

# Diamond Operator as a Spinless Square Root of d' Alembertian

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**Abstract:** Dirac equation includes the  $4 \times 4$  complex differential operator matrix, which is one of square roots of d' Alembertian with spin. We found another  $4 \times 4$  complex differential matrix as a spinless square root of d' Alembertian, which we call diamond operator. The extended Maxwell's equations including charge creation-annihilation field and the linear gravitational field equations including energy creation-annihilation field can be simply written by using the diamond operator. It is shown that the linear gravitational field equations derive Klein-Gordon equation, time independent Schrödinger equation, and the principle of quantum mechanics.

## I. Introduction

Dirac found a relativistic wave equation for electrons with a  $4 \times 4$  complex differential operator matrix as a square root of d' Alembertian.<sup>1)</sup> The equation is satisfied by Fermions with spin of half integer. Since elementary particles to mediate forces are Bosons, different relativistic wave equation is necessary to treat weak, strong, electromagnetic, and gravitational forces. We found a new  $4 \times 4$  complex differential operator matrix, we call it diamond operator, as a spinless square root of d' Alembertian, which enable us to treat Bosons including four forces. Recently, we have reported that the extended Maxwell's equations including charge creation-annihilation scalar field can treat generation-recombination of electron-hole pairs in semiconductors and the similar equation for the linear gravitational field with energy creation-annihilation scalar field can derive Klein-Gordon and time independent Schrödinger equations.<sup>2-8)</sup> It is shown that the diamond operator successfully and simply describes the extended Maxwell's and linear gravitational field equations.

## II. Dirac equation

Dirac equation is given by

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0, \quad (1)$$

where  $\hbar$  is Dirac constant,  $\psi$  is a wave function,  $m$  is a mass,  $c$  is light speed in free space, and

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (2)$$

$$\gamma^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad (3)$$

$$\gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad (4)$$

$$\gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (5)$$

$\sigma_1$ ,  $\sigma_2$ , and  $\sigma_3$  are following Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (6)$$

Since  $\psi$  satisfies

$$\square\psi + \frac{m^2c^2}{\hbar^2}\psi = 0, \quad (7)$$

$\gamma^\mu\partial_\mu$  satisfies

$$\left(\gamma^\mu\partial_\mu\right)^2 = \square, \quad (8)$$

where  $\square$  denotes d' Alembertian defined by

$$\square \equiv \partial^\mu\partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2. \quad (9)$$

Therefore,  $\gamma^\mu\partial_\mu$  is a square root of d' Alembertian with spin given by

$$\gamma^\mu \partial_\mu = \begin{pmatrix} \partial_0 & 0 & \partial_3 & \partial_1 - \partial_2 \\ 0 & \partial_0 & \partial_1 + i\partial_2 & -\partial_3 \\ -\partial_3 & -\partial_1 + i\partial_2 & -\partial_0 & 0 \\ -\partial_1 - i\partial_2 & \partial_3 & 0 & -\partial_0 \end{pmatrix}. \quad (10)$$

It should be noticed that  $\gamma^\mu \partial_\mu$  is not symmetrical for space axes for the purpose to include spin.

### III. Diamond operator

We define the diamond operator  $\diamond$  as

$$\diamond \equiv \delta^\mu \partial_\mu, \quad (11)$$

where

$$\delta^0 = \begin{pmatrix} -I_2 & 0 \\ 0 & -I_2 \end{pmatrix}^* = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}^*, \quad (12)$$

$$\delta^1 = \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}^*, \quad (13)$$

$$\delta^2 = \begin{pmatrix} 0 & i\sigma_3 \\ -i\sigma_3 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}^*, \quad (14)$$

$$\delta^3 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}^* = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}^*. \quad (15)$$

In the above equation, \* denotes the complex conjugate operator which satisfies

$$*A = A^*, \quad (16)$$

where  $A$  is a complex scalar, vector, or matrix, and  $A^*$  is the complex conjugate of  $A$ . The diamond operator satisfies

$$\diamond^2 = \square. \quad (17)$$

(11) – (15) give

$$\diamond = \begin{pmatrix} -\partial_0 & -i\partial_3 & i\partial_2 & -i\partial_1 \\ i\partial_3 & -\partial_0 & -i\partial_1 & -i\partial_2 \\ -i\partial_2 & i\partial_1 & -\partial_0 & -i\partial_3 \\ i\partial_1 & i\partial_2 & i\partial_3 & -\partial_0 \end{pmatrix}^*. \quad (18)$$

Therefore, this operator is symmetric for three space axes and does not include spin.

For electromagnetic and gravitational force, the

four current  $C$  and the four field  $F$  satisfy

$$gC = \diamond F. \quad (19)$$

In (19),  $g$  is a coupling constant and

$$C \equiv \begin{pmatrix} \mathbf{C} \\ iC_0 \end{pmatrix}, \quad (20)$$

$$F \equiv \begin{pmatrix} \mathbf{D} + i\mathbf{R} \\ -iS \end{pmatrix}, \quad (21)$$

where  $\mathbf{D}$ ,  $\mathbf{R}$ , and  $S$  are divergent, rotational, and scalar fields, respectively. (19)-(21) give

$$gC = \nabla \times \mathbf{R} - \partial_0 \mathbf{D} + \nabla S, \quad (22)$$

$$gC_0 = \nabla \cdot \mathbf{D} - \partial_0 S, \quad (23)$$

$$\nabla \times \mathbf{D} + \partial_0 \mathbf{R} = 0, \quad (24)$$

$$\nabla \cdot \mathbf{R} = 0. \quad (25)$$

The four field  $\bar{F}$  with gauge parameter  $\xi$  and four potential  $A$  satisfy

$$\bar{F} = \diamond A, \quad (26)$$

where

$$\bar{F} \equiv \begin{pmatrix} \mathbf{D} + i\mathbf{R} \\ -i\xi S \end{pmatrix}, \quad (27)$$

$$A \equiv \begin{pmatrix} \mathbf{A} \\ iA_0 \end{pmatrix}. \quad (28)$$

(26)-(28) give

$$\mathbf{D} = -\partial_0 \mathbf{A} - \nabla A_0, \quad (29)$$

$$\mathbf{R} = \nabla \times \mathbf{A}, \quad (30)$$

$$\xi S = -\nabla \cdot \mathbf{A} - \partial_0 A_0. \quad (31)$$

$C$ ,  $F$ , and  $A$  satisfy the following Lorentz transformation for the boost with velocity  $v$  along  $x$ -axis,

$$\begin{pmatrix} C_1' \\ C_2' \\ C_3' \\ iC_0' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & i\beta \\ \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{i\beta}{\sqrt{1-\beta^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-i\beta}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{1}{\sqrt{1-\beta^2}} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ iC_0 \end{pmatrix}, \quad (32)$$

$$\begin{pmatrix} F_1' \\ F_2' \\ F_3' \\ iF_0' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-\beta^2}} & \frac{i\beta}{\sqrt{1-\beta^2}} & 0 \\ 0 & \frac{-i\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ iF_0 \end{pmatrix}, \quad (33)$$

$$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \\ iA_0' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{i\beta}{\sqrt{1-\beta^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-i\beta}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{1}{\sqrt{1-\beta^2}} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ iA_0 \end{pmatrix}, \quad (34)$$

where  $\beta = v/c$ .

#### IV. Extended Maxwell's equations

Maxwell's equations cannot treat generation-recombination of charge pairs by the charge conservation equation

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \quad (35)$$

Because current  $\mathbf{J}$  and charge concentration  $\rho$  should satisfy the following equation in semiconductors,<sup>9,10)</sup>

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = GR. \quad (36)$$

where  $GR$  is charge generation-recombination rate. In order to obtain the extended Maxwell's equations,  $C$  and  $F$  are substituted by four current  $J$ , electric and magnetic fields  $\mathbf{E}$  and  $\mathbf{B}$ , and charge creation-annihilation field  $N$  as

$$C = J \equiv \begin{pmatrix} \mathbf{J} \\ ic\rho \end{pmatrix}, \quad (37)$$

$$F = \begin{pmatrix} \mathbf{E}/c + i\mathbf{B} \\ -iN \end{pmatrix}. \quad (38)$$

Since the coupling constant  $g$  is substituted by permeability  $\mu$ , the extended Maxwell's equations are written by

$$\mathbf{J} = \frac{1}{\mu} \nabla \times \mathbf{B} - \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu} \nabla N, \quad (39)$$

$$\rho = \varepsilon \nabla \cdot \mathbf{E} - \varepsilon \frac{\partial N}{\partial t}, \quad (40)$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad (41)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (42)$$

where  $\varepsilon$  is permittivity which satisfies  $\varepsilon \mu = 1/c^2$ . (1) and (2) give the following current continuity equation,

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = -\frac{1}{\mu} \square N. \quad (43)$$

Therefore  $-\square N/\mu$  is charge creation-annihilation rate. By using (31), The charge creation-annihilation field  $N$  satisfies

$$\xi N = -\nabla \cdot \mathbf{A} - \partial_0 A_0. \quad (44)$$

$N$  is equivalent to Nakanishi-Lautrup field<sup>11, 12)</sup> except  $\square N \neq 0$  in the region where charges are created or annihilated.

#### V. Linear Gravitational field and quantum mechanics

Einstein's gravitational equation is given by

$$G_{\mu\nu} = \kappa T_{\mu\nu} \quad (45)$$

where  $G_{\mu\nu}$  is Einstein tensor and  $\kappa$  is Einstein's gravitational constant.  $T_{\mu\nu}$  is momentum density tensor written by

$$T_{\mu\nu} = -\rho v_\mu v_\nu, \quad (46)$$

where  $v_\mu$  and  $v_\nu$  are  $\mu$  and  $\nu$  component of the velocity. When the momentum density is enough small, metric tensor  $g_{\mu\nu}$  is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (47)$$

where  $\eta_{\mu\nu}$  and  $h_{\mu\nu}$  are tensors which satisfy

$$\eta_{\mu\nu} \equiv \begin{cases} 1 & (\mu = \nu = 1, 2, 3) \\ 0 & (\mu \neq \nu) \\ -1 & (\mu = \nu = 0) \end{cases}, \quad (48)$$

$$|h_{\mu\nu}| \ll 1. \quad (49)$$

Here we define  $\bar{h}_{\mu\nu}$  as

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h^\lambda{}_\lambda. \quad (50)$$

In Lorentz gauge condition of  $\bar{h}^{\mu\alpha}{}_{,\alpha} = 0$ , we obtain

$$-\square \bar{h}_{\mu\nu} = 2\kappa T_{\mu\nu}. \quad (51)$$

The above equation is regarded as the wave equation for linear gravitational field.<sup>13)</sup> In order to obtain Lorentz vector, we assume small volume  $\Omega$ . Then the gravitational vector potential  $A_g$  and gravitational current  $C_g$  are given by

$$A_{g\mu} \equiv \frac{1}{2\kappa c} \bar{h}_{\mu 0} \Omega, \quad (52)$$

$$C_{g\mu} \equiv -\frac{1}{c} T_{\mu 0} \Omega = \rho v_\mu \Omega. \quad (53)$$

Therefore

$$C_{g\mu} = \square A_{g\mu}. \quad (54)$$

Then the gravitational fields  $F_g = (\mathbf{D}_g + i\mathbf{R}_g, -iS_g)$  and  $\bar{F}_g = (\mathbf{D}_g + i\mathbf{R}_g, -i\xi S_g)$  satisfy

$$C_{g\mu} = \diamond F_{g\mu}, \quad (55)$$

$$\bar{F}_{g\mu} = \diamond A_{g\mu}, \quad (56)$$

where  $\mathbf{D}_g$ ,  $\mathbf{R}_g$ , and  $S_g$  are the divergent, rotational, and scalar fields of the linear gravitational field.

Since the four current vector  $C_g$  is equivalent to the four momentum vector  $P = (\mathbf{P}, iU/c)$ , where  $\mathbf{P}$  and  $U$  denote 3D momentum and energy,

$$\mathbf{P} = \nabla \times \mathbf{R}_g - \partial_0 \mathbf{D}_g + \nabla S_g, \quad (57)$$

$$\frac{U}{c} = \nabla \cdot \mathbf{D}_g - \partial_0 S_g. \quad (58)$$

Since  $\mathbf{D}_g$  and  $\mathbf{R}_g$  can be regarded as much smaller than  $S_g$  in atomic scale, we obtain

$$\mathbf{P} = \nabla S_g, \quad (59)$$

$$U = -\frac{\partial S_g}{\partial t}. \quad (60)$$

Here we call  $S_g$  energy creation-annihilation field, because energy creation-annihilation rate  $\sigma$  is defined by

$$\sigma \equiv c^2 \nabla \cdot \mathbf{P} + \frac{\partial U}{\partial t} = -c^2 \square S_g. \quad (61)$$

When we define the wave function  $\phi$  as

$$\phi = \exp\left(\frac{iS_g}{\hbar}\right), \quad (62)$$

we obtain

$$\square \phi = \left( \frac{-U^2 + P^2 c^2}{c^2 \hbar^2} + \frac{i}{\hbar} \square S_g \right) \phi. \quad (63)$$

In the case of  $\square S_g = 0$ , we obtain Klein-Gordon equation of

$$\square \phi + \frac{m^2 c^2}{\hbar^2} \phi = 0. \quad (64)$$

If we assume that the above equation is satisfied even in the case of  $\square S_g \neq 0$  under the potential  $V$ , we obtain by using the total energy  $E = U + V$

$$(E - V)^2 = P^2 c^2 + m^2 c^4 - i\hbar \sigma. \quad (65)$$

The above equation suggests the principle of quantum mechanics, that it is equivalent to classical mechanics when the absolute value of  $\hbar \sigma$  is much smaller than  $P^2 c^2$ , otherwise the imaginary part of energy creation-annihilation field creates or annihilates quantized interactive energy depending on the potential  $V$ .

If  $P^2 c^2$  and the absolute value of  $\hbar \sigma$  are much smaller than  $m^2 c^4$ , we obtain

$$U = \sqrt{P^2 c^2 + m^2 c^4 - i\hbar \sigma} \approx mc^2 + \frac{P^2 + i\hbar \square S_g}{2m}. \quad (66)$$

When we assume  $V$  and  $S_g$  do not depend on time and the total energy is given by  $E = U - mc^2 + V$ , we obtain

$$E = \frac{P^2 - i\hbar \nabla^2 S_g}{2m} + V. \quad (67)$$

Since  $\nabla^2 \phi$  is given by

$$\nabla^2 \phi = \left( -\frac{P^2}{\hbar^2} + \frac{i}{\hbar} \nabla^2 S_g \right) \phi, \quad (68)$$

the following time independent Schrödinger equation is obtained

$$E\phi = -\frac{\hbar^2}{2m} \nabla^2 \phi + V\phi. \quad (69)$$

#### IV. Conclusion

We found the diamond operator, which is a  $4 \times 4$  complex differential operator matrix as a square root of d'Alembertian without spin, although Dirac equation's operator matrix includes spin. The extended Maxwell's and the linear gravitational field equations are simply written by using the diamond operator. The linear gravitational field equations derive Klein-Gordon equation, time independent Schrödinger equation, and the principle of quantum mechanics. It was found that imaginary part of the energy creation-annihilation field creates or annihilates quantized interactive energy depending on the potential.

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## Appendix

Since products of complex numbers  $a + ib$  and  $c + id$  is written by

$$(a + ib)(c + id) = ac - bd + i(bc + ad), \quad (\text{A1})$$

$(a + ib)$  and  $(c + id)$  can be regarded as matrix and vector, respectively as

$$a + ib = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad (\text{A2})$$

$$c + id \equiv \begin{pmatrix} c \\ d \end{pmatrix}. \quad (\text{A3})$$

Because

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac - bd \\ bc + ad \end{pmatrix}. \quad (\text{A4})$$

Therefore, imaginary unit  $i$  can be regarded as the  $2 \times 2$  matrix of

$$i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (\text{A5})$$

The complex conjugate operator  $*$  can be also regarded as the  $2 \times 2$  matrix of

$$* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\text{A6})$$

because

$$*(a + ib) = a - ib = \begin{pmatrix} a \\ -b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \quad (\text{A7})$$

Then product of  $i$  and  $*$  is also regarded as the following  $2 \times 2$  matrix

$$i* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (\text{A8})$$

Here

$$(*)^2 = (i*)^2 = -i^2 = 1. \quad (\text{A9})$$

When we define the following bracket operator

$$[A, B] = \begin{cases} 1 & (AB = BA) \\ 0 & (AB \neq \pm BA) \\ -1 & (AB = -BA) \end{cases}, \quad (\text{A10})$$

we obtain

$$[i, *] = [* , i*] = [i* , i] = -1. \quad (\text{A11})$$

Here, we define  $n \times n$  real basis matrices  $b_n^\mu$  ( $\mu = 0, 1, \dots, n^2-1$ ) as the real matrices whose linear combination can give all of  $n \times n$  real matrices and

square are equal to  $\pm I_n$ . Then,  $1, i, *$ , and  $i*$  are equivalent to the  $2 \times 2$  real basis matrices  $b_2^\mu$  given by  $b_2^0 = I_2$  and

$$b_2^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b_2^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b_2^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\text{A12})$$

where

$$(b_2^0)^2 = (b_2^1)^2 = -(b_2^2)^2 = (b_2^3)^2 = 1, \quad (\text{A13})$$

and for  $\mu = 0, 1, 2, 3$  and  $j, k = 1, 2, 3$ ,

$$[b_2^0, b_2^\mu] = 1, \quad [b_2^j, b_2^k]_{j \neq k} = -1. \quad (\text{A14})$$

Now we define partial product of matrices  $A$  and  $B$  as

$$A \times B \equiv \begin{pmatrix} b_{11}A & \cdot & \cdot & \cdot & b_{1n}A \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{m1}A & \cdot & \cdot & \cdot & b_{mn}A \end{pmatrix}, \quad (\text{A15})$$

where

$$B \equiv \begin{pmatrix} b_{11} & \cdot & \cdot & \cdot & b_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{m1} & \cdot & \cdot & \cdot & b_{mn} \end{pmatrix}. \quad (\text{A16})$$

Then  $4 \times 4$  real basis matrices can be written by

$$b_4^\lambda = b_2^\mu \times b_2^\nu. \quad (\text{A17})$$

If another  $4 \times 4$  real matrix is written by

$$b_4^\alpha = b_2^\beta \times b_2^\gamma, \quad (\text{A18})$$

the following relation is satisfied

$$b_4^\lambda \cdot b_4^\alpha = (b_2^\mu \cdot b_2^\beta) \times (b_2^\nu \cdot b_2^\gamma). \quad (\text{A19})$$

Then

$$[b_4^\lambda \cdot b_4^\alpha] = [b_2^\mu \cdot b_2^\beta] \cdot [b_2^\nu \cdot b_2^\gamma]. \quad (\text{A20})$$

Since  $\gamma^\mu$  and  $\delta^\nu$  are given by

$$\gamma^0 = b_2^0 \times b_2^3, \quad (\text{A21})$$

$$\gamma^1 = -b_2^1 \times b_2^2, \quad (\text{A22})$$

$$\gamma^2 = -ib_2^2 \times b_2^2, \quad (\text{A23})$$

$$\gamma^3 = -b_2^3 \times b_2^2, \quad (\text{A24})$$

and

$$\delta^0 = -b_2^0 \times b_2^0 *, \quad (\text{A25})$$

$$\delta^1 = ib_2^1 \times b_2^2 *, \quad (\text{A26})$$

$$\delta^2 = -ib_2^3 \times b_2^2 *, \quad (\text{A27})$$

$$\delta^3 = ib_2^2 \times b_2^0 *, \quad (\text{A28})$$

we obtain

$$(\gamma^0)^2 = (\delta^0)^2 = -(\gamma^\mu)_{\mu \neq 0}^2 = -(\delta^\mu)_{\mu \neq 0}^2 = 1, \quad (\text{A29})$$

$$[\gamma^\mu, \gamma^\nu]_{\mu \neq \nu} = [\delta^\mu, \delta^\nu]_{\mu \neq \nu} = -1. \quad (\text{A30})$$

The diamond operator matrix without spin needs the additional commutation relation of  $*$  and  $i*$  compared with Dirac operator matrix with spin. It seems to be a kind of spontaneous symmetry breaking.