is much higher than electroweak scale

$$\upsilon_{11}, \upsilon_{23} \gg \upsilon_t, \upsilon_{\nu_\tau}, \upsilon_\tau. \tag{96}$$

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[3]As stated earlier, different flavor projection assignments to relatively weaker interactions are also possible. For example, the following ansatz,

$$P^1 = P_q\zeta^0 + P_l\zeta^+, \tag{93}$$

$$P^2 = P_q\zeta^- + P_l\zeta^0, \tag{94}$$

$$P^3 = P_q\zeta^+ + P_l\zeta^-, \tag{95}$$

can lead to h_t Higgs boson decaying into τ lepton and Majorana mass terms mixing ν_R and $\nu_{\tau R}$ neutrinos.

Therefore, very small effective masses are generated for neutrinos, known as seesaw mechanism.

Dirac masses within a given condensation cohort (v_t, v_{ν_τ}, or v_τ) are proportional to four-fermion coupling constants. As demonstrated in last subsection, the coupling constants are hierarchical. As a result, Dirac masses follow the hierarchy pattern as shown in table 3, where masses in each column are in descending order. Each Higgs boson contributes to masses of four fermions in the same column. Note that the unknown neutrino masses in the table are meant to denote Dirac masses, rather than the significantly smaller seesaw effective masses.

Top Quark Higgs		Tau Neutrino Higgs		Tau Lepton Higgs	
t	173,000	ν_τ	?	τ	1,780
ν	?	c	1,280	s	96
b	4,180	μ	106	ν_μ	?
e	0.51	d	4.6	u	2.2

Table 3: Three species of composite electroweak Higgs bosons and corresponding fermion cohorts. Fermion masses are in units of MeV. Error margins of fermion masses are not shown.

Since the coupling constants g_t, g_{ν_τ}, and g_τ are of the same order, the mass discrepancies between top quark, tau neutrino, and tau lepton are driven primarily by the condensation magnitudes v_t, v_{ν_τ}, and v_τ. Knowing that top quark is much heavier than tau lepton, we can deduce

$$v_t \gg v_\tau. \tag{97}$$

Hence top condensation dwarfs tau lepton condensation in terms of contributing to electroweak scale.

Without knowing Dirac mass of tau neutrino, we can only estimate the relative magnitude of v_{ν_τ} based on circumstantial evidences. Given that charm quark mass is much larger than strange quark mass, we hypothesize that

$$v_{\nu_\tau} \gg v_\tau. \tag{98}$$

Consequently, tau neutrino condensation might play a substantial role in electroweak scale saturation, as originally envisioned by Martin [10]. Composite Higgs boson h_{ν_τ} has a sizable branching ratio into charm quark. We anticipate detection of Higgs boson h_{ν_τ} via the charm quark decay channel, in addition to the 125 GeV Higgs boson [37, 38],

which most likely corresponds to top condensation. Also, in light of the strong connection between muon and Higgs boson h_{ν_τ}, it is worthwhile to investigate tau neutrino condensation's contribution to the muon anomalous magnetic moment [39, 40].

3.6 Antisymmetric Condensation

Two-fermion condensation might also involve an antisymmetric tensor component such as

$$\Lambda^{-2}i\hat{q}_L^3 i\bar{t}_R \quad \rightarrow \quad \upsilon_{tAT}\gamma_1\gamma_2 P_+\zeta^0, \tag{99}$$

in addition to the scalar(pseudoscalar) component $\upsilon_t P_+\zeta^0$ discussed in last subsection.

The magnitude of this condensation υ_{tAT} could be extremely small compared with the scalar counterpart υ_t, rendering υ_{tAT} related effects unobservable in laboratories. The miniscule ratio of $\upsilon_{tAT}/\upsilon_t$ is in line with naturalness principle, since υ_{tAT} breaks Lorentz (and rotational) symmetry on top of breaking electroweak symmetry.

The ethereal antisymmetric condensation might manifest itself as 'dark torsion' (or 'dark spin current') [18] via interaction with gravitational spin connection. Corrections to torsion and Lorentz violation effects could in turn modify the behavior of gravity on galactic and cosmological scales [41]. The antisymmetric condensation (99) suggests a preferred direction in the universe, which might be reflected as large-scale anisotropies in the cosmic microwave background (CMB) [42–46].

Note that the inter-linkage between gravity field and electroweak Higgs field has also been studied in a geometrical 5D approach [47], which deduces all the known interactions from an induced symmetry breaking of the non-unitary GL(4)-group of diffeomorphisms.

4   Conclusion

The Higgs sector can be regarded as an effective description of the low energy physics represented by composite boson fields. A challenge facing the composite Higgs model is to account for the vast range of fermion masses which span five orders of magnitude. The current paper is an effort towards explaining the fermion mass hierarchies in the context of composite electroweak Higgs bosons.

Our approach is based on the framework of six-dimensional Clifford algebra $\mathcal{C}\ell_{0,6}$ and ternary Clifford algebra element ζ. Standard model fermions can be represented by algebraic spinors of $\mathcal{C}\ell_{0,6}$, while ternary ζ related flavor projection operators dictate allowable flavor-mixing interactions. The Lagrangian of the algebraic spinors accommodates local gauge symmetries $SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_{B-L}$. If the larger $SU(4)$ gauge is taken into consideration as well, the particular flavor projection operator assignment permits gauge-field driven transformations of first generations quarks into second generation leptons, instead of into first generation leptons. Given the heavy mass of muon, the

tree-level amplitude for proton decay into meson and positive muon would therefore be suppressed.

We propose that there are two global chiral symmetries $U(1)_\alpha$ and $U(1)_\beta$, in addition to the local gauge symmetries. The underlying model with eight-fermion interactions is $U(1)_\alpha$ and $U(1)_\beta$ invariant. By virtue of dynamical symmetry breaking mechanism, eight-fermion interactions are strong enough to induce four-fermion condensations. These condensations break the global chiral symmetries, while leaving the local gauge symmetries intact. The symmetry-breaking coupling constants of four-fermion interactions are resulted from the four-fermion condensations. Due to the explicit symmetry breaking originated from quantum anomaly and instanton effects, the pseudo-Nambu-Goldstone bosons associated with the four-fermion condensations acquire masses in a similar way as the axion. Since the four-fermion condensations are electroweak singlets, they are more in line with the invisible axion models. It is worthwhile to study further the four-fermion condensations as a possible solution to strong CP problem and dark matter candidate.

Two Majorana and three electroweak two-fermion condensations are engendered by four-fermion interactions. Jointly, they break the remaining local gauge symmetries further down to $SU(3)_c \times U(1)_{em}$. Because of the flavor projection operator properties, three or more electroweak condensations are required to generate Dirac masses for all three generations of fermions. We adopt the minimalist approach by allowing for top quark, tau neutrino, and tau lepton condensations only. Each electroweak condensation gives rise to Dirac masses of four different fermions. In accordance with naturalness principle, the global chiral symmetries $U(1)_\alpha$ and $U(1)_\beta$ are instrumental in determining the relative magnitudes of four-fermion coupling constants, and consequently establishing fermion mass hierarchies.

Both top quark and tau neutrino condensations play a significant role in electroweak scale saturation. Tau neutrino condensation may also contribute substantially to the muon anomalous magnetic moment. In addition to the 125 GeV Higgs boson observed at the Large Hadron Collider, we anticipate detection of tau neutrino Higgs boson via the charm quark decay channel. On the other hand, a feeble antisymmetric condensation breaks Lorentz symmetry on top of breaking electroweak symmetry. It might be gravitationally relevant and reflected as large-scale CMB anisotropies.

Acknowledgments

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