THE BINARY GOLDBACH CONJECTURE: A PROOF FOR THE EXISTENCE OF PRIME SUMS FOR ALL 2N

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DEFINITION 1

All elements of the set of odd integers from 3 to 2N-3 – set M – are either Composite (C) or Prime (P)**.**

DEFINITION 2

The paired sums of elements can be of three types:

1) *Prime + Prime*
2) *Prime + Composite*
3) *Composite + Composite*

DEFINITION 3

Let M = the number of elements in set M.
Let Y = the number of P elements in set M.
∴ $M - Y$ = the number of C elements in set M.
Let X = the number of C elements that form C+C pairs.

ARGUMENT

Proposition A. If there are no P+P paired elements, then all paired elements form X (C+C) and $2Y$ (P+C).
If conjecture is false:

$$\boldsymbol{M = X + 2Y}$$

Proposition B. If there are P+P paired elements, then 2Y is greater than the number of P+P and P+C elements.
If conjecture is true:

$$\boldsymbol{M < X + 2Y}$$

∴ ***Proving X > M – 2Y for all 2N proves the conjecture.***

Lemma 1: The factorization of 2N affects the rate of composite pairing – the more highly composite 2N is, the greater the proportion of C+C pairs – but a calculation can be made for a minimum X in all cases, regardless of the specific factorization of 2N**.**

Lemma 2: We can itemize C elements by their least prime factor; least prime factors are bounded by √2N.

For large 2N:

The proportion of each least prime factor composite is:

$$\boldsymbol{C_p} \cong \frac{1}{p_n} \times \left(\frac{p_{n-1}-1}{p_{n-1}}\right)\left(\frac{p_{n-2}-1}{p_{n-2}}\right)\dots\left(\frac{p_2-1}{p_2}\right) = \frac{\mathbf{1}}{\boldsymbol{p_n}} \times \prod_{\mathbf{2}}^{\boldsymbol{n-1}}\left(\frac{\boldsymbol{p_n}-\mathbf{1}}{\boldsymbol{p_n}}\right)$$

The proportion of remaining elements (primes, and composite elements that have a greater least prime factor) is:

$$\boldsymbol{R_p} \cong \frac{p_n-1}{p_n} \times \prod_{2}^{n-1}\left(\frac{p_n-1}{p_n}\right) = \prod_{\mathbf{2}}^{\boldsymbol{n}}\left(\frac{\boldsymbol{p_n}-\mathbf{1}}{\boldsymbol{p_n}}\right)$$

i.e., where $C_3 \propto \frac{1}{3}$, $R_3 \propto \frac{2}{3}$; for all $C_p \propto \frac{1}{p}$, $R_p \propto \frac{p-1}{p}$

Subtracting Y from the $M \times R_p$ ratio gives the approximation for C elements with a least prime factor greater than a given p*:*

$$\boldsymbol{G_p} \cong [M \times R_p] - Y = \left[\boldsymbol{M} \times \prod_{\mathbf{2}}^{\boldsymbol{n}}\left(\frac{\boldsymbol{p_n}-\mathbf{1}}{\boldsymbol{p_n}}\right)\right] - \boldsymbol{Y}$$

Lemma 3: For all composites as sorted by their least prime factor, G_p is the approximate number of coprime composites that are available to pair with $M \times C_p$ elements (for all p coprime with 2N). That is, $|G_p|$ elements will pair with $|M \times C_p|$ elements in a specific ratio, given by F_p:

$$\boldsymbol{F_p} \cong 2 \times \frac{1}{p_n-1} \times \left(\frac{p_{n-1}-2}{p_{n-1}-1}\right)\left(\frac{p_{n-2}-2}{p_{n-2}-1}\right)\dots\left(\frac{p_2-2}{p_2-1}\right) = \frac{\mathbf{2}}{\boldsymbol{p_n}-\mathbf{1}} \times \prod_{\mathbf{2}}^{\boldsymbol{n-1}}\frac{\boldsymbol{p_n}-\mathbf{2}}{\boldsymbol{p_n}-\mathbf{1}}$$

Combining the G_p elements with their F_p pairing ratios gives:

$$\boldsymbol{X_p} \geq \boldsymbol{G_p} \times \boldsymbol{F_p} = \boldsymbol{M}[\boldsymbol{R_p} \times \boldsymbol{F_p}] - \boldsymbol{Y}\,[\boldsymbol{F_p}]$$

The total X is the sum of all composite pairings to p_n.

$$\boldsymbol{X} \cong \sum_{\mathbf{2}}^{\boldsymbol{n}} \boldsymbol{X_{p_n}} = \boldsymbol{M} \sum_{\mathbf{2}}^{\boldsymbol{n}} \boldsymbol{R_{p_n}} \boldsymbol{F_{p_n}} - \boldsymbol{Y} \sum_{\mathbf{2}}^{\boldsymbol{n}} \boldsymbol{F_{p_n}}$$

This gives a minimum X in terms of M and Y. Since proving $X > M - 2Y$ proves the conjecture, we have a path to determining its validity.

Let the M coefficient for the X sum be s.
Let the Y coefficient for the X sum be g.

To test whether $X > M - 2Y$, then:

$$sM - gY > M - 2Y$$

$$Y(2 - g) > M(1 - s)$$

Dividing the M coefficient gives:

$$Y(2-g)/(1-s) > M$$

This is identical with:

$$\boldsymbol{Y \times \prod_{1}^{n} \frac{p_n}{p_n - 1} > M}$$

Given by the Prime Number Theorem, $Y \approx \frac{2M}{\ln 2M}$ *for large M, we can define the inequality as:*

$$\boldsymbol{\frac{2M}{\ln 2M} \times \prod_{1}^{n} \frac{p_n}{p_n - 1} \gg M}$$

NB: As the number of primes is always larger than the PNT estimate, using the PNT approximation on the greater side of the inequality will remain logically consistent for any size M.

Thus:

$$\boldsymbol{\prod_{1}^{n} \frac{p_n}{p_n - 1} \gg \frac{\ln 2M}{2}}$$

And since $2M \approx p_n^2$, the inequality can be formalized as:

$$\boldsymbol{\prod_{1}^{n} \frac{p_n}{p_n - 1} \gg \ln p_n}$$

This reduces the logic to a single variable. If the above inequality holds for all p_n, the conjecture is true.

TO INFINITY

The slope and spacing of the functions at both average and maximum $p_{n+1} - p_n$ gaps, or any sequence of gaps, indicate whether the $\boldsymbol{\ln p_n}$ curve could overtake the $\boldsymbol{\prod_1^n \frac{p_n}{p_n - 1}}$ curve. Both the derivatives and antiderivatives of each curve indicate infinite divergence. For relatively large individual gaps between primes, the absolute increase in $\boldsymbol{\ln p_n}$ is greater than the absolute increase in $\boldsymbol{\prod_1^n \frac{p_n}{p_n - 1}}$, so at some points – that is, over some $p_{n+1} - p_n$ – the gap is reduced. However, the *average growth* of the curves along the *average gap of primes* ensures that the curves diverge. This is because:

a) the net absolute gap in the curves, at every point, requires a prime gap or sequence of primes gaps several multiples of p_n for the $\boldsymbol{\ln p_n}$ curve to increase above the $\boldsymbol{\prod_1^n \frac{p_n}{p_n - 1}}$ curve;
b) individual $p_{n+1} - p_n$ gaps are significantly smaller than p_n for large p; and
c) the average prime gap of $\ln p_n$ is small relative to p_n, such that it ensures a net increase in the gap between the curves over any p_n increase.

This ensures $X > M - 2Y$ for all 2N, and the conjecture is true to infinity.