General Solutions and Exact Solutions of the Problems of Definite Solutions of Mathematical Physics Equations

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Abstract
In this paper, using proposed three new transformation methods we have solved general solutions and exact solutions of the problems of definite solutions of the Laplace equation, Poisson equation, Schrödinger Equation, the homogeneous and non-homogeneous wave equations, Helmholtz equation and heat equation. In the process of solving, we find that in the more general case, general solutions of partial differential equations have various forms such as basic general solution, series general solution, transformational general solution, generalized series general solution and so on.

keywords:
transformation methods, general solution; exact solutions; problems of definite solutions; mathematical physics equations.

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Introduction

Since the theory of partial differential equations (PDEs) has been established nearly 300 years, there is no universal and effective method for solving PDEs, and the existing methods have many limitations. Three new transformation methods proposed in this paper cannot solve all the PDEs, but the types of equations that can be solved effectively are extremely rich. As typical cases, we have solved general solutions and exact solutions of the problems of definite solutions of various important mathematical physics equations in this paper.

1. New principles and methods I

1.1. An axiom and a conjecture

It is well known that using algebraic methods to solve some algebraic equations may get extraneous root, that is, the correct algebraic operation might not get the correct result; so any conclusion from correct logic is not always the correct conclusion, conclusive verification is an indispensable link to ensure correct results, so we first put forward a verification axiom:

Validation Axioms. *Any conclusion by the correct logic, which has not been corroborated, is not always the correct conclusion.*

We will follow the validation axiom to verify any result obtained in this paper to ensure that they are correct.

Here we analyze one of the simplest PDEs

$$u_x = 0. \tag{1}$$

The general solution of this equation seems to be easy to write, that is

$$u = f\left(y\right), \tag{2}$$

where f is an arbitrary first differentiable functions, according to the general solution, it is easy to obtain its infinite particular solutions, such as $u = y^2, u = y^3 + siny$ and so on.

General solution (2) is clearly correct, but it is complete?

When we propose Eq. (1), we actually ignore a condition that cannot be ignored: Eq. (1) is formed in which space? If in $\mathbb{R}^2$, (2) is clearly complete; if in $\mathbb{R}^3$ whose independent variables are x, y, z, clearly the general solution is

$$u = f\left(y, z\right), \tag{3}$$

So Eq. (2) is not the complete general solution in $\mathbb{R}^3$, if in $\mathbb{R}^4$ whose independent variables are x, y, z, t, the general solution of Eq. (1) is:

$$u = f\left(y, z, t\right), \tag{4}$$

This can be analogized to arbitrary $\mathbb{R}^n$ space, $(n \geq 2)$. That is, in different dimensionality space the general solutions of Eq. (1) are different.

It is necessary to further clarify that some particular solutions of PDEs are independent with space dimensionality, and some particular solutions are related to it. As for Eq. (1), the special solution $u = y^2, u = y^3 + siny$ are correct in any $\mathbb{R}^n$ space, $(n \geq 2)$, and $u = y^2 + z, u = y^3 + sinz$ are correct in $\mathbb{R}^n$,$(n \geq 3)$), and they are wrong in $\mathbb{R}^2$.

In order to strengthen this understanding, let us analyze another typical case

$$u_{xy} = 0. \tag{5}$$

In $\mathbb{R}^2$, its general solution is

$$u = f_1(x) + f_2(y). \tag{6}$$

In $\mathbb{R}^3$ whose independent variables are x, y, z, the general solution of Eq. (5) is

$$u = f_1(x, z) + f_2(y, z). \tag{7}$$

In $\mathbb{R}^4$ whose independent variables are x, y, z, t, the general solution of Eq. (5) is

$$u = f_1(x, z, t) + f_2(y, z, t). \tag{8}$$

where f_1 and f_2 are arbitrary second differentiable functions.

Based on the above two simple and profound cases, we propose a new guess:

Guess 1: *If a PDE has a general solution in $\mathbb{R}^n$, $(n \geq 2)$, then the general solution is related to n.*

General solutions of common PDEs are obviously not easy to be solved in different dimension spaces. In this paper, the new methods will solve this problem effectively.

1.2. The law of arbitrary function

General solutions of PDEs contain arbitrary functions, we will use the principle of function correlation to get the laws of arbitrary functions. Here we need to explain that: the essence of any function $f(x_1, \cdots x_n)$ interrelated with any constant c is

$$f(x_1, \ldots x_n) \cdot 0 + c = c, \tag{9}$$

is not the solution of $f(x_1, \cdots x_n) = c$, because if so, then any two functions are related, but this is wrong.

Now we propose the concept of **Equivalent Function** .

Definition 1. *In the domain D, $(D \subset \mathbb{R}^n)$, if $u_1, u_2, \cdots u_m$ are independent of each other, $v_1, v_2, \cdots v_m$ are also independent of each other, and u_i is related to $v_1, v_2, \cdots v_m$, or v_i is related to $u_1, u_2, \cdots u_m$, $(i = 1, 2, \cdots m \leq n)$, we call $f(u_1, u_2, \cdots u_m)$ and $g(v_1, v_2, \cdots v_m)$ equivalent, or they are equivalent functions, write as*

$$f(u_1, u_2, \ldots u_m) \leftrightarrow g(v_1, v_2, \ldots v_m), \tag{10}$$

where f and g are arbitrary functions with m independent variables.

The interpretation of definition 1 as follows:
Assuming u_i is related to $v_1, v_2, \cdots v_m$, namely $u_i = h_i(v_1, v_2, \cdots v_m)$, then

$$\begin{aligned} f(u_1, u_2, \ldots u_m) &= f(h_1(v_1, v_2, \ldots v_m), h_2(v_1, v_2, \ldots v_m), \ldots h_m(v_1, v_2, \ldots v_m)) \\ &= g(v_1, v_2, \ldots v_m). \end{aligned}$$

So $f(u_1, u_2, \cdots u_m)$ is equivalent to $g(v_1, v_2, \cdots v_m)$. If v_i is related to $u_1, u_2, \cdots u_m$,the proof is similar. □

According to Definition 1, we can get Theorem 1.

Theorem 1. *In the domain D, $(D \subset \mathbb{R}^n)$, if $x_1, x_2, \cdots x_n$ are independent of each other, and $y_1, y_2, \cdots y_n$ are also independent of each other, then*

$$f\left(x_1, x_2, \ldots x_n\right) \leftrightarrow g\left(y_1, y_2, \ldots y_n\right), \tag{11}$$

where f and g are arbitrary functions with n independent variables.

Proof. According to the principle of function correlation, in $\mathbb{R}^n$, if $x_1, x_2, \cdots x_n$ are independent of each other, y_i must be related to $x_1, x_2, \cdots x_n$, that is $y_i = y_i(x_1, x_2, \cdots x_n)$; $y_1, y_2, \cdots y_n$ are also independent of each other, so x_i must be related to $y_1, y_2, \cdots y_n$ too, $(i = 1, 2, \cdots n)$, according to Definition 1

$$f\left(x_1, x_2, \ldots x_n\right) \leftrightarrow g\left(y_1, y_2, \ldots y_n\right)$$

where f and g are arbitrary functions with n independent variables. □

1.3. Z_1 Transformation

In order to obtain general solutions or exact solutions of some PDEs, we propose Z_1 Transformation:

Z_1 **Transformation.** *In the domain D, $(D \subset \mathbb{R}^n)$,any established mth-order PDE with n space variables $F(x_1, \cdots x_n, u, u_{x_1}, \cdots u_{x_n}, u_{x_1 x_2}, \cdots) = 0$,set $y_i = y_i(x_1, \cdots x_n)$ and $u = f(y_1, \cdots y_l)$ are both undetermined mth-differentiable functions $(u, y_i \in C^m(D), 1 \leq l \leq n)$, $y_1, y_2 \cdots y_l$ are independent of each other, then substitute $u = f(y_1, \cdots y_l)$ and its partial derivatives into $F = 0$,*
1. In case of working out $y_i = y_i(x_1, \cdots x_n)$ and $f(y_1, \cdots y_l)$, then $u = f(y_1, \cdots y_l)$ is the solution of $F = 0$,
2. In case of dividing out $u = f(y_1, \cdots y_l)$ and its partial derivative, also working out $y_i = y_i(x_1, \cdots x_n)$, then $u = f(y_1, \cdots y_l)$ is the solution of $F = 0$, and f is an arbitrary mth-differentiable function,
3. In case of dividing out $u = f(y_1, \cdots y_l)$ and its partial derivative, also getting $k = 0$, but in fact $k \neq 0$, then $u = f(y_1, \cdots y_l)$ is not the solution of $F = 0$, and f is an arbitrary mth-differentiable function.

In Z_1 Transformation, $y_i = y_i(x_1, \cdots x_n)$ and $u = f(y_1, \cdots y_l)$ are both undetermined, $y_i(x_1, \cdots x_n)$ may be an unknown function completely or has a determinate form with unknown constants, the solution of $f(y_1, \cdots y_l)$ may be an arbitrary or a certain mth-differentiable function, the solution of $y_i(x_1, \cdots x_n)$ and $f(y_1, \cdots y_l)$ may not be single, etc., which are determined by the PDE and the specific solution process.

In Z_1 Transformation, if $l = n$, the transformation belongs to the independent variable transformation, then we can set $u(x_1, \cdots x_n) = u(y_1, \cdots y_n)$.

Using Z_1 Transformation, we can get general solutions or exact solutions of many PDEs, such as:

Example 1.1. In $\mathbb{R}^n$, using Z_1 Transformation to get the general solution of

$$a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_{x_3} = A\left(x_1, x_2, \ldots x_n\right), \tag{12}$$

where a_1, a_2 and a_3 are arbitrary known constants and $A(x_1, x_2, \cdots x_n)$ is any known function.

Distinctly Eq. (12) cannot be solved by the characteristic equation method. According to Z_1 Transformation, set $u(x_1, x_2, \cdots x_n) = u(y_1, y_2, y_3, x_4, x_5, \cdots x_n)$, $A(x_1, x_2, \cdots x_n) = A(y_1, y_2, y_3, x_4, x_5, \cdots x_n)$, and

$$y_1 = c_1x_1 + c_2x_2 + c_3x_3 \ , y_2 = c_4x_1 + c_5x_2 + c_6x_3, y_3 = c_7x_1 + c_8x_2 + c_9x_3, \tag{13}$$

where $c_1 - c_9$ are undetermined constants, and set

$$\frac{\partial\left(y_1, y_2, y_3, x_4, x_5, \ldots x_n\right)}{\partial\left(x_1, x_2, x_3, x_4, x_5, \ldots x_n\right)} \neq 0,$$

namely

$$-c_3c_5c_7 + c_2c_6c_7 + c_3c_4c_8 - c_1c_6c_8 - c_2c_4c_9 + c_1c_5c_9 \neq 0, \tag{14}$$

From (13), we get

$$\begin{cases} x_1 = -\frac{-c_6c_8y_1+c_5c_9y_1+c_3c_8y_2-c_2c_9y_2-c_3c_5y_3+c_2c_6y_3}{c_3c_5c_7-c_2c_6c_7-c_3c_4c_8+c_1c_6c_8+c_2c_4c_9-c_1c_5c_9} \\ x_2 = -\frac{c_6c_7y_1-c_4c_9y_1-c_3c_7y_2+c_1c_9y_2+c_3c_4y_3-c_1c_6y_3}{c_3c_5c_7-c_2c_6c_7-c_3c_4c_8+c_1c_6c_8+c_2c_4c_9-c_1c_5c_9} \\ x_3 = -\frac{-c_5c_7y_1+c_4c_8y_1+c_2c_7y_2-c_1c_8y_2-c_2c_4y_3+c_1c_5y_3}{c_3c_5c_7-c_2c_6c_7-c_3c_4c_8+c_1c_6c_8+c_2c_4c_9-c_1c_5c_9} \end{cases}. \tag{15}$$

So

$$\begin{aligned} &a_1u_{x_1} + a_2u_{x_2} + a_3u_{x_3} \\ &= \left(a_1c_1 + a_2c_2 + a_3c_3\right)u_{y_1} + \left(a_1c_4 + a_2c_5 + a_3c_6\right)u_{y_2} + \left(a_1c_7 + a_2c_8 + a_3c_9\right)u_{y_3}. \end{aligned}$$

Set

$$a_1c_1 + a_2c_2 + a_3c_3 = a_1c_4 + a_2c_5 + a_3c_6 = 0.$$

We obtain

$$c_1 = \frac{-a_2c_2 - a_3c_3}{a_1}, c_4 = \frac{-a_2c_5 - a_3c_6}{a_1}. \tag{16}$$

Then

$$a_1u_{x_1} + a_2u_{x_2} + a_3u_{x_3} = \left(a_1c_7 + a_2c_8 + a_3c_9\right)u_{y_3} = A\left(y_1, y_2, y_3, x_4, x_5, \ldots x_n\right).$$

So the general solution of Eq. (12) is:

$$u = f\left(y_1, y_2, x_4, x_5, \ldots x_n\right) + \frac{\int A\left(y_1, y_2, y_3, x_4, x_5, \ldots x_n\right)dy_3}{a_1c_7 + a_2c_8 + a_3c_9}, \tag{17}$$

where f is an arbitrary first differentiable functions, and

$$y_1 = \frac{-a_2c_2 - a_3c_3}{a_1}x_1 + c_2x_2 + c_3x_3,$$

$$y_2 = \frac{-a_2c_5 - a_3c_6}{a_1}x_1 + c_5x_2 + c_6x_3,$$

c_2, c_3 and $c_5 - c_9$ are arbitrary constants that satisfy Eq. (14). In a certain number field, an arbitrary constant with some limit is called **relatively arbitrary constant;** an arbitrary constant without any limit is called **absolutely arbitrary constant;** such as the solution of an algebraic equation is

$$x = \frac{1}{a - b}.$$

If a, b may be arbitrary constants, they can only be relatively arbitrary constants, because $a \neq b$. If the solution of an algebraic equation is

$$x = a - b.$$

If a, b can be any constants, then they are absolutely arbitrary constants.

According to the above conclusions,

$$a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_{x_3} = 0, \tag{18}$$

the general solution of Eq. (18) in $\mathbb{R}^n$ is:

$$u = f\left(\frac{-a_2c_2 - a_3c_3}{a_1}x_1 + c_2x_2 + c_3x_3, \frac{-a_2c_5 - a_3c_6}{a_1}x_1 + c_5x_2 + c_6x_3, x_4, x_5, \ldots x_n\right). \tag{19}$$

That is, the general solution is related to the spatial dimension n. According to (19) we can get the general solution of Eq. (18) in $\mathbb{R}^3$ is:

$$u = f\left(\frac{-a_2c_2 - a_3c_3}{a_1}x_1 + c_2x_2 + c_3x_3, \frac{-a_2c_5 - a_3c_6}{a_1}x_1 + c_5x_2 + c_6x_3\right). \tag{20}$$

Using the characteristic equation method, the general solution of Eq. (18) in $\mathbb{R}^3$ is

$$u = f\left(\frac{-a_2}{a_1}x_1 + x_2, \frac{a_3}{a_1}x_1 + x_3\right), \tag{21}$$

(21) is a special case of (20) as $c_2 = c_6 = 1, c_3 = c_5 = 0$, so using the characteristic equation method can only get the incomplete general solution of Eq. (18) in $\mathbb{R}^3$, and cannot obtain the general solution of Eq. (18) in $\mathbb{R}^n$.

In $\mathbb{R}^n$, consider the follow equation

$$a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_m u_{x_m} = A\left(x_1, x_2, \ldots x_n\right), \tag{22}$$

where $a_1, a_2 \cdots a_m$ are arbitrary known constants, $(m \leq n)$, and $A(x_1, x_2, \cdots x_n)$ is any known function. According to the previous analysis, we can see that the characteristic equation method cannot solve the general solution of Eq. (22) and the complete general solution of $a_1u_{x_1} + a_2u_{x_2} + \ldots + a_mu_{x_m} = 0$.

According to Z_1 Transformation, set $u(x_1, x_2, \cdots x_n) = u(y_1, y_2, \cdots y_m, x_{m+1}, x_{m+2}, \cdots x_n)$, $A(x_1, x_2, \cdots x_n) = A(y_1, y_2, \cdots y_m, x_{m+1}, x_{m+2}, \cdots x_n)$ and

$$\begin{cases} y_1 = c_1x_1 + c_2x_2 + \ldots + c_mx_m \\ y_2 = c_{m+1}x_1 + c_{m+2}x_2 + \ldots + c_{2m}x_m, \\ \cdots \\ y_m = c_{m(m-1)+1}x_1 + c_{m(m-1)+2}x_2 + \ldots + c_{m^2}x_m \end{cases}$$

and

$$\frac{\partial\left(y_1, y_2, \ldots y_m, x_{m+1}, x_{m+2}, \ldots x_n\right)}{\partial\left(x_1, x_2, \ldots x_n\right)} \neq 0,$$

Similar to Eq. (17), we can solve the general solution of Eq. (22).

Example 1.2. In $\mathbb{R}^4$, using Z_1 Transformation to get the exact solution of

$$a_1(u)u_{tt} + a_2(u)u_{xx} + a_3(u)u_{yy} + a_4(u)u_{zz} = a_5(u), \tag{23}$$

where $a_i(u)$ is any known function, $(i = 1, 2, \cdots 5)$.

According to Z_1 Transformation, set

$$u(x, y, z, t) = f(v) = f(k_1t + k_2x + k_3y + k_4z + k_5),$$

where $k_1 - k_5$ are constants to be determined, f is an undetermined unary function, then

$$\begin{aligned} & a_1(u)\, u_{tt} + a_2(u)\, u_{xx} + a_3(u)\, u_{yy} + a_4(u)\, u_{zz} \\ & = k_1^2 a_1(f)\, f_v'' + k_2^2 a_2(f)\, f_v'' + k_3^2 a_3(f)\, f_v'' + k_4^2 a_4(f)\, f_v'' = a_5(f), \end{aligned}$$

namely

$$f_v'' = \frac{a_5(f)}{k_1^2 a_1(f) + k_2^2 a_2(f) + k_3^2 a_3(f) + k_4^2 a_4(f)}.$$

So the particular solution of Eq. (23) is

$$v = k_6 \pm \int \left(k_7 + 2 \int \frac{a_5(u)}{k_1^2 a_1(u) + k_2^2 a_2(u) + k_3^2 a_3(u) + k_4^2 a_4(u)} \right)^{-\frac{1}{2}} du, \tag{24}$$

where $k_1 - k_7$ are arbitrary constant.

Nonlinear wave equation

$$u_{tt} - a^2 \triangle u = g(u) \tag{25}$$

is a special case of (23), the law of the equation has been the hotspot of the study [1-3], according to (24) we can get that the exact solution is

$$v = k_6 \pm \int \left(k_7 + 2 \int \frac{g(u)}{k_1^2 - a^2 \left(k_2^2 + k_3^2 + k_4^2\right)} \right)^{-\frac{1}{2}} du. \tag{26}$$

In this paper, we also need a new theorem.

Theorem 2. *In the domain D, $(D \subset \mathbb{R}^n)$, if $v_i(x_1, x_2, \cdots x_n)$ is an first differentiable functions , $(i \in \{1, 2, \cdots m\}, m \leq n)$, then:*

$$\begin{aligned} & \frac{\partial \int \ldots \int f(v_1, v_2, \ldots v_m)\, dv_1 dv_2 \ldots dv_m}{\partial x_i} \\ & = \sum_{j=1}^{m} v_{j x_i} \int \cdots \int f(v_1, v_2, \ldots v_m)\, dv_1 dv_2 \ldots dv_{j-1} dv_{j+1} \ldots dv_m, \end{aligned}$$

where f is an arbitrary integrable function with m independent variables.

Proof. Set

$$u(x_1, x_2, \ldots x_n) = u(v_1, v_2, \ldots v_m) = \int \cdots \int f(v_1, v_2, \ldots v_m)\, dv_1 dv_2 \ldots dv_m.$$

So

$$u_{v_j} = \int \cdots \int f(v_1, v_2, \ldots v_m)\, dv_1 dv_2 \ldots dv_{j-1} dv_{j+1} \ldots dv_m$$

Then

$$\begin{aligned} u_{x_i} &= \frac{\partial \int \ldots \int f(v_1, v_2, \ldots v_m)\, dv_1 dv_2 \ldots dv_m}{\partial x_i} = \sum_{j=1}^{m} u_{v_j} v_{j x_i} \\ &= \sum_{j=1}^{m} v_{j x_i} \int \cdots \int f(v_1, v_2, \ldots v_m)\, dv_1 dv_2 \ldots dv_{j-1} dv_{j+1} \ldots dv_m. \end{aligned}$$

So theorem 2 proved. □

2. Solutions of Mathematical Physics Equation I

Before studying solutions of mathematical physics equation, we first use Z_1 Transformation to obtain general solutions or exact solutions of three typical PDEs.

Example 2.1. In $\mathbb{R}^n$, using Z_1 Transformation to get the exact solution of

$$a_1\left(u_{x_1}^{(m)}\right)^r + a_2\left(u_{x_2}^{(m)}\right)^r + \ldots + a_n\left(u_{x_n}^{(m)}\right)^r + a_{n+1}\left(u_{x_2x_3}^{(pq)}\right)^r = 0, \tag{27}$$

where

$$u_{x_i}^{(m)} \equiv \frac{\partial^m u}{\partial x_i^m}, u_{x_2x_3}^{(pq)} \equiv \frac{\partial^{p+q} u}{\partial x_2^p \partial x_3^q},$$

$a_i, (i = 1, 2, \cdots n+1)$ are arbitrary known constants, $r \geq 1, 1 \leq p+q = m$, the left of Eq. (27) could be added any number and types of $\left(u_{x_1x_2\ldots x_n}^{(i_1i_2\ldots i_n)}\right)^r$ with any constant coefficient,$(i_1 + i_2 + \cdots + i_n = m)$, since the similar calculation method, for facilitating writing there is only the $a_{n+1}\left(u_{x_2x_3}^{(pq)}\right)^r$ in Eq. (27).

By Z_1 Transformation, set $u(x_1, \cdots x_n) = f(v), v(x_1, \cdots x_n) = k_1x_1 + k_2x_2 + \cdots + k_nx_n + k_{n+1}$, where $k_1, k_2, \cdots k_{n+1}$ are unascertained constants and f is an undetermined unary mth-differentiable function,then

$$\begin{aligned} a_1\left(u_{x_1}^{(m)}\right)^r &+a_2\left(u_{x_2}^{(m)}\right)^r + \cdots + a_n\left(u_{x_n}^{(m)}\right)^r + a_{n+1}\left(u_{x_2x_3}^{(pq)}\right)^r \\ &= \left(a_1k_1^{mr} + a_2k_2^{mr} + \cdots + a_nk_n^{mr} + a_{n+1}k_2^{pr}k_3^{qr}\right)\left(f_v^{(m)}\right)^r = 0. \end{aligned}$$

The first case is

$$\left(f_v^{(m)}\right)^r = 0, \tag{28}$$

according to Z_1 Transformation the solution of Eq. (27) is

$$u = f(v) = c_{m-1}v^{m-1} + c_{m-2}v^{m-2} + \ldots + c_1v, \tag{29}$$

where $v(x_1, \cdots x_n) = k_1x_1 + k_2x_2 + \cdots + k_nx_n + k_{n+1}$, $c_1 - c_{m-1}$ and $k_1 - k_{n+1}$ are all arbitrary constants.
Since v contains arbitrary constants k_{n+1}, so there is no arbitrary constants c_0 in (29).

The second case is

$$a_1k_1^{mr} + a_2k_2^{mr} + \ldots a_nk_n^{mr} + a_{n+1}k_2^{pr}k_3^{qr} = 0, \tag{30}$$

if m and r are both odd, then

$$k_1 = \left(-\frac{a_2k_2^{mr} + a_3k_3^{mr} + \ldots + a_nk_n^{mr} + a_{n+1}k_2^{pr}k_3^{qr}}{a_1}\right)^{\frac{1}{mr}}, \tag{31}$$

where $k_2 - k_{n+1}$ are all arbitrary constants. By Z_1 Transformation the solution of Eq. (27) is

$$u = f\left((-\frac{a_2k_2^{mr} + \ldots + a_nk_n^{mr} + a_{n+1}k_2^{pr}k_3^{qr}}{a_1})^{\frac{1}{mr}} x_1 + k_2x_2 + \ldots + k_nx_n + k_{n+1}\right), \tag{32}$$

where f is an arbitrary unary mth-differentiable function.
If there is at least one even number among m and r in Eq. (27), then

$$k_1 = \pm\Big(-\frac{a_2k_2^{mr} + a_3k_3^{mr} + \ldots + a_nk_n^{mr} + a_{n+1}k_2^{pr}k_3^{qr}}{a_1}\Big)^{\frac{1}{mr}}. \tag{33}$$

By Z_1 Transformation, except (29) and (32) another solution of Eq. (27) is

$$u = f\left(-(-\frac{a_2k_2^{mr} + \ldots + a_nk_n^{mr} + a_{n+1}k_2^{pr}k_3^{qr}}{a_1})^{\frac{1}{mr}} x_1 + k_2x_2 + \ldots + k_nx_n + k_{n+1}\right). \tag{34}$$

In the case of $r = 1$, Eq. (27) becomes linear equation

$$a_1u_{x_1}^{(m)} + a_2u_{x_2}^{(m)} + \ldots + a_nu_{x_n}^{(m)} + a_{n+1}u_{x_2x_3}^{(pq)} = 0. \tag{35}$$

If m is odd, by (29) and (32) the solution of Eq. (35) is

$$\begin{aligned} u = f&\left(\left(-\tfrac{a_2k_2^m + \ldots + a_nk_n^m + a_{n+1}k_2^pk_3^q}{a_1}\right)^{\frac{1}{m}} x_1 + k_2x_2 + \ldots + k_nx_n + k_{n+1}\right) \\ &+ c_{m-1}v^{m-1} + c_{m-2}v^{m-2} + \ldots + c_1v, \end{aligned} \tag{36}$$

where $v = C_1x_1 + C_2x_2 + \cdots + C_nx_n + C_{n+1}, C_1 - C_{n+1}$ are arbitrary constants. If m is even, by (29), (32) and (34) the solution of Eq. (35) is

$$\begin{aligned} u = f_1&\left(\left(-\tfrac{a_2k_2^m + \ldots + a_nk_n^m + a_{n+1}k_2^pk_3^q}{a_1}\right)^{\frac{1}{m}} x_1 + k_2x_2 + \ldots + k_nx_n + k_{n+1}\right) \\ &+ f_2\left(-\left(-\tfrac{a_2l_2^m + \ldots + a_nl_n^m + a_{n+1}l_2^pl_3^q}{a_1}\right)^{\frac{1}{m}} x_1 + l_2x_2 + \ldots + l_nx_n + l_{n+1}\right) \\ &+ c_{m-1}v^{m-1} + c_{m-2}v^{m-2} + \ldots + c_1v \end{aligned} \tag{37}$$

where f_1 and f_2 are arbitrary unary mth-differentiable functions, $k_2 - k_{n+1}$ and $l_2 - l_{n+1}$ are arbitrary constants. In Appendix A we proved that if $k_1, l_1 \neq 0$ and $k_1, l_1 \nrightarrow 0$ in (37), c_1v can be described by f_1 and f_2.

If $m = 2, r = p = q = 1$, Eq. (27) becomes

$$a_1u_{x_1}^{(2)} + a_2u_{x_2}^{(2)} + \ldots + a_nu_{x_n}^{(2)} + a_{n+1}u_{x_2x_3} = 0. \tag{38}$$

According to (37), the general solution of Eq. (38) is

$$\begin{aligned} u = f_1&\left(\left(-\frac{a_2k_2^2 + \ldots a_nk_n^2 + a_{n+1}k_2k_3}{a_1}\right)^{\frac{1}{2}} x_1 + k_2x_2 + \ldots + k_nx_n + k_{n+1}\right) \\ &+ f_2\left(-\left(-\frac{a_2l_2^2 + \ldots a_nl_n^2 + a_{n+1}l_2l_3}{a_1}\right)^{\frac{1}{2}} x_1 + l_2x_2 + \ldots + l_nx_n + l_{n+1}\right) + c_1v. \end{aligned} \tag{39}$$

(39) can be written as

$$\begin{aligned} u = \sum_{i=1}^{s}&\left(f_{1_i}((-\frac{a_2k_{i_2}^2 + \ldots + a_nk_{i_n}^2 + a_{n+1}k_{i_2}k_{i_3}}{a_1})^{\frac{1}{2}} x_1 + k_{i_2}x_2 + \ldots + k_{i_n}x_n + k_{i_{n+1}}\right) \\ &+ f_{2_i}\left(-(-\frac{a_2l_{i_2}^2 + \ldots + a_nl_{i_n}^2 + a_{n+1}l_{i_2}l_{i_3}}{a_1})^{\frac{1}{2}} x_1 + l_{i_2}x_2 + \ldots + l_{i_n}x_n + l_{i_{n+1}})\right) + c_1v \end{aligned} \tag{40}$$

where f_{1_i} and f_{2_i} are arbitrary mth-differentiable functions, $k_{i_2}-k_{i_{n+1}}$ and $l_{i_2}-l_{i_{n+1}}$ are arbitrary determined constants.

Since Eq. (40) can have infinitely many series of functions, we call it a **series general solution** , and Eq. (39) is the **basic general solution** .

Consider the following Cauchy problem of Eq. (38)

$$u\left(0, x_2, \ldots x_n\right)=\sum_{i=1}^s \varphi_i\left(k_{i_2} x_2+k_{i_3} x_3+\ldots+k_{i_n} x_n+k_{i_{n+1}}\right), \tag{41}$$

$$u_{x_1}\left(0, x_2, \ldots x_n\right)=\sum_{i=1}^s \psi_i\left(k_{i_2} x_2+k_{i_3} x_3+\ldots+k_{i_n} x_n+k_{i_{n+1}}\right), \tag{42}$$

where $1 \leq s \leq \infty$, x_1 sometimes equal to time t. In (40), set $c_1=0, k_{i_j}=l_{i_j},(i=1,2, \cdots s, j=2,3, \cdots n+1)$, by further calculation which is in Appendix B, the exact solution of Eq. (38) in the conditions of (41) and (42) is

$$\begin{aligned} u= & \frac{1}{2} \sum_{i=1}^s\left(\varphi_i\left(k_{i_1} x_1+k_{i_2} x_2+\ldots+k_{i_n} x_n+k_{i_{n+1}}\right)\right. \\ & +\varphi_i\left(-k_{i_1} x_1+k_{i_2} x_2+\ldots+k_{i_n} x_n+k_{i_{n+1}}\right) \\ & \left.+\frac{1}{k_{i_1}} \int_{-k_{i_1} x_1+k_{i_2} x_2+\ldots+k_{i_n} x_n+k_{i_{n+1}}}^{k_{i_1} x_1+k_{i_2} x_2+\ldots+k_{i_n} x_n+k_{i_{n+1}}} \psi\left(\xi_i\right) d \xi_i\right) \end{aligned} \tag{43}$$

where

$$k_{i_1}=\left(-\left(a_2 k_{i_2}^2+\ldots+a_n k_{i_n}^2+a_{n+1} k_{i_2} k_{i_3}\right) / a_1\right)^{\frac{1}{2}}. \tag{44}$$

Example 2.2. In $\mathbb{R}^3$, using Z_1 Transformation to get the general solution of

$$u_{xx}+u_{yy}+u_{zz}+2 u_{xy}-2 u_{yz}-2 u_{zx}=A(x, y, z), \tag{45}$$

where $A(x,y,z)$ is any known function.

According to Z_1 Transformation, set $u(x,y,z)=u(p,q,r), A(x,y,z)=A(p,q,r)$, and

$$p=k_1 x+k_2 y+k_3 z, q=k_4 x+k_5 y+k_6 z, r=k_7 x+k_8 y+k_9 z, \tag{46}$$

where k_1-k_9 are undetermined constants, and set

$$\frac{\partial(p, q, r)}{\partial(x, y, z)}=-k_3 k_5 k_7+k_2 k_6 k_7+k_3 k_4 k_8-k_1 k_6 k_8-k_2 k_4 k_9+k_1 k_5 k_9 \neq 0. \tag{47}$$

By Eq. (46), we have

$$x=-\frac{-r k_3 k_5+r k_2 k_6+q k_3 k_8-p k_6 k_8-q k_2 k_9+p k_5 k_9}{k_3 k_5 k_7-k_2 k_6 k_7-k_3 k_4 k_8+k_1 k_6 k_8+k_2 k_4 k_9-k_1 k_5 k_9}, \tag{48}$$

$$y=-\frac{r k_3 k_4-r k_1 k_6-q k_3 k_7+p k_6 k_7+q k_1 k_9-p k_4 k_9}{k_3 k_5 k_7-k_2 k_6 k_7-k_3 k_4 k_8+k_1 k_6 k_8+k_2 k_4 k_9-k_1 k_5 k_9}, \tag{49}$$

$$z=\frac{r k_2 k_4-r k_1 k_5-q k_2 k_7+p k_5 k_7+q k_1 k_8-p k_4 k_8}{k_3 k_5 k_7-k_2 k_6 k_7-k_3 k_4 k_8+k_1 k_6 k_8+k_2 k_4 k_9-k_1 k_5 k_9}. \tag{50}$$

Then

$$\begin{aligned}
&u_{xx}+u_{yy}+u_{zz}+2u_{xy}-2u_{yz}-2u_{zx}\\
&=\left(k_1^2+k_2^2+k_3^2+2k_1k_2-2k_2k_3-2k_1k_3\right)u_{pp}\\
&+\left(k_4^2+k_5^2+k_6^2+2k_4k_5-2k_5k_6-2k_4k_6\right)u_{qq}\\
&+\left(k_7^2+k_8^2+k_9^2+2k_7k_8-2k_8k_9-2k_7k_9\right)u_{rr}\\
&+2\left(k_1k_4+k_2k_5+k_3k_6+k_1k_5+k_2k_4-k_2k_6-k_3k_5-k_1k_6-k_3k_4\right)u_{pq}\\
&+2\left(k_1k_7+k_2k_8+k_3k_9+k_1k_8+k_2k_7-k_2k_9-k_3k_8-k_1k_9-k_3k_7\right)u_{pr}\\
&+2\left(k_4k_7+k_5k_8+k_6k_9+k_4k_8+k_5k_7-k_5k_9-k_6k_8-k_4k_9-k_6k_7\right)u_{qr}\\
&=A\left(p,q,r\right).
\end{aligned}\tag{51}$$

Set

$$k_1^2+k_2^2+k_3^2+2k_1k_2-2k_2k_3-2k_1k_3=k_4^2+k_5^2+k_6^2+2k_4k_5-2k_5k_6-2k_4k_6=0.$$

We get

$$k_3=k_1+k_2, k_6=k_4+k_5.$$

So

$$\begin{aligned}
&k_1k_4+k_2k_5+k_3k_6+k_1k_5+k_2k_4-k_2k_6-k_3k_5-k_1k_6-k_3k_4\\
&=k_1k_7+k_2k_8+k_3k_9+k_1k_8+k_2k_7-k_2k_9-k_3k_8-k_1k_9-k_3k_7\\
&=k_4k_7+k_5k_8+k_6k_9+k_4k_8+k_5k_7-k_5k_9-k_6k_8-k_4k_9-k_6k_7=0
\end{aligned}$$

Note that we cannot further set $k_7^2+k_8^2+k_9^2+2k_7k_8-2k_8k_9-2k_7k_9=0$, otherwise $k_9=k_7+k_8$ and $-k_3k_5k_7+k_2k_6k_7+k_3k_4k_8-k_1k_6k_8-k_2k_4k_9+k_1k_5k_9=0$.
Thus

$$u_{xx}+u_{yy}+u_{zz}+2u_{xy}-2u_{yz}-2u_{zx}=\left(k_7^2+k_8^2+k_9^2+2k_7k_8-2k_8k_9-2k_7k_9\right)u_{rr}=A\left(p,q,r\right).$$

So the general solution of Eq. (45) is:

$$u=f_1\left(p,q\right)+rf_2\left(p,q\right)+B\left(x,y,z\right),\tag{52}$$

where

$$p=k_1x+k_2y+\left(k_1+k_2\right)z, q=k_4x+k_5y+\left(k_4+k_5\right)z, r=k_7x+k_8y+k_9z,\tag{53}$$

$$-k_1k_5k_7+k_2k_4k_7+k_2k_4k_8-k_1k_5k_8-k_2k_4k_9+k_1k_5k_9\neq 0,\tag{54}$$

$$B\left(x,y,z\right)=\frac{\iint A\left(p,q,r\right)drdr}{k_7^2+k_8^2+k_9^2+2k_7k_8-2k_8k_9-2k_7k_9},\tag{55}$$

f_1 and f_2 are arbitrary second differentiable functions, k_1,k_2,k_4,k_5 and k_7-k_9 are relatively arbitrary constants which satisfy Eq. (54).

It can be verified that

$$u=f_1\left(p,q\right)+rf_2\left(p,q\right)\tag{56}$$

is the general solution of

$$u_{xx}+u_{yy}+u_{zz}+2u_{xy}-2u_{yz}-2u_{zx}=0.\tag{57}$$

Using theorem 2, we can verify

$$u=\frac{\iint A\left(p,q,r\right)drdr}{k_7^2+k_8^2+k_9^2+2k_7k_8-2k_8k_9-2k_7k_9}\tag{58}$$

is a special solution of Eq. (45).

Example 2.3. In $\mathbb{R}^3$, using Z_1 Transformation to get the general solution of

$$a_1u_{xx} + a_2u_{yy} + a_3u_{zz} = A\left(x, y, z\right), \tag{59}$$

where $a_1 - a_3$ are arbitrary known constants and $A(x, y, z)$ is any known function.

According to Z_1 Transformation, set $u(x, y, z) = u(p, q, r), A(x, y, z) = A(p, q, r)$ and

$$p = k_1x + k_2y + k_3z, q = k_4x + k_5y + k_6z, r = k_7x + k_8y + k_9z, \tag{46}$$

where $k_1 - k_9$ are undetermined constants, and set

$$\frac{\partial\left(p, q, r\right)}{\partial\left(x, y, z\right)} = -k_3k_5k_7 + k_2k_6k_7 + k_3k_4k_8 - k_1k_6k_8 - k_2k_4k_9 + k_1k_5k_9 \neq 0. \tag{47}$$

By Eq. (46), we get

$$x = -\frac{-rk_3k_5 + rk_2k_6 + qk_3k_8 - pk_6k_8 - qk_2k_9 + pk_5k_9}{k_3k_5k_7 - k_2k_6k_7 - k_3k_4k_8 + k_1k_6k_8 + k_2k_4k_9 - k_1k_5k_9}, \tag{48}$$

$$y = -\frac{rk_3k_4 - rk_1k_6 - qk_3k_7 + pk_6k_7 + qk_1k_9 - pk_4k_9}{k_3k_5k_7 - k_2k_6k_7 - k_3k_4k_8 + k_1k_6k_8 + k_2k_4k_9 - k_1k_5k_9}, \tag{49}$$

$$z = \frac{rk_2k_4 - rk_1k_5 - qk_2k_7 + pk_5k_7 + qk_1k_8 - pk_4k_8}{k_3k_5k_7 - k_2k_6k_7 - k_3k_4k_8 + k_1k_6k_8 + k_2k_4k_9 - k_1k_5k_9}. \tag{50}$$

Then

$$\begin{aligned} & a_1u_{xx} + a_2u_{yy} + a_3u_{zz} \\ & = \left(a_1k_1^2 + a_2k_2^2 + a_3k_3^2\right)u_{pp} + \left(a_1k_4^2 + a_2k_5^2 + a_3k_6^2\right)u_{qq} + \left(a_1k_7^2 + a_2k_8^2 + a_3k_9^2\right)u_{rr} \\ & + 2\left(a_1k_1k_4 + a_2k_2k_5 + a_3k_3k_6\right)u_{pq} + 2\left(a_1k_1k_7 + a_2k_2k_8 + a_3k_3k_9\right)u_{pr} \\ & + 2\left(a_1k_4k_7 + a_2k_5k_8 + a_3k_6k_9\right)u_{qr} \\ & = A\left(p, q, r\right). \end{aligned} \tag{60}$$

There are many methods for solving Eq. (60), a typical method is to set

$$a_1k_4^2 + a_2k_5^2 + a_3k_6^2 = a_1k_7^2 + a_2k_8^2 + a_3k_9^2 = 0. \tag{61}$$

So

$$k_4 = \pm\sqrt{\frac{-a_2k_5^2 - a_3k_6^2}{a_1}}, k_7 = \pm\sqrt{\frac{-a_2k_8^2 - a_3k_9^2}{a_1}}.$$

Then set

$$a_1k_4k_7 + a_2k_5k_8 + a_3k_6k_9 = 0. \tag{62}$$

Hence

$$\begin{aligned} & a_1k_4k_7 + a_2k_5k_8 + a_3k_6k_9 = \pm a_1\sqrt{\frac{-a_2k_5^2 - a_3k_6^2}{a_1}}\sqrt{\frac{-a_2k_8^2 - a_3k_9^2}{a_1}} + a_2k_5k_8 + a_3k_6k_9 = 0 \\ & \Longrightarrow \left(a_2k_5^2 + a_3k_6^2\right)\left(a_2k_8^2 + a_3k_9^2\right) = \left(a_2k_5k_8 + a_3k_6k_9\right)^2 \\ & \Longrightarrow k_5k_9 = k_6k_8. \end{aligned}$$

Namely

$$k_8 = kk_9, k_5 = kk_6.$$

It can be verified that if $k_6k_9>0$, k_4 and k_7 must satisfy $k_4k_7<0$; if $k_6k_9<0$, k_4 and k_7 must satisfy $k_4k_7>0$; otherwise q and r are function correlation.

For the unity and convenience, we set $k_6k_9>0$, namely

$$k_4=\sqrt{\frac{-a_2k_5^2-a_3k_6^2}{a_1}}, k_7=-\sqrt{\frac{-a_2k_8^2-a_3k_9^2}{a_1}}, k_8=kk_9, k_5=kk_6, k_6k_9>0. \tag{63}$$

Using the eliminant to solve

$$\begin{cases} a_1k_4^2+a_2k_5^2+a_3k_6^2=0 \\ a_1k_7^2+a_2k_8^2+a_3k_9^2=0, \\ a_1k_4k_7+a_2k_5k_8+a_3k_6k_9=0 \end{cases}$$

we will get the same conclusion.
Further order

$$a_1k_1k_4+a_2k_2k_5+a_3k_3k_6=a_1k_1k_7+a_2k_2k_8+a_3k_3k_9=0 \tag{64}$$

So

$$a_1k_1k_4+a_2k_2k_5+a_3k_3k_6=a_1k_1\sqrt{\frac{-a_2k^2k_6^2-a_3k_6^2}{a_1}}+a_2kk_2k_6+a_3k_3k_6=0$$

$$\Longrightarrow k_1=\frac{-a_2kk_2k_6-a_3k_3k_6}{a_1\left|k_6\right|\sqrt{\frac{-a_2k^2-a_3}{a_1}}}$$

$$a_1k_1k_7+a_2k_2k_8+a_3k_3k_9=-a_1k_1\sqrt{\frac{-a_2k^2k_9^2-a_3k_9^2}{a_1}}+a_2kk_2k_9+a_3k_3k_9=0$$

$$\Longrightarrow k_1=\frac{a_2kk_2k_9+a_3k_3k_9}{a_1\left|k_9\right|\sqrt{\frac{-a_2k^2-a_3}{a_1}}}$$

If $k_6,k_9>0$, then

$$k_1=\frac{-a_2kk_2k_6-a_3k_3k_6}{a_1\left|k_6\right|\sqrt{\frac{-a_2k^2-a_3}{a_1}}}=\frac{-a_2kk_2-a_3k_3}{a_1\sqrt{\frac{-a_2k^2-a_3}{a_1}}}=\frac{a_2kk_2k_9+a_3k_3k_9}{a_1\left|k_9\right|\sqrt{\frac{-a_2k^2-a_3}{a_1}}}=\frac{a_2kk_2+a_3k_3}{a_1\sqrt{\frac{-a_2k^2-a_3}{a_1}}}$$

So $k_1=0$, and

$$a_2kk_2+a_3k_3=0\Longrightarrow k_2=\frac{-a_3k_3}{a_2k}$$

If $k_6,k_9<0$, by the similar calculation, the conclusion is same.

If

$$k_4=\sqrt{\frac{-a_2k_5^2-a_3k_6^2}{a_1}}, k_7=\sqrt{\frac{-a_2k_8^2-a_3k_9^2}{a_1}}, k_8=kk_9, k_5=kk_6, k_6k_9<0.$$

Then

$$a_1k_1k_4+a_2k_2k_5+a_3k_3k_6=a_1k_1\sqrt{\frac{-a_2k^2k_6^2-a_3k_6^2}{a_1}}+a_2kk_2k_6+a_3k_3k_6=0$$

$$\Longrightarrow k_1=\frac{-a_2kk_2k_6-a_3k_3k_6}{a_1\left|k_6\right|\sqrt{\frac{-a_2k^2-a_3}{a_1}}},$$

$$a_1k_1k_7 + a_2k_2k_8 + a_3k_3k_9 = a_1k_1\sqrt{\frac{-a_2k^2k_9^2 - a_3k_9^2}{a_1}} + a_2kk_2k_9 + a_3k_3k_9 = 0$$

$$\Longrightarrow k_1 = \frac{-a_2kk_2k_9 - a_3k_3k_9}{a_1\left|k_9\right|\sqrt{\frac{-a_2k^2-a_3}{a_1}}}.$$

If $k_6 > 0, k_9 < 0$, then

$$k_1 = \frac{-a_2kk_2k_6 - a_3k_3k_6}{a_1\left|k_6\right|\sqrt{\frac{-a_2k^2-a_3}{a_1}}} = \frac{-a_2kk_2 - a_3k_3}{a_1\sqrt{\frac{-a_2k^2-a_3}{a_1}}} = \frac{-a_2kk_2k_9 - a_3k_3k_9}{a_1\left|k_9\right|\sqrt{\frac{-a_2k^2-a_3}{a_1}}} = \frac{a_2kk_2 + a_3k_3}{a_1\sqrt{\frac{-a_2k^2-a_3}{a_1}}}$$
$$= 0$$

So

$$a_2kk_2 + a_3k_3 = 0 \Longrightarrow k_2 = \frac{-a_3k_3}{a_2k}$$

If $k_6 < 0, k_9 > 0$, by the similar calculation, the conclusion is same. Thus

$$k_1 = 0, k_2 = \frac{-a_3k_3}{a_2k} \tag{65}$$

Then Eq. (60) can be simplified as

$$\left(a_2k_2^2 + a_3k_3^2\right)u_{pp} = A\left(p,q,r\right). \tag{66}$$

The general solution of Eq. (59) seems to be written as

$$u = f_1\left(q,r\right) + pf_2\left(q,r\right) + B\left(x,y,z\right), \tag{67}$$

where

$$B\left(x,y,z\right) = \frac{\iint A\left(p,q,r\right)dpdp}{a_2k_2^2 + a_3k_3^2}, \tag{68}$$

f_1 and f_2 are arbitrary 2th-differentiable functions, under the condition of (47), (63) and (65), there are four relatively arbitrary constants among $k_2 - k_9$, such as k_3, k_5, k_6 and k_8, $(k_5, k_6, k_8 \neq 0)$.

By theorem 2, it can be verified that $u = B(x,y,z)$ is the correct special solution of Eq. (59), but $u = f_1(q,r), u = pf_2(q,r)$ and $u = f_1(q,r)+pf_2(q,r)$ are not the solutions of $a_1u_{xx}+a_2u_{yy}+a_3u_{zz} = 0$. Because (63) can be verified the extraneous root of simultaneous Eq. (61) and (62) by the substitution of various numerical value, so (67) is not the correct general solution of Eq. (59). This phenomenon, which is caused by extraneous roots of algebraic equations, is called the **excrescent general solution** ,and we will find that it is more common.

By Z_1 Transformation, we get the special solution of Eq. (59):

$$u = \frac{\iint A\left(p,q,r\right)dpdp}{a_2k_2^2 + a_3k_3^2}, \tag{69}$$

where

$$p = k_2y + k_3z, q = k_4x + k_5y + k_6z, r = k_7x + k_8y + k_9z, \tag{70}$$

$$-k_3k_5k_7 + k_2k_6k_7 + k_3k_4k_8 - k_2k_4k_9 \neq 0. \tag{71}$$

According to (69), we can find and verify

$$u = \frac{\iint A\left(p,q,r\right)dqdq}{a_1k_4^2 + a_2k_5^2 + a_3k_6^2}, \tag{72}$$

$$u=\frac{\iint A(p,q,r)\,drdr}{a_1k_7^2+a_2k_8^2+a_3k_9^2}, \tag{73}$$

are all the special solutions of Eq. (59), and can further find the more concise special solution is

$$u=B(x,y,z)=\frac{\iint A(p,y,z)\,dpdp}{a_1k_1^2+a_2k_2^2+a_3k_3^2}, \tag{74}$$

where

$$p=k_1x+k_2y+k_3z,\quad (k_1,k_2,k_3\neq 0) \tag{75}$$

In front, we mentioned there were many methods for solving Eq. (60), such as set

$$a_1k_4^2+a_2k_5^2+a_3k_6^2=a_1k_7^2+a_2k_8^2+a_3k_9^2=a_1k_4k_7+a_2k_5k_8+a_3k_6k_9=0.$$

We get

$$\left(a_1k_1^2+a_2k_2^2+a_3k_3^2\right)u_{pp}+2\left(a_1k_1k_4+a_2k_2k_5+a_3k_3k_6\right)u_{pq}+2\left(a_1k_1k_7+a_2k_2k_8+a_3k_3k_9\right)u_{pr}$$
$$=A(p,q,r).$$

Then set $w=u_p$, the general solution of Eq. (59) seems can be solved, but it can be verified that we may obtain a variety of special solutions by Eq. (60), and only obtain excrescent general solutions of Eq. (59).

According to (39),

$$a_1u_{xx}+a_2u_{yy}+a_3u_{zz}=0. \tag{76}$$

The general solution of Eq. (76) is

$$u=f_1(v_1)+f_2(v_2)+v_3, \tag{77}$$

where

$$v_1=\left(-\frac{a_2c_1^2+a_3c_2^2}{a_1}\right)^{\frac{1}{2}}x+c_1y+c_2z+c_3, \tag{78}$$

$$v_2=-\left(-\frac{a_2c_4^2+a_3c_5^2}{a_1}\right)^{\frac{1}{2}}x+c_4y+c_5z+c_6, \tag{79}$$

$$v_3=c_7x+c_8y+c_9z+c_{10}, \tag{80}$$

f_1 and f_2 are arbitrary 2th-differentiable functions, c_1-c_{10} are absolutely arbitrary constants. So the general solutions of Eq. (59) is

$$u=f_1(v_1)+f_2(v_2)+v_3+B(x,y,z). \tag{81}$$

2.1. Laplace equation

Laplace equation is importantly used not only in classical electrodynamics, thermodynamics and fluid dynamics etc., but also in the modern theory of the invisible [4, 5]. In recent decades a research hotspot is using many numerical methods for solving Laplace's equation in various geometries and boundary conditions, such as the moment methods [6], quasi-reversibility methods [7, 8], finite difference methods [9] and so on.

In $\mathbb{R}^3$, The form of Laplace equation in Cartesian coordinate system is

$$u_{xx}+u_{yy}+u_{zz}=0 \tag{82}$$

Eq. (82) is a special case of Eq. (38) and (76), according to (39) and (40), its basic general solution and series general solution can be get respectively

$$u = f_1\left(x\sqrt{-k_1^2-k_2^2}+k_1y+k_2z+k_3\right)+f_2\left(-x\sqrt{-k_4^2-k_5^2}+k_4y+k_5z+k_6\right)+k_7x+k_8y$$
$$+k_9z+k_{10}, \tag{83}$$

$$u=\sum_{i=1}^{s}(f_{1_i}\ (x\sqrt{-k_{1_i}^2-k_{2_i}^2}+k_{1_i}y+k_{2_i}z+k_{3_i})+f_{2_i}\ (-x\sqrt{-k_{4_i}^2-k_{5_i}^2}+k_{4_i}y+k_{5_i}z+k_{6_i})$$
$$+k_{7_i}x+k_{8_i}y+k_{9_i}z+k_{10_i}) \tag{84}$$

where f_1, f_2, f_{1_i} and f_{2_i} are arbitrary second differentiable functions, $(1 \le s \le \infty)$, $k_1 - k_{10}$ are absolutely arbitrary constants, and $k_{1_i} - k_{10_i}$ are arbitrary determined constants.

(83) and (84) can be abbreviated as

$$u = f_1(v_1) + f_2(v_2) + v_3, \tag{85}$$

$$u = \sum_{i=1}^{s}\left(f_{1_i}\ (v_{1_i}) + f_{2_i}\ (v_{2_i}) + v_{3_i}\right), \tag{86}$$

where $v_1 = x\sqrt{-k_1^2-k_2^2}+k_1y+k_2z+k_3$,$v_2 = -x\sqrt{-k_4^2-k_5^2}+k_4y+k_5z+k_6$,$v_3 = k_7x+k_8y+k_9z+k_{10}$,$v_{1_i} = x\sqrt{-k_{1_i}^2-k_{2_i}^2}+k_{1_i}y+k_{2_i}z+k_{3_i}$,$v_{2_i} = -x\sqrt{-k_{4_i}^2-k_{5_i}^2}+k_{4_i}y+k_{5_i}z+k_{6_i}$,and $v_{3_i} = k_{7_i}x+k_{8_i}y+k_{9_i}z+k_{10_i}$.

Assuming Eq. (82) satisfies the following boundary conditions

$$u(0,y,z) = \sum_{i=1}^{s}\varphi_i\left(k_{i_1}y+k_{i_2}z+k_{i_3}\right), u_x(0,y,z) = \sum_{i=1}^{s}\psi_i\left(k_{i_1}y+k_{i_2}z+k_{i_3}\right). \tag{87}$$

According to (43) and (44), the exact solution of Eq. (82) on the conditions of (87) is

$$u = \frac{1}{2}\sum_{i=1}^{s}(\varphi_i(x\sqrt{-k_{i_1}^2-k_{i_2}^2}+k_{i_1}y+k_{i_2}z+k_{i_3})+\varphi_i(-x\sqrt{-k_{i_1}^2-k_{i_2}^2}+k_{i_1}y+k_{i_2}z+k_{i_3})$$
$$+\frac{1}{\sqrt{-k_{i_1}^2-k_{i_2}^2}}\int_{-x\sqrt{-k_{i_1}^2-k_{i_2}^2}+k_{i_1}y+k_{i_2}z+k_{i_3}}^{x\sqrt{-k_{i_1}^2-k_{i_2}^2}+k_{i_1}y+k_{i_2}z+k_{i_3}}\psi(\xi_i)d\xi_i) \tag{88}$$

Solutions of some PDEs have special transformation laws, that is, if we know a solution, we can get another solution by the transformational rule. For example, solutions of Laplace equation have two important laws [10]:

Suppose u(x,y,z) is a solution of the Laplace equation. Then the functions

$$u_1 = \frac{A}{r}u\left(\frac{x}{r^2},\frac{y}{r^2},\frac{z}{r^2}\right), r = \sqrt{x^2+y^2+z^2} \tag{89}$$

$$u_2 = \frac{A}{\sqrt{R}}u\left(\frac{x-ar^2}{R},\frac{y-ar^2}{R},\frac{z-ar^2}{R}\right), R = 1-2(ax+by+cz)+\left(a^2+b^2+c^2\right)r^2 \tag{90}$$

where A, a, b and c are arbitrary constants, are also solutions of this equation.

According to these laws, the basic general solution (83) of the Laplace equation will be transformed into

$$\begin{aligned} u=&\frac{A}{r}(f_1(\frac{x}{r^2}\sqrt{-k_1^2-k_2^2}+\frac{k_1y}{r^2}+\frac{k_2z}{r^2}+k_3)+f_2(-\frac{x}{r^2}\sqrt{-k_4^2-k_5^2}+\frac{k_4y}{r^2}+\frac{k_5z}{r^2}+k_6)\\ &+\frac{k_7x}{r^2}+\frac{k_8y}{r^2}+\frac{k_9z}{r^2}+k_{10}), \end{aligned} \tag{91}$$

$$\begin{aligned} u=&\frac{A}{\sqrt{R}}(f_1\left(\frac{x-ar^2}{R}\sqrt{-k_1^2-k_2^2}+\frac{k_1}{R}\left(y-ar^2\right)+\frac{k_2}{R}\left(z-ar^2\right)+k_3\right)\\ &+f_2\left(\frac{-x+ar^2}{R}\sqrt{-k_4^2-k_5^2}+\frac{k_4}{R}\left(y-ar^2\right)+\frac{k_5}{R}\left(z-ar^2\right)+k_6\right)\\ &+\frac{k_7}{R}\left(x-ar^2\right)+\frac{k_8}{R}\left(y-ar^2\right)+\frac{k_9}{R}\left(z-ar^2\right)+k_{10}) \end{aligned} \tag{92}$$

where f_1 and f_2 are arbitrary second differentiable functions, k_1-k_{10} are absolutely arbitrary constants. According to (91) and (92), we can see that the structure of the general solution of the Laplace equation has become

$$u=v_0\left(f_1\left(v_1\right)+f_2\left(v_2\right)+v_3\right), \tag{93}$$

which is different from Eq. (85), so there is an important new question, whether or not are Eq. (83), (91) and (92) independent of each other? If they do not dependent on each other, according to the superposition principle, the number of arbitrary functions in the general solution of Laplace equation will be greater than 2.

The problem is not over here. Eq. (91) and (92) can deduce infinite new solutions by (89) and (90), so the structure of the general solution of the Laplace equation has evolved into

$$u=\sum_{i=1}^{s}v_{0_i}\left(f_{1_i}\ \left(v_{1_i}\right)+f_{2_i}\ \left(v_{2_i}\right)+v_{3_i}\right),\left(1\le s\le\infty\right) \tag{94}$$

where f_{1_i} and f_{2_i} are arbitrary second differentiable functions, and

$$\begin{aligned} &v_{0_1}=1, v_{1_1}=x\sqrt{-k_1^2-k_2^2}+k_1y+k_2z+k_3,\\ &v_{2_1}=-x\sqrt{-k_4^2-k_5^2}+k_4y+k_5z+k_6, v_{3_1}=k_7x+k_8y+k_9z+k_{10}. \end{aligned} \tag{95}$$

When $i\ge 2$, by (89) and (90) we can see v_{j_i} has two choice, $(0\le j\le 3)$, so the general solution of the Laplace equation is an infinite function series which contains infinite number of arbitrary constants. In order to classify these general solutions, we call (91) and (92) the **transformational general solution** , and (94) a **generalized series general solution.**

The transformation laws of the solution of Laplacian equation may be more than (89) and (90), so the concrete form of transformational general solution and generalized series general solution may be enriched.

2.2. Poisson equation

In $\mathbb{R}^3$,consider the following Poisson equation

$$\triangle u=A\left(x,y,z\right). \tag{96}$$

where $\triangle$ is the Laplace operator and $A(x,y,z)$ is any known function.
Eq. (96) is a special case of Eq. (59), that is

$$a_1=a_2=a_3=1,$$

according to (74), (75) and (78-81), its basic general solution is

$$\begin{aligned} u = {} & f_1\left(x\sqrt{-k_1^2-k_2^2}+k_1y+k_2z+k_3\right)+f_2\left(-x\sqrt{-k_4^2-k_5^2}+k_4y+k_5z+k_6\right) \\ & +k_7x+k_8y+k_9z+\frac{\iint A(p,y,z)\,dpdp}{a_1k_{10}^2+a_2k_{11}^2+a_3k_{12}^2}, \end{aligned} \tag{97}$$

where

$$p = k_{10}x+k_{11}y+k_{12}z, \tag{98}$$

f_1 and f_2 are arbitrary second differentiable functions, k_1-k_9 are absolute arbitrary constants, and $k_{10}-k_{12}$ are relative arbitrary constants which are not equal to zero.

The series general solution of Eq. (96) is

$$\begin{aligned} u = {} & \sum_{i=1}^{s}\left(f_{i_1}\left(x\sqrt{-k_{i_1}^2-k_{i_2}^2}+k_{i_1}y+k_{i_2}z+k_{i_3}\right)+f_{i_2}\left(-x\sqrt{-k_{i_4}^2-k_{i_5}^2}+k_{i_4}y+k_{i_5}z+k_{i_6}\right)\right) \\ & +k_7x+k_8y+k_9z+\frac{\iint A(p,y,z)\,dpdp}{a_1k_{10}^2+a_2k_{11}^2+a_3k_{12}^2}, \end{aligned} \tag{99}$$

where f_{i_1} and f_{i_2} are arbitrary second differentiable functions, $k_{i_1}-k_{i_6}$ are arbitrary determined constants,

Currently, using numerical methods to analyse Poisson equation is a hot research area [11]. Assuming Eq. (96) satisfies the following boundary conditions

$$u(0,y,z) = q(y,z)+\sum_{i=1}^{s}\varphi_i\left(k_{i_1}y+k_{i_2}z+k_{i_3}\right), \tag{100}$$

$$u_x(0,y,z) = q_x(y,z)+\sum_{i=1}^{s}\psi_i\left(k_{i_1}y+k_{i_2}z+k_{i_3}\right), \tag{101}$$

where φ_i, ψ_i and q are known functions, and

$$q(x,y,z) = k_7x+k_8y+k_9z+\frac{\iint A(p,y,z)\,dpdp}{a_1k_{10}^2+a_2k_{11}^2+a_3k_{12}^2}, \tag{102}$$

k_7-k_{12} are known constants.

By (99), set $k_{i_1}=k_{i_4}, k_{i_2}=k_{i_5}$ and $k_{i_3}=k_{i_6}$, similar to the calculation of (43) we get

$$\begin{aligned} & f_{i_1}\left(k_{i_1}y+k_{i_2}z+k_{i_3}\right)+f_{i_2}\left(k_{i_1}y+k_{i_2}z+k_{i_3}\right)=\varphi_i\left(k_{i_1}y+k_{i_2}z+k_{i_3}\right), \\ & f_{i_1}\left(k_{i_1}y+k_{i_2}z+k_{i_3}\right)-f_{i_2}\left(k_{i_1}y+k_{i_2}z+k_{i_3}\right) \\ & =\frac{1}{\sqrt{-k_{i_1}^2-k_{i_2}^2}}\int_{k_{i_1}y_0+k_{i_2}z_0+k_{i_3}}^{k_{i_1}y+k_{i_2}z+k_{i_3}}\psi_i(\xi_i)d\xi_i+f_{i_1}(k_{i_1}y_0+k_{i_2}z_0+k_{i_3}) \\ & -f_{i_2}(k_{i_1}y_0+k_{i_2}z_0+k_{i_3}). \end{aligned}$$

By the further calculation, the exact solution of Eq. (106) on the conditions of (100-102) is

$$\begin{aligned} & u(x,y,z) \\ & =q(x,y,t)+\frac{1}{2}\sum_{i=1}^{s}(\varphi_i(x\sqrt{-k_{i_1}^2-k_{i_2}^2}+k_{i_1}y+k_{i_2}z+k_{i_3}) \\ & +\varphi_i(-x\sqrt{-k_{i_1}^2-k_{i_2}^2}+k_{i_1}y+k_{i_2}z+k_{i_3}) \\ & +\frac{1}{\sqrt{-k_{i_1}^2-k_{i_2}^2}}\int_{-x\sqrt{-k_{i_1}^2-k_{i_2}^2}+k_{i_1}y+k_{i_2}z+k_{i_3}}^{x\sqrt{-k_{i_1}^2-k_{i_2}^2}+k_{i_1}y+k_{i_2}z+k_{i_3}}\psi_i(\xi_i)d\xi_i) \end{aligned} \tag{103}$$

2.3. 2D wave equation

In $\mathbb{R}^3$, the form of 2D wave equation in Cartesian coordinate system is

$$u_{tt} - a^2 u_{xx} - a^2 u_{yy} = 0. \tag{104}$$

Eq. (104) is an especial case of Eq. (38), by (39) its basic general solution can be obtained

$$\begin{aligned} u &= f_1\left(k_1 x + k_2 y + at\sqrt{k_1^2 + k_2^2} + k_3\right) \\ &+ f_2\left(k_4 x + k_5 y - at\sqrt{k_4^2 + k_5^2} + k_6\right) + k_7 x + k_8 y + k_9 t + k_{10} \\ &= g\left(\frac{k_1 x}{\sqrt{k_1^2+k_2^2}} + \frac{k_2 y}{\sqrt{k_1^2+k_2^2}} + at + g_0\right) \\ &+ h\left(\frac{k_4 x}{\sqrt{k_4^2+k_5^2}} + \frac{k_5 y}{\sqrt{k_4^2+k_5^2}} - at + h_0\right) + k_7 x + k_8 y + k_9 t + k_{10} \\ &= g\left(x\cos\theta + y\sin\theta + at + g_0\right) \\ &+ h\left(x\cos\varphi + y\sin\varphi - at + h_0\right) + k_7 x + k_8 y + k_9 t + k_{10}, \end{aligned} \tag{105}$$

where f_1, f_2, g and h are arbitrary unary second differentiable functions, $k_1 - k_{10}, \theta, \varphi, g_0$ and h_0 are arbitrary constants. $g(x\cos\theta + y\sin\theta + at + g_0)$ is a parallel wave with the speed a, the angle between x axis and spread direction of g is θ which is arbitrary.

The series general solution of Eq. (104) is

$$\begin{aligned} u(x, y, t) =& \sum_i (g_i(x\cos\theta_i + y\sin\theta_i + at + g_{i0}) + \\ & h_i(x\cos\varphi_i + y\sin\varphi_i - at + h_{i0})) + k_7 x + k_8 y + k_9 t + k_{10}, \end{aligned} \tag{106}$$

where g_i and h_i are arbitrary unary second differentiable functions, $\theta_i, \varphi_i, g_{i0}$ and h_{i0} are arbitrary determined constants.

A research hotspot is using numerical methods to study the 2D wave equation [12]. Consider the following initial value problem of Eq. (104)

$$\begin{aligned} u\left(x, y, 0\right) &= \sum_i \varphi_i\left(k_{i_1} x + k_{i_2} y + k_{i_3}\right), \\ u_t\left(x, y, 0\right) &= \sum_i \psi_i\left(k_{i_1} x + k_{i_2} y + k_{i_3}\right). \end{aligned} \tag{107}$$

Similar to the solving method of (43), the exact solution of Eq. (104) on the conditions of (107) can be got

$$\begin{aligned} u =& \frac{1}{2}\sum_i (\varphi_i(k_{i_1} x + k_{i_2} y - at\sqrt{k_{i_1}^2 + k_{i_2}^2} + k_{i_3}) \\ & + \varphi_i(k_{i_1} x + k_{i_2} y + at\sqrt{k_{i_1}^2 + k_{i_2}^2} + k_{i_3}) \\ & + \frac{1}{a\sqrt{k_{i_1}^2 + k_{i_2}^2}} \int_{k_{i_1} x + k_{i_2} y - at\sqrt{k_{i_1}^2 + k_{i_2}^2} + k_{i_3}}^{k_{i_1} x + k_{i_2} y + at\sqrt{k_{i_1}^2 + k_{i_2}^2} + k_{i_3}} \psi_i\left(\xi\right) d\xi). \end{aligned} \tag{108}$$

Solutions of homogeneous 2D wave equation have two transformational laws as follows [10]:

Suppose $u(x,y,z)$ is a solution of the 2D wave equation. Then the functions

$$u_1 = \frac{A}{\sqrt{|r^2-a^2t^2|}} u\left(\frac{x}{r^2-a^2t^2}, \frac{y}{r^2-a^2t^2}, \frac{t}{r^2-a^2t^2}\right), \tag{109}$$

$$u_2 = \frac{A}{\sqrt{R}} u\left(\frac{x+B_1\left(a^2t^2-r^2\right)}{R}, \frac{y+B_2\left(a^2t^2-r^2\right)}{R}, \frac{at+B_3\left(a^2t^2-r^2\right)}{aR}\right), \tag{110}$$

$$r = \sqrt{x^2+y^2}, \quad R = 1-2\left(B_1x+B_2y-aB_3t\right)+\left(B_1^2+B_2^2-B_3^2\right)\left(r^2-a^2t^2\right), \tag{111}$$

where A, v, B_1, B_2 and B_3 are arbitrary constants, are also solutions of this equation.

According to (109, 110), the transformational general solutions of Eq. (104) are

$$\begin{aligned} u =&\frac{A}{\sqrt{|r^2-a^2t^2|}}(f_1(\frac{k_1x+k_2y+at\sqrt{k_1^2+k_2^2}}{r^2-a^2t^2}+k_3)+f_2(\frac{k_4x+k_5y-at\sqrt{k_4^2+k_5^2}}{r^2-a^2t^2}+k_6) \\ &+\frac{k_7x+k_8y+k_9t}{r^2-a^2t^2}+k_{10}) \end{aligned} \tag{112}$$

$$\begin{aligned} u =& \\ &\frac{A}{\sqrt{R}}(f_1(\frac{k_1(x+B_1(a^2t^2-r^2))}{R}+\frac{k_2(y+B_2(a^2t^2-r^2))}{R}+\frac{at+B_3(a^2t^2-r^2)}{aR}\sqrt{k_1^2+k_2^2}+k_3) \\ &+f_2(\frac{k_4(x+B_1(a^2t^2-r^2))}{R}+\frac{k_5(y+B_2(a^2t^2-r^2))}{R}-\frac{at+B_3(a^2t^2-r^2)}{aR}\sqrt{k_4^2+k_5^2}+k_6) \\ &+\frac{k_7(x+B_1(a^2t^2-r^2))}{R}+\frac{k_8(y+B_2(a^2t^2-r^2))}{R}+\frac{k_9(at+B_3(a^2t^2-r^2))}{aR}+k_{10}) \end{aligned} \tag{113}$$

Eq. (112, 113) can deduce infinite new solutions by (109, 110), according to the principle of superposition, the generalized series general solution of 2D wave equation can also be written as

$$u = \sum_{i=1}^{s} v_{0_i}\left(f_{1_i}\ \left(v_{1_i}\right)+f_{2_i}\ \left(v_{2_i}\right)+v_{3_i}\right), (1 \leq s \leq \infty) \tag{94}$$

In $\mathbb{R}^3$, The form of the nonhomogeneous 2D wave equation in Cartesian coordinate system is

$$u_{tt}-a^2u_{xx}-a^2u_{yy} = A\left(t,x,y\right), \tag{114}$$

(114) is a special case of Eq. (59), namely

$$t = z, a_1 = a_2 = -a^2, a_3 = 1. \tag{115}$$

According to (74), (75) and (78-81), its basic general solution is

$$\begin{aligned} u =& f_1\left(k_1x+k_2y+at\sqrt{k_1^2+k_2^2}+k_3\right)+f_2\left(k_4x+k_5y-at\sqrt{k_4^2+k_5^2}+k_6\right) \\ &+k_7x+k_8y+k_9t+k_{10}+\frac{\iint A\left(p,y,t\right)dpdp}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2}, \end{aligned} \tag{116}$$

where

$$p = k_{11}x+k_{12}y+k_{13}t, \quad (k_{11},k_{12},k_{13} \neq 0) \tag{117}$$

f_1 and f_2 are arbitrary second differentiable functions, k_1-k_{10} are absolutely arbitrary constants, and $k_{11}-k_{13}$ are relatively arbitrary constants.

The series general solution of Eq. (114) is

$$u(x,y,z)=\sum_{i=1}^{s}\left(f_{i1}(k_{i_1}x+k_{i_2}y+at\sqrt{k_{i_1}^2+k_{i_2}^2}+k_{i_3})+f_{i2}(k_{i_4}x+k_{i_5}y-at\sqrt{k_{i_4}^2+k_{i_5}^2}+k_{i_6})\right)$$
$$+k_7x+k_8y+k_9t+k_{10}+\frac{\iint A(p,y,t)\,dpdp}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2}\quad,(1\le s<\infty),\tag{118}$$

where f_{1_i} and f_{2_i} are arbitrary second differentiable functions, $k_{i_1}-k_{i_6}$ are arbitrary determined constants.

Consider the following initial value problem of Eq. (114)

$$u(0,x,y)=q(x,y)+\sum_{i=1}^{s}\varphi_i(k_{i_1}x+k_{i_2}y+k_{i_3}),\tag{119}$$

$$u_t(0,x,y)=q_t(x,y)+\sum_{i=1}^{s}\psi_i(k_{i_1}x+k_{i_2}y+k_{i_3}),\tag{120}$$

where φ_i,ψ_i and q are known functions, and

$$q(x,y,t)=k_7x+k_8y+k_9t+k_{10}+\frac{\iint A(p,y,t)\,dpdp}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2},\tag{121}$$

k_7-k_{13} are known constants.

Similar to the solving method of (103), the exact solution of Eq. (114) on the conditions of (119-121) is

$$\begin{aligned}&u(x,y,z)\\&=q(x,y,t)+\frac{1}{2}\sum_{i=1}^{s}(\varphi_i(at\sqrt{k_{i_1}^2+k_{i_2}^2}+k_{i_1}x+k_{i_2}y+k_{i_3})\\&+\varphi_i(-at\sqrt{k_{i_1}^2+k_{i_2}^2}+k_{i_1}x+k_{i_2}y+k_{i_3})\\&+\frac{1}{\sqrt{k_{i_1}^2+k_{i_2}^2}}\int_{-at\sqrt{k_{i_1}^2+k_{i_2}^2}+k_{i_1}x+k_{i_2}y+k_{i_3}}^{at\sqrt{k_{i_1}^2+k_{i_2}^2}+k_{i_1}x+k_{i_2}y+k_{i_3}}\psi_i(\xi_i)\,d\xi_i)\end{aligned}\tag{122}$$

The general solution and the exact solution of the problem of definite solution of the non-homogeneous 1D wave equation can be obtained similarly, which is in Appendix C.

2.4. Acoustic wave equation

In $\mathbb{R}^4$,The form of acoustic wave equation is

$$p_{tt}-c_0^2\triangle p=0\tag{123}$$

where p is the sound pressure and c_0 is the sound speed. Eq. (123) is a special case of Eq. (38), according to (39) its basic general solution in Cartesian coordinate system is

$$
\begin{aligned}
p =& f_1\left(k_1x+k_2y+k_3z+c_0t\sqrt{k_1^2+k_2^2+k_3^2}+k_4\right)\\
&+f_2\left(k_5x+k_6y+k_7z-c_0t\sqrt{k_5^2+k_6^2+k_7^2}+k_8\right)+k_9x+k_{10}y+k_{11}z+k_{12}t+k_{13}\\
&=g\left(\frac{k_1x}{\sqrt{k_1^2+k_2^2+k_3^2}}+\frac{k_2y}{\sqrt{k_1^2+k_2^2+k_3^2}}+\frac{k_3z}{\sqrt{k_1^2+k_2^2+k_3^2}}+c_0t+\frac{k_4}{\sqrt{k_1^2+k_2^2+k_3^2}}\right)\\
&+h\left(\frac{k_5x}{\sqrt{k_5^2+k_6^2+k_7^2}}+\frac{k_6y}{\sqrt{k_5^2+k_6^2+k_7^2}}+\frac{k_7z}{\sqrt{k_5^2+k_6^2+k_7^2}}-c_0t+\frac{k_8}{\sqrt{k_5^2+k_6^2+k_7^2}}\right)\\
&+k_9x+k_{10}y+k_{11}z+k_{12}t+k_{13}\\
&=g(x\sin\theta\cos\varphi+y\sin\theta\sin\varphi+z\cos\theta+c_0t+g_0)\\
&+h(x\sin\phi\cos\psi+y\sin\phi\sin\psi+z\cos\phi-c_0t+h_0)+k_9x+k_{10}y+k_{11}z+k_{12}t+k_{13},
\end{aligned}
\tag{124}
$$

where f_1, f_2, g and h are arbitrary second differentiable functions, $k_1-k_{10}, \theta, \varphi, g_0$ and h_0 are arbitrary constants. $g(x\sin\theta\cos\varphi+y\sin\theta\sin\varphi+z\cos\theta+c_0t+g_0)$ is a parallel wave with the speed c_0, θ is the angle between z axis and spread direction of g, φ is the angle between x axis and the projection in xy plane of spread direction of g.

The series general solution of Eq. (123) is

$$
\begin{aligned}
p =& \sum_i \left(g_i(x\sin\theta_i\cos\varphi_i+y\sin\theta_i\sin\varphi_i+z\cos\theta_i+c_0t+g_{i0}\right)\\
&+h_i(x\sin\phi_i\cos\psi_i+y\sin\phi_i\sin\psi_i+z\cos\phi_i-c_0t+h_{i0}))\\
&+k_9x+k_{10}y+k_{11}z+k_{12}t+k_{13},
\end{aligned}
\tag{125}
$$

where g_i and h_i are arbitrary second differentiable functions, $\theta_i, \varphi_i, g_{i0}$,and h_{i0} are arbitrary determined constants.

The solutions of Eq. (123) have transformational laws which similar to (109, 110), and a similar discussion can be made.

Consider the following initial value problem of Eq. (123)

$$
p\left(x,y,z,0\right)=\sum_i \varphi_i\left(k_{i_1}x+k_{i_2}y+k_{i_3}z+k_{i_4}\right), \tag{126}
$$

$$
p_t\left(x,y,z,0\right)=\sum_i \psi_i\left(k_{i_1}x+k_{i_2}y+k_{i_3}z+k_{i_4}\right). \tag{127}
$$

Similar to the solving method of (43), the exact solution of Eq. (123) on the conditions of (126) and (127) is

$$
\begin{aligned}
p =& \frac{1}{2}\sum_i (\varphi_i(k_{i_1}x+k_{i_2}y+k_{i_3}z+c_0t\sqrt{k_{i_1}^2+k_{i_2}^2+k_{i_3}^2}+k_{i_4})\\
&+\varphi_i(k_{i_1}x+k_{i_2}y+k_{i_3}z-c_0t\sqrt{k_{i_1}^2+k_{i_2}^2+k_{i_3}^2}+k_{i_4})\\
&+\frac{1}{c_0\sqrt{k_{i_1}^2+k_{i_2}^2+k_{i_3}^2}}\int_{k_{i_1}x+k_{i_2}y+k_{i_3}z-c_0t\sqrt{k_{i_1}^2+k_{i_2}^2+k_{i_3}^2}+k_{i_4}}^{k_{i_1}x+k_{i_2}y+k_{i_3}z+c_0t\sqrt{k_{i_1}^2+k_{i_2}^2+k_{i_3}^2}+k_{i_4}}\psi_i(\xi)d\xi)
\end{aligned}
\tag{128}
$$

In $\mathbb{R}^4$,consider the following nonhomogeneous acoustic wave equation

$$
p_{tt}-c_0^2\triangle p=A(x,y,z,t) \tag{129}
$$

where $A(x,y,z,t)$ is any known function. According to Z_1 Transformation, set $p(x,y,z,t) = p(X,Y,Z,T), A(x,y,z,t) = A(X,Y,Z,T)$, and

$$T = k_1t + k_2x + k_3y + k_4z \tag{130}$$

$$X = k_5t + k_6x + k_7y + k_8z \tag{131}$$

$$Y = k_9t + k_{10}x + k_{11}y + k_{12}z \tag{132}$$

$$Z = k_{13}t + k_{14}x + k_{15}y + k_{16}z \tag{133}$$

where $k_1 - k_{16}$ are undetermined constants, and set

$$\begin{aligned}
&\frac{\partial\,(X,Y,Z,T)}{\partial\,(x,y,z,t)} = E\\
&= k_4k_7k_{10}k_{13} - k_3k_8k_{10}k_{13} - k_4k_6k_{11}k_{13} + k_2k_8k_{11}k_{13} + k_3k_6k_{12}k_{13} - k_2k_7k_{12}k_{13}\\
&- k_4k_7k_9k_{14} + k_3k_8k_9k_{14} + k_4k_5k_{11}k_{14} - k_1k_8k_{11}k_{14} - k_3k_5k_{12}k_{14} + k_1k_7k_{12}k_{14}\\
&+ k_4k_6k_9k_{15} - k_2k_8k_9k_{15} - k_4k_5k_{10}k_{15} + k_1k_8k_{10}k_{15} + k_2k_5k_{12}k_{15} - k_1k_6k_{12}k_{15}\\
&- k_3k_6k_9k_{16} + k_2k_7k_9k_{16} + k_3k_5k_{10}k_{16} - k_1k_7k_{10}k_{16} - k_2k_5k_{11}k_{16} + k_1k_6k_{11}k_{16} \neq 0
\end{aligned}$$

Then

$$x = -\frac{B}{C}, y = -\frac{D}{E}, z = \frac{F}{E}, t = \frac{G}{E},$$

$$\begin{aligned}
B = &((-k_4k_5 + k_1k_8)(-k_3k_9 + k_1k_{11}) - (-k_3k_5 + k_1k_7)(-k_4k_9 + k_1k_{12}))((-k_4k_5 + k_1k_8)\\
&(-Zk_1 + Tk_{13}) - (-Xk_1 + Tk_5)(-k_4k_{13} + k_1k_{16})) - ((-k_4k_5 + k_1k_8)(-Yk_1 + Tk_9)\\
&- (-Xk_1 + Tk_5)(-k_4k_9 + k_1k_{12}))((-k_4k_5 + k_1k_8)(-k_3k_{13} + k_1k_{15}) - (-k_3k_5 + k_1k_7)\\
&(-k_4k_{13} + k_1k_{16}))
\end{aligned}$$

$$\begin{aligned}
C = &((-k_4k_5 + k_1k_8)(-k_3k_9 + k_1k_{11}) - (-k_3k_5 + k_1k_7)(-k_4k_9 + k_1k_{12}))((-k_4k_5 + k_1k_8)\\
&(-k_2k_{13} + k_1k_{14}) - (-k_2k_5 + k_1k_6)(-k_4k_{13} + k_1k_{16})) - ((-k_4k_5 + k_1k_8)(-k_2k_9 + k_1k_{10})\\
&- (-k_2k_5 + k_1k_6)(-k_4k_9 + k_1k_{12}))((-k_4k_5 + k_1k_8)(-k_3k_{13} + k_1k_{15}) - (-k_3k_5 + k_1k_7)\\
&(-k_4k_{13} + k_1k_{16}))
\end{aligned}$$

$$\begin{aligned}
D = &-Zk_4k_6k_9 + Zk_2k_8k_9 + Zk_4k_5k_{10} - Zk_1k_8k_{10} - Zk_2k_5k_{12} + Zk_1k_6k_{12} + Yk_4k_6k_{13}\\
&- Yk_2k_8k_{13} - Xk_4k_{10}k_{13} + Tk_8k_{10}k_{13} + Xk_2k_{12}k_{13} - Tk_6k_{12}k_{13} - Yk_4k_5k_{14} + Yk_1k_8k_{14}\\
&+ Xk_4k_9k_{14} - Tk_8k_9k_{14} - Xk_1k_{12}k_{14} + Tk_5k_{12}k_{14} + Yk_2k_5k_{16} - Yk_1k_6k_{16} - Xk_2k_9k_{16}\\
&+ Tk_6k_9k_{16} + Xk_1k_{10}k_{16} - Tk_5k_{10}k_{16}
\end{aligned}$$

$$\begin{aligned}
F = &-Zk_3k_6k_9 + Zk_2k_7k_9 + Zk_3k_5k_{10} - Zk_1k_7k_{10} - Zk_2k_5k_{11} + Zk_1k_6k_{11} + Yk_3k_6k_{13}\\
&- Yk_2k_7k_{13} - Xk_3k_{10}k_{13} + Tk_7k_{10}k_{13} + Xk_2k_{11}k_{13} - Tk_6k_{11}k_{13} - Yk_3k_5k_{14} + Yk_1k_7k_{14}\\
&+ Xk_3k_9k_{14} - Tk_7k_9k_{14} - Xk_1k_{11}k_{14} + Tk_5k_{11}k_{14} + Yk_2k_5k_{15} - Yk_1k_6k_{15} - Xk_2k_9k_{15}\\
&+ Tk_6k_9k_{15} + Xk_1k_{10}k_{15} - Tk_5k_{10}k_{15}
\end{aligned}$$

$$\begin{aligned}
G = &Zk_4k_7k_{10} - Zk_3k_8k_{10} - Zk_4k_6k_{11} + Zk_2k_8k_{11} + Zk_3k_6k_{12} - Zk_2k_7k_{12} - Yk_4k_7k_{14}\\
&+ Yk_3k_8k_{14} + Xk_4k_{11}k_{14} - Tk_8k_{11}k_{14} - Xk_3k_{12}k_{14} + Tk_7k_{12}k_{14} + Yk_4k_6k_{15} - Yk_2k_8k_{15}\\
&- Xk_4k_{10}k_{15} + Tk_8k_{10}k_{15} + Xk_2k_{12}k_{15} - Tk_6k_{12}k_{15} - Yk_3k_6k_{16} + Yk_2k_7k_{16} + Xk_3k_{10}k_{16}\\
&- Tk_7k_{10}k_{16} - Xk_2k_{11}k_{16} + Tk_6k_{11}k_{16}
\end{aligned}$$

So

$$\begin{aligned}
u_{tt}-a^2\Delta u &= \left(k_1^2-a^2k_2^2-a^2k_3^2-a^2k_4^2\right)u_{TT}+\left(k_5^2-a^2k_6^2-a^2k_7^2-a^2k_8^2\right)u_{XX}\\
&+\left(k_9^2-a^2k_{10}^2-a^2k_{11}^2-a^2k_{12}^2\right)u_{YY}+\left(k_{13}^2-a^2k_{14}^2-a^2k_{15}^2-a^2k_{16}^2\right)u_{ZZ}\\
&+2\left(k_1k_5-a^2k_2k_6-a^2k_3k_7-a^2k_4k_8\right)u_{TX}\\
&+2\left(k_1k_9-a^2k_2k_{10}-a^2k_3k_{11}-a^2k_4k_{12}\right)u_{TY}\\
&+2\left(k_1k_{13}-a^2k_2k_{14}-a^2k_3k_{15}-a^2k_4k_{16}\right)u_{TZ}\\
&+2\left(k_5k_9-a^2k_6k_{10}-a^2k_7k_{11}-a^2k_8k_{12}\right)u_{XY}\\
&+2\left(k_5k_{13}-a^2k_6k_{14}-a^2k_7k_{15}-a^2k_8k_{16}\right)u_{XZ}\\
&+2\left(k_9k_{13}-a^2k_{10}k_{14}-a^2k_{11}k_{15}-a^2k_{12}k_{16}\right)u_{YZ}\\
&=A\left(X,Y,Z,T\right).
\end{aligned}\tag{134}$$

Similar to the solving method of (74), the particular solution of Eq. (129) is

$$p=B\left(x,y,z,t\right)=\frac{\iint A\left(x,y,z,T\right)dTdT}{k_1^2-c_0^2k_2^2-c_0^2k_3^2-c_0^2k_4^2},\tag{135}$$

where

$$T=k_1t+k_2x+k_3y+k_4z,\quad \left(k_1,k_2,k_3,k_4\neq 0\right).$$

It can be verified that we only obtain excrescent general solutions of Eq. (129) by (134). According to (124) and (135), the basic general solution of Eq. (129) may be written as

$$\begin{aligned}
p &= f_1\left(k_1x+k_2y+k_3z+c_0t\sqrt{k_1^2+k_2^2+k_3^2}+k_4\right)\\
&+f_2\left(k_5x+k_6y+k_7z-c_0t\sqrt{k_5^2+k_6^2+k_7^2}+k_8\right)\\
&+k_9x+k_{10}y+k_{11}z+k_{12}t+\frac{\iint A\left(x,y,z,T\right)dTdT}{k_{13}^2-c_0^2k_{14}^2-c_0^2k_{15}^2-c_0^2k_{16}^2},
\end{aligned}\tag{136}$$

where k_1-k_{12} are absolutely arbitrary constants, $k_{13}-k_{16}$ are relatively arbitrary constants, and $T=k_{13}t+k_{14}x+k_{15}y+k_{16}z$.

The series general solution of Eq. (129) is

$$\begin{aligned}
p &= \sum_{i=1}^{s}(f_{i_1}(k_{i_1}x+k_{i_2}y+k_{i_3}z+c_0t\sqrt{k_{i_1}^2+k_{i_2}^2+k_{i_3}^2}+k_{i_4})\\
&+f_{i_2}(k_{i_5}x+k_{i_6}y+k_{i_7}z-c_0t\sqrt{k_{i_5}^2+k_{i_6}^2+k_{i_7}^2}+k_{i_8}))\\
&+k_9x+k_{10}y+k_{11}z+k_{12}t+\frac{\iint A\left(x,y,z,T\right)dTdT}{k_{13}^2-c_0^2k_{14}^2-c_0^2k_{15}^2-c_0^2k_{16}^2},
\end{aligned}\tag{137}$$

Consider the following initial value problem of Eq. (129)

$$u\left(0,x,y,z\right)=q\left(x,y,z\right)+\sum_{i=1}^{s}\varphi_i\left(k_{i_1}x+k_{i_2}y+k_{i_3}z+k_{i_4}\right),\tag{138}$$

$$u_t\left(0,x,y,z\right)=q_t\left(x,y,z\right)+\sum_{i=1}^{s}\psi_i\left(k_{i_1}x+k_{i_2}y+k_{i_3}z+k_{i_4}\right),\tag{139}$$

where φ_i, ψ_i and q are known functions, and

$$q(x,y,z,t) = k_9 x + k_{10} y + k_{11} z + k_{12} t + \frac{\iint A(x,y,z,T)\, dT dT}{k_{13}^2 - c_0^2 k_{14}^2 - c_0^2 k_{15}^2 - c_0^2 k_{16}^2}, \tag{140}$$

$k_9 - k_{16}$ are known constants.

According to (109), set $k_{i_1} = k_{i_4}, k_{i_2} = k_{i_5}, k_{i_3} = k_{i_6}$, similar to the solving method of (103), the exact solution of Eq. (129) on the conditions of (138-140) is

$$\begin{aligned} u(x,y,z,t) = {} & q(x,y,z,t) + \frac{1}{2}\sum_i (\varphi_i(k_{i_1}x + k_{i_2}y + k_{i_3}z + c_0 t\sqrt{k_{i_1}^2 + k_{i_2}^2 + k_{i_3}^2} + k_{i_4}) \\ & + \varphi_i(k_{i_1}x + k_{i_2}y + k_{i_3}z - c_0 t\sqrt{k_{i_1}^2 + k_{i_2}^2 + k_{i_3}^2} + k_{i_4}) \\ & + \frac{1}{c_0\sqrt{k_{i_1}^2 + k_{i_2}^2 + k_{i_3}^2}} \int_{k_{i_1}x + k_{i_2}y + k_{i_3}z - c_0 t\sqrt{k_{i_1}^2 + k_{i_2}^2 + k_{i_3}^2} + k_{i_4}}^{k_{i_1}x + k_{i_2}y + k_{i_3}z + c_0 t\sqrt{k_{i_1}^2 + k_{i_2}^2 + k_{i_3}^2} + k_{i_4}} \psi_i(\xi)\, d\xi). \end{aligned} \tag{141}$$

3. New principles and methods II

In Z_1 Transformation, $y_i(x_1, \cdots x_n)$ and f are both undetermined $(i = 1, 2, \cdots l)$. To solve some PDEs we may be required to set f pending and $y_i(x_1, \cdots x_n)$ known, so put forward Z_2 Transformation.

Z_2 **Transformation.** *In the domain D, $(D \subset \mathbb{R}^n)$, any established mth-order PDE with n space variables $F(x_1, \cdots x_n, u, u_{x_1}, \cdots u_{x_n}, u_{x_1 x_2}, \cdots) = 0$, set $y_i = y_i(x_1, \cdots x_n)$ known and $u = f(y_1, \cdots y_l)$ undetermined $(u, y_i \in C^m(D), i \in \{1, 2, \cdots l\}, 1 \le l \le n)$, $y_1, y_2, \cdots y_l$ are independent of each other, then substitute $u = f(y_1, \cdots y_l)$ and its partial derivatives into $F = 0$*
1. In case of working out $f(y_1, \cdots y_l)$, then $u = f(y_1, \ldots y_l)$ is the solution of $F = 0$,
2. In case of dividing out $u = f(y_1, \ldots y_l)$ and its partial derivative, also getting $0 = 0$, then $u = f(y_1, \cdots y_l)$ is the solution of $F = 0$, and f is an arbitrary mth-differentiable function,
3. In case of dividing out $u = f(y_1, \cdots y_l)$ and its partial derivative, also getting $k = 0$, but in fact $k \neq 0$, then $u = f(y_1, \cdots y_l)$ is not the solution of $F = 0$, and f is an arbitrary mth-differentiable function.

We will research the application of Z_2 Transformation later in this paper. Through the comparison, we can find that the traveling wave method and the solitary wave method are the concrete applications of Z_1 and Z_2 Transformation.

Now we study an important compound law of multivariate functions. In $\mathbb{R}^n$ space $(n \ge 2)$, assuming u, g, h and y_i are smooth functions, and set

$$u = g(x_1, x_2, \ldots x_n)\, h(y_1, y_2, \ldots y_l), (1 \le l \le n),$$

where $y_i = y_i(x_1, x_2, \cdots x_n)$, and $y_1, y_2, \cdots y_l$ are independent of each other, then

$$\begin{aligned} du &= u_{x_1}dx_1 + u_{x_2}dx_2 + \ldots + u_{x_n}dx_n = hdg + gdh \\ &= \left(hg_{x_1} + g\sum_{i=1}^{l} h_{y_i} y_{i x_1}\right) dx_1 + \left(hg_{x_2} + g\sum_{i=1}^{l} h_{y_i} y_{i x_2}\right) dx_2 + \ldots \\ &+ \left(hg_{x_n} + g\sum_{i=1}^{l} h_{y_i} y_{i x_n}\right) dx_n. \end{aligned}$$

So

$$u_{x_j} = hg_{x_j} + g\sum_{i=1}^{l} h_{y_i} y_{ix_j} \tag{142}$$

According to (142), we can get

$$u_{x_jx_k} = hg_{x_jx_k} + g\sum_{i=1}^{l} h_{y_i} y_{ix_jx_k} + g_{x_j}\sum_{i=1}^{l} h_{y_i} y_{ix_k} + g_{x_k}\sum_{i=1}^{l} h_{y_i} y_{ix_j} + g\sum_{i=1}^{l}\sum_{s=1}^{l} h_{y_iy_s} y_{ix_j} y_{sx_k} \tag{143}$$

and

$$u_{x_jx_j} = hg_{x_jx_j} + g\sum_{i=1}^{l} h_{y_i} y_{ix_jx_j} + 2g_{x_j}\sum_{i=1}^{l} h_{y_i} y_{ix_j} + g\sum_{i=1}^{l}\sum_{s=1}^{l} h_{y_iy_s} y_{ix_j} y_{sx_j} \tag{144}$$

Higher order law may be deduced analogously.

According to the above laws we present Z_3 Transformation.

Z_3 **Transformation.** *In the domain D, $(D \subset \mathbb{R}^n)$, any established mth-order PDE with n space variables $F(x_1, \cdots x_n, u, u_{x_1}, \cdots u_{x_n}, u_{x_1x_2}, \cdots) = 0$, setting $g(x_1, \cdots x_n), h(y_1, \cdots y_l)$ and $y_i = y_i(x_1, \cdots x_n)$ are all undetermined function, $y_1, y_2, \cdots y_l$ are independent of each other, $(g, h, y_i \in C^m(D), i \in \{1, 2, \cdots l\}, 1 \leq l \leq n)$, then substitute $u = gh(y_1, \cdots y_l)$ and its partial derivatives into $F = 0$,*
1. In case of working out h, g and y_i, then $u = gh(y_1, \cdots y_l)$ is the solution of $F = 0$,
2. In case of dividing out h and its partial derivative, also working out g and y_i, then $u = gh(y_1, \cdots y_l)$ is the solution of $F = 0$, and h is an arbitrary mth-differentiable function,
3. In case of getting $k = 0$, but in fact $k \neq 0$, then $u = gh(y_1, \cdots y_l)$ is not the solution of $F = 0$.

In Z_3 Transformation $y_i(x_1, \cdots x_n)$ and $g(x_1, \cdots x_n)$ may be unknown completely or have definite forms with unknown constants, the solution of h, y_i and g may not be single.

To solve some PDEs we may be required to set $h, y_i(x_1, \cdots x_n)$ undetermined and g known or set g, h undetermined and $y_i(x_1, \cdots x_n)$ known and so on. The forms of these laws are similar to Z_3 Transformation, we will not present here.

4. Solutions of Mathematical Physics Equation II

4.1. Helmholtz equation

Before research Helmholtz equation, we first consider a PDE as follows

$$a_1u_{xx} + a_2u_{yy} + a_3u_{zz} + a_4u_{xy} + a_5u_{yz} + a_6u_{zx} = a_7, \tag{145}$$

where $a_i = a_i(x, y, z, u), (i = 1, 2, \cdots 7)$, according to Z_1 Transformation, set

$$u(x, y, z) = f(v) = f(k_1x + k_2y + k_3z + k_4), \tag{146}$$

where $k_1 - k_4$ are constants to be determined, f is an undetermined unary function, then

$$\begin{aligned} &a_1u_{xx} + a_2u_{yy} + a_3u_{zz} + a_4u_{xy} + a_5u_{yz} + a_6u_{zx} \\ &= k_1^2a_1f_v'' + k_2^2a_2f_v'' + k_3^2a_3f_v'' + k_1k_2a_4f_v'' + k_2k_3a_5f_v'' + k_1k_3a_6f_v'' \\ &= a_7. \end{aligned}$$

Namely

$$f_v'' = \frac{a_7}{k_1^2a_1 + k_2^2a_2 + k_3^2a_3 + k_1k_2a_4 + k_2k_3a_5 + k_1k_3a_6}. \tag{147}$$

If $\frac{a_7}{k_1^2a_1+k_2^2a_2+k_3^2a_3+k_1k_2a_4+k_2k_3a_5+k_1k_3a_6}$ can be converted into $g(v)$ or $h(f)$, it can be further computed, set

$$a_i\left(x,y,z,u\right)=a_i\left(v\right),\left(i=1,2,\ldots 7\right), \tag{148}$$

So the particular solution of Eq. (145) on the condition of (148) is

$$u\left(x,y,z\right)=\iint\frac{a_7dvdv}{k_1^2a_1+k_2^2a_2+k_3^2a_3+k_1k_2a_4+k_2k_3a_5+k_1k_3a_6}+C_1v+C_2, \tag{149}$$

where C_1 and C_2 are arbitrary constant, k_1-k_4 are determinate constants. For instance

$$u_{xx}+\left(k_1x+k_2y+k_3z+k_4\right)^m u_{yy}+\left(k_1x+k_2y+k_3z+k_4\right)^n u_{zz}=\sin\left(k_1x+k_2y+k_3z+k_4\right). \tag{150}$$

According to (149) its particular solution is

$$u\left(x,y,z\right)=\iint\frac{\sin v dvdv}{k_1^2+k_2^2v^m+k_3^2v^n}+C_1v+C_2,$$

where $v(x,y,z)=k_1x+k_2y+k_3z+k_4$. Set

$$a_i\left(x,y,z,u\right)=a_i\left(u\right),\left(i=1,2,\ldots 7\right). \tag{151}$$

From (146)-(147) we have

$$\begin{aligned} f_v'' &= \frac{a_7}{k_1^2a_1+k_2^2a_2+k_3^2a_3+k_1k_2a_4+k_2k_3a_5+k_1k_3a_6} \\ \Longrightarrow v &= C_1\pm\int\left(C_2+2\int\frac{a_7df}{k_1^2a_1+k_2^2a_2+k_3^2a_3+k_1k_2a_4+k_2k_3a_5+k_1k_3a_6}\right)^{-\frac{1}{2}}df, \end{aligned}$$

where k_1-k_4, C_1 and C_2 are arbitrary constant. Namely

$$a_1\left(u\right)u_{xx}+a_2\left(u\right)u_{yy}+a_3\left(u\right)u_{zz}+a_4\left(u\right)u_{xy}+a_5\left(u\right)u_{yz}+a_6\left(u\right)u_{zx}=a_7\left(u\right). \tag{152}$$

The particular solution of Eq. (152) is

$$v=C_1\pm\int\left(C_2+2\int\frac{a_7du}{k_1^2a_1+k_2^2a_2+k_3^2a_3+k_1k_2a_4+k_2k_3a_5+k_1k_3a_6}\right)^{-\frac{1}{2}}du. \tag{153}$$

The solving method of Eq. (145) can be extended to any similar PDEs with n space variables. Emden-Fowler equation [13, 14], Klein-Gordon equation [15, 16] and sine-Gordon equation [17] are special cases of Eq. (145), which are the hotspots of current research.

Consider the following PDE

$$a_1u_{xx}+a_2u_{yy}+a_3u_{zz}+k^2u=0 \tag{154}$$

It's a special case of Eq. (152), according to (153)

$$\begin{aligned} v &= C_1\pm\int\left(C_2-2\int\frac{k^2udu}{k_1^2a_1+k_2^2a_2+k_3^2a_3}\right)^{-\frac{1}{2}}du \\ &= C_1\pm\frac{\sqrt{k_1^2a_1+k_2^2a_2+k_3^2a_3}\arcsin\left(C_3u\right)}{k} \\ \Rightarrow u &= \frac{1}{C_3}\sin\left(\frac{\pm k\left(v-C_1\right)}{\sqrt{k_1^2a_1+k_2^2a_2+k_3^2a_3}}\right) \\ &= \pm C_4\sin\left(\frac{C_5+k\left(k_1x+k_2y+k_3z\right)}{\sqrt{k_1^2a_1+k_2^2a_2+k_3^2a_3}}\right). \end{aligned}$$

Since C_4 is an arbitrary constant, so the particular solution of Eq. (154) can be written as

$$u\left(x,y,z\right)=C_4\sin\left(\frac{C_5+k\left(k_1x+k_2y+k_3z\right)}{\sqrt{k_1^2a_1+k_2^2a_2+k_3^2a_3}}\right), \tag{155}$$

where k_1-k_3, C_4 and C_5 are arbitrary constant.

We use Z_3 Transformation to obtain the general solution of Eq. (154), set

$$u\left(x,y,z\right)=g\left(x,y,z\right)h\left(w\right)=g\left(x,y,z\right)h\left(l_1x+l_2y+l_3z+l_4\right), \tag{156}$$

where $w(x,y,z)=l_1x+l_2y+l_3z+l_4$, l_1-l_4 are undetermined parameters, $h(w)$ and $g(x,y,z)$ are undetermined second differentiable functions, so

$$\begin{aligned}&a_1u_{xx}+a_2u_{yy}+a_3u_{zz}+k^2u\\&=a_1hg_{xx}+2a_1l_1g_xh_w^{'}+a_1l_1^2gh_w^{''}+a_2hg_{yy}+2a_2l_2g_yh_w^{'}\\&+a_2l_2^2gh_w^{''}+a_3hg_{zz}+2a_3l_3g_zh_w^{'}+a_3l_3^2gh_w^{''}+k^2gh.\end{aligned}$$

Namely

$$\left(a_1l_1^2+a_2l_2^2+a_3l_3^2\right)gh_w^{''}+2\left(a_1l_1g_x+a_2l_2g_y+a_3l_3g_z\right)h_w^{'}+\left(a_1g_{xx}+a_2g_{yy}+a_3g_{zz}+k^2g\right)h=0. \tag{157}$$

Set $h(w)$ an arbitrary unary second differentiable function, according to (157) we obtain

$$a_1l_1^2+a_2l_2^2+a_3l_3^2=0\Rightarrow l_1=\pm\sqrt{\frac{-a_2l_2^2-a_3l_3^2}{a_1}}, \tag{158}$$

$$a_1l_1g_x+a_2l_2g_y+a_3l_3g_z=0, \tag{159}$$

$$a_1g_{xx}+a_2g_{yy}+a_3g_{zz}+k^2g=0. \tag{160}$$

By (155) the particular solution of Eq. (160) is

$$g\left(x,y,z\right)=C_4\sin\left(\frac{C_5+k\left(k_1x+k_2y+k_3z\right)}{\sqrt{k_1^2a_1+k_2^2a_2+k_3^2a_3}}\right). \tag{161}$$

Substituting from (161) into (159) we get

$$\begin{aligned}&a_1l_1g_x+a_2l_2g_y+a_3l_3g_z\\&=\frac{a_1l_1C_4kk_1+a_2l_2C_4kk_2+a_3l_3C_4kk_3}{\sqrt{k_1^2a_1+k_2^2a_2+k_3^2a_3}}\cos\left(\frac{C_5+k\left(k_1x+k_2y+k_3z\right)}{\sqrt{k_1^2a_1+k_2^2a_2+k_3^2a_3}}\right)=0\\&\Longrightarrow a_1l_1C_4kk_1+a_2l_2C_4kk_2+a_3l_3C_4kk_3=0.\end{aligned}$$

Namely

$$k_1=\frac{-a_2k_2l_2-a_3k_3l_3}{a_1l_1}. \tag{162}$$

Then

$$\begin{aligned}&u\left(x,y,z\right)=g\left(x,y,z\right)h\left(w\right)\\&=\sin\left(\frac{C_5+k\left(k_1x+k_2y+k_3z\right)}{\sqrt{k_1^2a_1+k_2^2a_2+k_3^2a_3}}\right)h\left(l_1x+l_2y+l_3z+l_4\right)\\&=\sin\left(\frac{C_5a_1l_1-k\left(a_2k_2l_2+a_3k_3l_3\right)x+ka_1l_1\left(k_2y+k_3z\right)}{\sqrt{\left(a_2k_2l_2+a_3k_3l_3\right)^2+\left(a_2k_2^2+a_3k_3^2\right)a_1^2l_1^2}}\right)h\left(l_1x+l_2y+l_3z+l_4\right).\end{aligned}$$

So the general solution of Eq. (154) is

$$\begin{aligned} u = \sin & \left(\frac{l_5 - k\left(a_2 k_2 l_2 + a_3 k_3 l_3\right) x + k\sqrt{-a_1 a_2 l_2^2 - a_1 a_3 l_3^2}\left(k_2 y + k_3 z\right)}{\sqrt{\left(a_2 k_2 l_2 + a_3 k_3 l_3\right)^2 - a_1\left(a_2 k_2^2 + a_3 k_3^2\right)\left(a_2 l_2^2 + a_3 l_3^2\right)}} \right) \\ h_1 & \left(\sqrt{\frac{-a_2 l_2^2 - a_3 l_3^2}{a_1}} x + l_2 y + l_3 z + l_4 \right) \\ + \sin & \left(\frac{l_{15} - k\left(a_2 k_{12} l_{12} + a_3 k_{13} l_{13}\right) x - k\sqrt{-a_1 a_2 l_{12}^2 - a_1 a_3 l_{13}^2}\left(k_{12} y + k_{13} z\right)}{\sqrt{\left(a_2 k_{12} l_{12} + a_3 k_{13} l_{13}\right)^2 - a_1\left(a_2 k_{12}^2 + a_3 k_{13}^2\right)\left(a_2 l_{12}^2 + a_3 l_{13}^2\right)}} \right) \\ h_2 & \left(-\sqrt{\frac{-a_2 l_{12}^2 - a_3 l_{13}^2}{a_1}} x + l_{12} y + l_{13} z + l_{14} \right), \end{aligned} \tag{163}$$

where h_1 and h_2 are arbitrary second differentiable functions, k_2,k_3,k_{12},k_{13},$l_2 - l_5$ and $l_{12} - l_{15}$ are arbitrary constants.

Consider the following 3D Helmholtz equation

$$u_{xx} + u_{yy} + u_{zz} + k^2 u = 0. \tag{164}$$

According to (163) we can get the general solution of Eq. (164) is

$$\begin{aligned} u = \sin & \left(\frac{l_5 - k\left(k_2 l_2 + k_3 l_3\right) x + k\sqrt{-l_2^2 - l_3^2}\left(k_2 y + k_3 z\right)}{\sqrt{\left(k_2 l_2 + k_3 l_3\right)^2 - \left(k_2^2 + k_3^2\right)\left(l_2^2 + l_3^2\right)}} \right) h_1\left(\sqrt{-l_2^2 - l_3^2} x + l_2 y + l_3 z + l_4 \right) \\ + \sin & \left(\frac{l_{15} - k\left(k_{12} l_{12} + k_{13} l_{13}\right) x - k\sqrt{-l_{12}^2 - l_{13}^2}\left(k_{12} y + k_{13} z\right)}{\sqrt{\left(k_{12} l_{12} + k_{13} l_{13}\right)^2 - \left(k_{12}^2 + k_{13}^2\right)\left(l_{12}^2 + l_{13}^2\right)}} \right) \\ h_2 & \left(-\sqrt{-l_{12}^2 - l_{13}^2} x + l_{12} y + l_{13} z + l_{14} \right), \end{aligned} \tag{165}$$

Consider the following 2D Helmholtz equation

$$u_{xx} + u_{yy} + k^2 u = 0. \tag{166}$$

By (163) the general solution of Eq. (166) could be got

$$\begin{aligned} u = \sin & \left(\frac{C_6 - k\left(a_2 k_2 l_2 + a_3 k_3 l_3\right) x + k\sqrt{-a_1 a_2 l_2^2 - a_1 a_3 l_3^2}\left(k_2 y + k_3 z\right)}{\sqrt{\left(a_2 k_2 l_2\right)^2 - \left(a_2 k_2^2\right)\left(a_2 l_2^2\right)}} \right) \\ h_1 & \left(\sqrt{\frac{-a_2 l_2^2}{a_1}} x + l_2 y + l_4 \right) \\ + \sin & \left(\frac{C_8 - k\left(a_2 k_{12} l_{12} + a_3 k_{13} l_{13}\right) x - k\sqrt{-a_1 a_2 l_{12}^2 - a_1 a_3 l_{13}^2}\left(k_{12} y + k_{13} z\right)}{\sqrt{\left(a_2 k_{12} l_{12}\right)^2 - \left(a_2 k_{12}^2\right)\left(a_2 l_{12}^2\right)}} \right) \\ h_2 & \left(-\sqrt{\frac{-a_2 l_{12}^2 - a_3 l_{13}^2}{a_1}} x + l_{12} y + l_{13} z + l_{14} \right). \end{aligned}$$

The denominator of the above equation is equal to zero, so we can preliminarily judge that Eq. (166) has no general solution.

For the 1D Helmholtz equation $u_{xx}+k^2u=0$, according to (163) we can get that the denominator is equal to zero, so it can be judged preliminarily that 1D Helmholtz equation has no general solution.

Currently analysing the Helmholtz equation is mainly used numerical methods [18-21]. Here we consider the following boundary value problem of Eq. (164)

$$u\left(0,y,z\right)=\sin\left(\sqrt{2}k\left(y+2z\right)\right)\varphi\left(y+z\right), \tag{167}$$

$$u_x\left(0,y,z\right)=\sqrt{-2}\sin\left(\sqrt{2}k\left(y+2z\right)\right)\phi'\left(x+y\right)+3ki\cos\left(\sqrt{2}k\left(y+2z\right)\right)\phi\left(x+y\right), \tag{168}$$

where φ, ϕ are known function, comparing (165) with (167) we obtain

$$k_2=k_{12}=l_2=l_3=l_{12}=l_{13}=1, k_3=k_{13}=2, l_4=l_{14}=C_6=C_8=0.$$

Namely

$$\begin{aligned}u=&\sin\left(3kix+\sqrt{2}k\left(y+2z\right)\right)h_1\left(\sqrt{-2}x+y+z\right)\\&+\sin\left(3kix-\sqrt{2}k\left(y+2z\right)\right)h_2\left(-\sqrt{-2}x+y+z\right).\end{aligned} \tag{169}$$

Then

$$\begin{aligned}u\left(0,y,z\right)=&\sin\left(\sqrt{2}k\left(y+2z\right)\right)h_1\left(y+z\right)-\sin\left(\sqrt{2}k\left(y+2z\right)\right)h_2\left(y+z\right)\\=&\sin\left(\sqrt{2}k\left(y+2z\right)\right)\varphi\left(y+z\right)\Longrightarrow h_1\left(y+z\right)-h_2\left(y+z\right)=\varphi\left(y+z\right),\end{aligned}$$

$$\begin{aligned}u_x\left(0,y,z\right)=&\sqrt{-2}\sin\left(\sqrt{2}k\left(y+2z\right)\right)\left(h_1'\left(y+z\right)+h_2'\left(y+z\right)\right)\\&+3ki\cos\left(\sqrt{2}k\left(y+2z\right)\right)\left(h_1\left(y+z\right)+h_2\left(y+z\right)\right)\\=&\sqrt{-2}\sin\left(\sqrt{2}k\left(y+2z\right)\right)\phi'\left(x+y\right)+3ki\cos\left(\sqrt{2}k\left(y+2z\right)\right)\phi\left(x+y\right)\\\Longrightarrow&\, h_1\left(y+z\right)+h_2\left(y+z\right)=\phi\left(x+y\right).\end{aligned}$$

Namely

$$h_1\left(y+z\right)-h_2\left(y+z\right)=\varphi\left(y+z\right) \tag{170}$$

$$h_1\left(y+z\right)+h_2\left(y+z\right)=\phi\left(x+y\right) \tag{171}$$

Then

$$\begin{aligned}&h_1\left(y+z\right)=\frac{1}{2}\left(\phi\left(y+z\right)+\varphi\left(y+z\right)\right)\\&\Longrightarrow h_1\left(\sqrt{-2}x+y+z\right)=\frac{1}{2}\left(\phi\left(\sqrt{-2}x+y+z\right)+\varphi\left(\sqrt{-2}x+y+z\right)\right),\\&h_2\left(y+z\right)=\frac{1}{2}\left(\phi\left(y+z\right)-\varphi\left(y+z\right)\right)\\&\Longrightarrow h_2\left(-\sqrt{-2}x+y+z\right)=\frac{1}{2}\left(\phi\left(-\sqrt{-2}x+y+z\right)-\varphi\left(-\sqrt{-2}x+y+z\right)\right).\end{aligned}$$

So the exact solution of Eq. (164) on the conditions of (167) and (168) can be get

$$\begin{aligned}u=&\frac{1}{2}\sin\left(3kix+\sqrt{2}k\left(y+2z\right)\right)\left(\phi\left(\sqrt{-2}x+y+z\right)+\varphi\left(\sqrt{-2}x+y+z\right)\right)\\&+\frac{1}{2}\sin\left(3kix-\sqrt{2}k\left(y+2z\right)\right)\left(\phi\left(\sqrt{-2}x+y+z\right)-\varphi\left(\sqrt{-2}x+y+z\right)\right).\end{aligned} \tag{172}$$

In $\mathbb{R}^3$, we use Z_3 Transformation to get the general solution of the nonhomogeneous Helmholtz equation.

$$u_{xx} + u_{yy} + u_{zz} + k^2 u = A\left(x, y, z\right), \tag{173}$$

where $A(x, y, z)$ is any known function. According to Z_3 Transformation, set

$$u\left(x, y, z\right) = g\left(x, y, z\right) h\left(p, q, r\right) \tag{174}$$

$$p = k_1 x + k_2 y + k_3 z, q = k_4 x + k_5 y + k_6 z, r = k_7 x + k_8 y + k_9 z, \tag{46}$$

and

$$-k_3 k_5 k_7 + k_2 k_6 k_7 + k_3 k_4 k_8 - k_1 k_6 k_8 - k_2 k_4 k_9 + k_1 k_5 k_9 \neq 0. \tag{47}$$

$$x = -\frac{-r k_3 k_5 + r k_2 k_6 + q k_3 k_8 - p k_6 k_8 - q k_2 k_9 + p k_5 k_9}{k_3 k_5 k_7 - k_2 k_6 k_7 - k_3 k_4 k_8 + k_1 k_6 k_8 + k_2 k_4 k_9 - k_1 k_5 k_9}, \tag{48}$$

$$y = -\frac{r k_3 k_4 - r k_1 k_6 - q k_3 k_7 + p k_6 k_7 + q k_1 k_9 - p k_4 k_9}{k_3 k_5 k_7 - k_2 k_6 k_7 - k_3 k_4 k_8 + k_1 k_6 k_8 + k_2 k_4 k_9 - k_1 k_5 k_9}, \tag{49}$$

$$z = \frac{r k_2 k_4 - r k_1 k_5 - q k_2 k_7 + p k_5 k_7 + q k_1 k_8 - p k_4 k_8}{k_3 k_5 k_7 - k_2 k_6 k_7 - k_3 k_4 k_8 + k_1 k_6 k_8 + k_2 k_4 k_9 - k_1 k_5 k_9}. \tag{50}$$

So

$$\begin{aligned}
& a_1 u_{xx} + a_2 u_{yy} + a_3 u_{zz} + a_4 u \\
& = \left(g_{xx} + g_{yy} + g_{zz} + k^2 g\right) h + 2\left(k_1 g_x + k_2 g_y + k_3 g_z\right) h_p + 2\left(k_4 g_x + k_5 g_y + k_6 g_z\right) h_q \\
& + 2\left(k_7 g_x + k_8 g_y + k_9 g_z\right) h_r + \left(k_1^2 + k_2^2 + k_3^2\right) g h_{pp} + \left(k_4^2 + k_5^2 + k_6^2\right) g h_{qq} \\
& + \left(k_7^2 + k_8^2 + k_9^2\right) g h_{rr} + 2\left(k_1 k_4 + k_2 k_5 + k_3 k_6\right) g h_{pq} + 2\left(k_1 k_7 + k_2 k_8 + k_3 k_9\right) g h_{pr} \\
& + 2\left(k_4 k_7 + k_5 k_8 + k_6 k_9\right) g h_{qr} = A\left(p, q, r\right).
\end{aligned} \tag{175}$$

Set

$$g_{xx} + g_{yy} + g_{zz} + k^2 g = 0, \tag{176}$$

$$k_1 g_x + k_2 g_y + k_3 g_z = 0, \tag{177}$$

$$k_4 g_x + k_5 g_y + k_6 g_z = 0, \tag{178}$$

$$k_7 g_x + k_8 g_y + k_9 g_z = 0. \tag{179}$$

By (155), the particular solution of Eq. (176) is

$$g\left(x, y, z\right) = c_4 \sin\left(\frac{c_5 + k\left(c_1 x + c_2 y + c_3 z\right)}{\sqrt{c_1^2 + c_2^2 + c_3^2}}\right), \tag{180}$$

where $c_1 - c_5$ are arbitrary constants. Substituting from (180) into (177-179) we get

$$\begin{aligned}
& k_1 g_x + k_2 g_y + k_3 g_z = \frac{a_1 k_1 c_4 k c_1 + a_2 k_2 c_4 k c_2 + a_3 k_3 c_4 k c_3}{\sqrt{c_1^2 a_1 + c_2^2 a_2 + c_3^2 a_3}} \cos\left(\frac{c_5 + k\left(c_1 x + c_2 y + c_3 z\right)}{\sqrt{c_1^2 a_1 + c_2^2 a_2 + c_3^2 a_3}}\right) = 0 \\
& \Longrightarrow a_1 k_1 c_4 k c_1 + a_2 k_2 c_4 k c_2 + a_3 k_3 c_4 k c_3 = 0.
\end{aligned}$$

Namely

$$c_1 = \frac{-k_2 c_2 - k_3 c_3}{k_1}. \tag{181}$$

$$k_4 g_x + k_5 g_y + k_6 g_z = \frac{a_1 k_4 c_4 k c_1 + a_2 k_5 c_4 k c_2 + a_3 k_6 c_4 k c_3}{\sqrt{c_1^2 a_1 + c_2^2 a_2 + c_3^2 a_3}} \cos\left(\frac{c_5 + k\left(c_1 x + c_2 y + c_3 z\right)}{\sqrt{c_1^2 a_1 + c_2^2 a_2 + c_3^2 a_3}}\right) = 0$$

$$\Longrightarrow a_1 k_4 c_4 k c_1 + a_2 k_5 c_4 k c_2 + a_3 k_6 c_4 k c_3 = 0.$$

So

$$c_1 = \frac{-k_5 c_2 - k_6 c_3}{k_4}. \tag{182}$$

$$k_7 g_x + k_8 g_y + k_9 g_z = \frac{a_1 k_7 c_4 k c_1 + a_2 k_8 c_4 k c_2 + a_3 k_9 c_4 k c_3}{\sqrt{c_1^2 a_1 + c_2^2 a_2 + c_3^2 a_3}} \cos\left(\frac{c_5 + k\left(c_1 x + c_2 y + c_3 z\right)}{\sqrt{c_1^2 a_1 + c_2^2 a_2 + c_3^2 a_3}}\right) = 0$$

$$\Longrightarrow a_1 k_7 c_4 k c_1 + a_2 k_8 c_4 k c_2 + a_3 k_9 c_4 k c_3 = 0.$$

Thus

$$c_1 = \frac{-k_8 c_2 - k_9 c_3}{k_7}. \tag{183}$$

So Eq. (175) is simplified as

$$\begin{aligned} & u_{xx} + u_{yy} + u_{zz} + k^2 u \\ & = \left(k_1^2 + k_2^2 + k_3^2\right) g h_{pp} + \left(k_4^2 + k_5^2 + k_6^2\right) g h_{qq} + \left(k_7^2 + k_8^2 + k_9^2\right) g h_{rr} \\ & + 2\left(k_1 k_4 + k_2 k_5 + k_3 k_6\right) g h_{pq} + 2\left(k_1 k_7 + k_2 k_8 + k_3 k_9\right) g h_{pr} \\ & + 2\left(k_4 k_7 + k_5 k_8 + k_6 k_9\right) g h_{qr} = A\left(p, q, r\right). \end{aligned} \tag{184}$$

Namely

$$\begin{aligned} & \left(k_1^2 + k_2^2 + k_3^2\right) h_{pp} + \left(k_4^2 + k_5^2 + k_6^2\right) h_{qq} + \left(k_7^2 + k_8^2 + k_9^2\right) h_{rr} + 2\left(k_1 k_4 + k_2 k_5 + k_3 k_6\right) h_{pq} \\ & + 2\left(k_1 k_7 + k_2 k_8 + k_3 k_9\right) h_{pr} + 2\left(k_4 k_7 + k_5 k_8 + k_6 k_9\right) h_{qr} = \frac{A\left(p, q, r\right)}{g}. \end{aligned} \tag{185}$$

Eq. (185) is completely similar to Eq. (60), using similar calculations we can obtain its excrescent general solutions and special solution which is

$$h = \frac{\iint \frac{A(p,q,r)}{g(p,q,r)} dp dp}{k_1^2 + k_2^2 + k_3^2}. \tag{186}$$

So the special solution of Eq. (173) is

$$u = gh = \frac{g \iint \frac{A(p,q,r)}{g(p,q,r)} dp dp}{k_1^2 + k_2^2 + k_3^2}. \tag{187}$$

Combining (165), we can get the basic general solution of Eq. (173).

For 2D nonhomogeneous Helmholtz equation

$$u_{xx} + u_{yy} + k^2 u = A\left(x, y\right).$$

By similar method we can get its particular solution.

4.2. Heat equation and diffusion equation

In $\mathbb{R}^4$,consider the following PDE

$$a_0 u_t + a_1 u_{xx} + a_2 u_{yy} + a_3 u_{zz} = 0, \tag{189}$$

where a_i are known constants. For solving its particular solution, by Z_1 Transformation we set

$$u(t,x,y,z) = f(v) = f(k_0t + k_1x + k_2y + k_3z + k_4), \tag{190}$$

where $v(t,x,y,z) = k_0t + k_1x + k_2y + k_3z + k_4, k_0 - k_4$ are constants to be determined, f is an undetermined second differentiable function. So

$$a_0u_t + a_1u_{xx} + a_2u_{yy} + a_3u_{zz} = a_0k_0f_v' + \left(a_1k_1^2 + a_2k_2^2 + a_3k_3^2\right)f_v'' = 0.$$

Set $w = f_v'$, then

$$\begin{aligned}
&a_0k_0f_v' + \left(a_1k_1^2 + a_2k_2^2 + a_3k_3^2\right)f_v'' = 0\\
&\Longrightarrow \left(a_1k_1^2 + a_2k_2^2 + a_3k_3^2\right)w_v' = -a_0k_0w\\
&\Longrightarrow w = k_7e^{\frac{-a_0k_0v}{a_1k_1^2+a_2k_2^2+a_3k_3^2}}\\
&\Longrightarrow f(v) = -k_7\frac{a_1k_1^2 + a_2k_2^2 + a_3k_3^2}{a_0k_0}e^{\frac{-a_0k_0v}{a_1k_1^2+a_2k_2^2+a_3k_3^2}} + k_6.
\end{aligned}$$

So the particular solution of Eq. (189) is

$$u(t,x,y,z) = k_5e^{\frac{-a_0k_0(k_0t+k_1x+k_2y+k_3z)}{a_1k_1^2+a_2k_2^2+a_3k_3^2}} + k_6, \tag{191}$$

where $k_0 - k_6$ are arbitrary constants.

In order to obtain the general solution of Eq. (189), according to Z_3 Transformation, we set

$$u(t,x,y,z) = gh(w) = g(t,x,y,z)\,h(l_0t + l_1x + l_2y + l_3z + l_4), \tag{192}$$

where $w(t,x,y,z) = l_0t + l_1x + l_2y + l_3z + l_4, l_0 - l_4$ are constants to be determined, h and g are undetermined second differentiable functions. Then

$$\begin{aligned}
&a_0u_t + a_1u_{xx} + a_2u_{yy} + a_3u_{zz}\\
&= \left(a_1l_1^2 + a_2l_2^2 + a_3l_3^2\right)gh_w'' + \left(a_0l_0g + 2a_1l_1g_x + 2a_2l_2g_y + 2a_3l_3g_z\right)h_w'\\
&+ \left(a_0g_t + a_1g_{xx} + a_2g_{yy} + a_3g_{zz}\right)h = 0.
\end{aligned} \tag{192}$$

Set $h(w)$ an arbitrary second differentiable function, according to (193) we get

$$a_1l_1^2 + a_2l_2^2 + a_3l_3^2 = 0 \Longrightarrow l_1 = \pm\sqrt{\frac{-a_2l_2^2 - a_3l_3^2}{a_1}}, \tag{194}$$

$$a_0l_0g + 2a_1l_1g_x + 2a_2l_2g_y + 2a_3l_3g_z = 0, \tag{195}$$

$$a_0g_t + a_1g_{xx} + a_2g_{yy} + a_3g_{zz} = 0. \tag{196}$$

By (191) the particular solution of Eq. (196) is

$$g(t,x,y,z) = k_5e^{\frac{-a_0k_0(k_0t+k_1x+k_2y+k_3z)}{a_1k_1^2+a_2k_2^2+a_3k_3^2}} + k_6, \tag{197}$$

Set $k_6 = 0$, and substituting from (197) into (195), then

$$\begin{aligned}
&a_0l_0g + 2a_1l_1g_x + 2a_2l_2g_y + 2a_3l_3g_z\\
&= a_0l_0k_5e^{\frac{-a_0k_0(k_0t+k_1x+k_2y+k_3z)}{a_1k_1^2+a_2k_2^2+a_3k_3^2}} - 2a_0k_0k_5e^{\frac{-a_0k_0(k_0t+k_1x+k_2y+k_3z)}{a_1k_1^2+a_2k_2^2+a_3k_3^2}}\frac{a_1l_1k_1 + a_2l_2k_2 + a_3l_3k_3}{a_1k_1^2 + a_2k_2^2 + a_3k_3^2} = 0.
\end{aligned}$$

We have

$$l_0 = 2k_0 \frac{a_1 l_1 k_1 + a_2 l_2 k_2 + a_3 l_3 k_3}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2}. \tag{198}$$

Therefore

$$\begin{aligned} u(x,y,z,t) &= g(x,y,z,t)\, h(w) = k_5 e^{\frac{-a_0 k_0 (k_0 t + k_1 x + k_2 y + k_3 z)}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2}} h(l_0 t + l_1 x + l_2 y + l_3 z + l_4) \\ &= e^{\frac{-a_0 k_0 (k_0 t + k_1 x + k_2 y + k_3 z)}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2}} h\left(\frac{2k_0 (a_1 l_1 k_1 + a_2 l_2 k_2 + a_3 l_3 k_3)\, t}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2} + l_1 x + l_2 y + l_3 z + l_4 \right). \end{aligned}$$

So the general solution of Eq. (189) is

$$\begin{aligned} u = {} & e^{\frac{-a_0 k_0 (k_0 t + k_1 x + k_2 y + k_3 z)}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2}} \\ & h_1 \left(\frac{2k_0 \left(k_1 \sqrt{-a_1 (a_2 l_2^2 + a_3 l_3^2)} + a_2 l_2 k_2 + a_3 l_3 k_3 \right) t}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2} + \sqrt{\frac{-a_2 l_2^2 - a_3 l_3^2}{a_1}} x + l_2 y + l_3 z + l_4 \right) \\ & + e^{\frac{-a_0 k_{10} (k_{10} t + k_{11} x + k_{12} y + k_{13} z)}{a_1 k_{11}^2 + a_2 k_{12}^2 + a_3 k_{13}^2}} h_2 (\frac{2k_{10} (-k_{11} \sqrt{-a_1 (a_2 l_{12}^2 + a_3 l_{13}^2)} + a_2 l_{12} k_{12} + a_3 l_{13} k_{13}) t}{a_1 k_{11}^2 + a_2 k_{12}^2 + a_3 k_{13}^2} \\ & - \sqrt{\frac{-a_2 l_{12}^2 - a_3 l_{13}^2}{a_1}} x + l_{12} y + l_{13} z + l_{14}), \end{aligned} \tag{199}$$

where h_1 and h_2 are arbitrary unary second differentiable functions, $k_0 - k_3$, $k_{10} - k_{13}$, $l_2 - l_4$ and $l_{12} - l_{14}$ are arbitrary constants.

The form of 3D heat equation and diffusion equation is

$$u_t - a^2 (u_{xx} + u_{yy} + u_{zz}) = 0. \tag{200}$$

According to (199) we can get the general solution of Eq. (200) is

$$\begin{aligned} u = {} & e^{\frac{k_0 (k_0 t + k_1 x + k_2 y + k_3 z)}{(k_1^2 + k_2^2 + k_3^2) a^2}} h_1 \left(\frac{2k_0 \left(\sqrt{-l_2^2 - l_3^2} k_1 + l_2 k_2 + l_3 k_3 \right)}{k_1^2 + k_2^2 + k_3^2} t + \sqrt{-l_2^2 - l_3^2} x + l_2 y + l_3 z + l_4 \right) \\ & + e^{\frac{k_{10} (k_{10} t + k_{11} x + k_{12} y + k_{13} z)}{(k_{11}^2 + k_{12}^2 + k_{13}^2) a^2}} \\ & h_2 \left(\frac{2k_{10} \left(-\sqrt{-l_{12}^2 - l_{13}^2} k_{11} + l_{12} k_{12} + l_{13} k_{13} \right)}{k_{11}^2 + k_{12}^2 + k_{13}^2} t - \sqrt{-l_{12}^2 - l_{13}^2} x + l_{12} y + l_{13} z + l_{14} \right). \end{aligned} \tag{201}$$

The form of 2D heat equation and diffusion equation is

$$u_t - a^2 (u_{xx} + u_{yy}) = 0. \tag{202}$$

By (199) the general solution of Eq. (202) could be got

$$\begin{aligned} u = {} & e^{\frac{k_0 (k_0 t + k_1 x + k_2 y)}{(k_1^2 + k_2^2) a^2}} h_1 \left(\frac{2k_0 (i l_2 k_1 + l_2 k_2)}{k_1^2 + k_2^2} t + i l_2 x + l_2 y + l_4 \right) \\ & + e^{\frac{k_{10} (k_{10} t + k_{11} x + k_{12} y)}{(k_{11}^2 + k_{12}^2) a^2}} h_2 \left(\frac{2k_{10} (-i l_{12} k_{11} + l_{12} k_{12})}{k_{11}^2 + k_{12}^2} t - i l_{12} x + l_{12} y + l_{14} \right). \end{aligned} \tag{203}$$

The form of 1D heat equation and diffusion equation is

$$u_t - a^2 u_{xx} = 0. \tag{204}$$

According to (199) we have

$$u = Ce^{\frac{k_0(k_0 t + k_1 x)}{a^2 k_1^2}}. \tag{205}$$

Therefore, it can be preliminarily determined that Eq. (204) has no general solution.

Nonlinear problem [22-25] and numerical methods [26-28] are the research hotspots of the heat equation, here we consider the following initial value problem of Eq. (200)

$$u(x,y,z,0) = e^{\frac{x+y+z}{a^2}} \left(\varphi_1\left(\sqrt{-2}x + y + z\right) + \varphi_2\left(-\sqrt{-2}x + y + z\right)\right). \tag{206}$$

Comparing (201) with (206) we get

$$k_1 = k_2 = k_3 = \frac{k_4}{3}, k_{11} = k_{12} = k_{13} = \frac{k_{14}}{3}, l_2 = l_3 = l_{12} = l_{13} = 1, l_5 = l_{15} = 0.$$

So

$$\begin{aligned} & u(x,y,z,t) = e^{\frac{x+y+z+3t}{a^2}} \\ & \left(h_1\left(\sqrt{-2}x + y + z + \left(4 + 2\sqrt{-2}\right)t\right) + h_2\left(-\sqrt{-2}x + y + z + \left(4 - 2\sqrt{-2}\right)t\right)\right). \end{aligned} \tag{207}$$

Then

$$\begin{aligned} & u(x,y,z,0) \\ & = e^{\frac{x+y+z}{a^2}} \left(\varphi_1\left(\sqrt{-2}x + y + z\right) + \varphi_2\left(-\sqrt{-2}x + y + z\right)\right) \\ & = e^{\frac{x+y+z}{a^2}} \left(h_1\left(\sqrt{-2}x + y + z\right) + h_2\left(-\sqrt{-2}x + y + z\right)\right) \\ & \Longrightarrow \varphi_1\left(\sqrt{-2}x + y + z\right) + \varphi_2\left(-\sqrt{-2}x + y + z\right) = h_1\left(\sqrt{-2}x + y + z\right) + h_2\left(-\sqrt{-2}x + y + z\right) \\ & \Longrightarrow \varphi_1\left(\sqrt{-2}x + y + z + \left(4 + 2\sqrt{-2}\right)t\right) = h_1\left(\sqrt{-2}x + y + z + \left(4 + 2\sqrt{-2}\right)t\right) \\ & \varphi_2\left(-\sqrt{-2}x + y + z + \left(4 - 2\sqrt{-2}\right)t\right) = h_2\left(-\sqrt{-2}x + y + z + \left(4 - 2\sqrt{-2}\right)t\right). \end{aligned}$$

Namely

$$\begin{aligned} h_1\left(\sqrt{-2}x + y + z + \left(4 + 2\sqrt{-2}\right)t\right) &= \varphi_1\left(\sqrt{-2}x + y + z + \left(4 + 2\sqrt{-2}\right)t\right), \\ h_2\left(-\sqrt{-2}x + y + z + \left(4 - 2\sqrt{-2}\right)t\right) &= \varphi_2\left(-\sqrt{-2}x + y + z + \left(4 - 2\sqrt{-2}\right)t\right). \end{aligned}$$

So the exact solution of Eq. (200) on the conditions of (206) can be get

$$\begin{aligned} & u(x,y,z,t) \\ & = e^{\frac{x+y+z+3t}{a^2}} \left(\varphi_1\left(\sqrt{-2}x + y + z + \left(4 + 2\sqrt{-2}\right)t\right) + \varphi_2\left(-\sqrt{-2}x + y + z + \left(4 - 2\sqrt{-2}\right)t\right)\right). \end{aligned} \tag{208}$$

In $\mathbb{R}^4$, we use Z_3 Transformation to get the general solution of the nonhomogeneous heat equation.

$$u_t - a^2\left(u_{xx} + u_{yy} + u_{zz}\right) = A(x,y,z,t), \tag{209}$$

where $A(x,y,z,t)$ is any known function. According to Z_3 Transformation, set

$$u(x,y,z,t) = g(x,y,z,t)\, h(X,Y,Z,T), \tag{210}$$

where

$$T = k_1 t + k_2 x + k_3 y + k_4 z, \tag{211}$$

$$X = k_5 t + k_6 x + k_7 y + k_8 z, \tag{212}$$

$$Y = k_9 t + k_{10} x + k_{11} y + k_{12} z, \tag{213}$$

$$Z = k_{13} t + k_{14} x + k_{15} y + k_{16} z, \tag{214}$$

$k_1 - k_{16}$ are undetermined constants, and set

$$\begin{aligned}
&\frac{\partial (X, Y, Z, T)}{\partial (x, y, z, t)} = E \\
&= k_4k_7k_{10}k_{13} - k_3k_8k_{10}k_{13} - k_4k_6k_{11}k_{13} + k_2k_8k_{11}k_{13} + k_3k_6k_{12}k_{13} - k_2k_7k_{12}k_{13} \\
&- k_4k_7k_9k_{14} + k_3k_8k_9k_{14} + k_4k_5k_{11}k_{14} - k_1k_8k_{11}k_{14} - k_3k_5k_{12}k_{14} + k_1k_7k_{12}k_{14} \\
&+ k_4k_6k_9k_{15} - k_2k_8k_9k_{15} - k_4k_5k_{10}k_{15} + k_1k_8k_{10}k_{15} + k_2k_5k_{12}k_{15} - k_1k_6k_{12}k_{15} \\
&- k_3k_6k_9k_{16} + k_2k_7k_9k_{16} + k_3k_5k_{10}k_{16} - k_1k_7k_{10}k_{16} - k_2k_5k_{11}k_{16} + k_1k_6k_{11}k_{16} \neq 0.
\end{aligned} \tag{215}$$

So

$$x = -\frac{B}{C}, y = -\frac{D}{E}, z = \frac{F}{E}, t = \frac{G}{E}, \tag{216}$$

$$\begin{aligned}
B = &((-k_4k_5 + k_1k_8)(-k_3k_9 + k_1k_{11}) - (-k_3k_5 + k_1k_7)(-k_4k_9 + k_1k_{12}))((-k_4k_5 + k_1k_8) \\
&(-Zk_1 + Tk_{13}) - (-Xk_1 + Tk_5)(-k_4k_{13} + k_1k_{16})) - ((-k_4k_5 + k_1k_8)(-Yk_1 + Tk_9) \\
&- (-Xk_1 + Tk_5)(-k_4k_9 + k_1k_{12}))((-k_4k_5 + k_1k_8)(-k_3k_{13} + k_1k_{15}) - (-k_3k_5 + k_1k_7) \\
&(-k_4k_{13} + k_1k_{16})),
\end{aligned}$$

$$\begin{aligned}
C = &((-k_4k_5 + k_1k_8)(-k_3k_9 + k_1k_{11}) - (-k_3k_5 + k_1k_7)(-k_4k_9 + k_1k_{12}))((-k_4k_5 + k_1k_8) \\
&(-k_2k_{13} + k_1k_{14}) - (-k_2k_5 + k_1k_6)(-k_4k_{13} + k_1k_{16})) - ((-k_4k_5 + k_1k_8)(-k_2k_9 + k_1k_{10}) \\
&- (-k_2k_5 + k_1k_6)(-k_4k_9 + k_1k_{12}))((-k_4k_5 + k_1k_8)(-k_3k_{13} + k_1k_{15}) - (-k_3k_5 + k_1k_7) \\
&(-k_4k_{13} + k_1k_{16})),
\end{aligned}$$

$$\begin{aligned}
D = &-Zk_4k_6k_9 + Zk_2k_8k_9 + Zk_4k_5k_{10} - Zk_1k_8k_{10} - Zk_2k_5k_{12} + Zk_1k_6k_{12} + Yk_4k_6k_{13} \\
&- Yk_2k_8k_{13} - Xk_4k_{10}k_{13} + Tk_8k_{10}k_{13} + Xk_2k_{12}k_{13} - Tk_6k_{12}k_{13} - Yk_4k_5k_{14} + Yk_1k_8k_{14} \\
&+ Xk_4k_9k_{14} - Tk_8k_9k_{14} - Xk_1k_{12}k_{14} + Tk_5k_{12}k_{14} + Yk_2k_5k_{16} - Yk_1k_6k_{16} - Xk_2k_9k_{16} \\
&+ Tk_6k_9k_{16} + Xk_1k_{10}k_{16} - Tk_5k_{10}k_{16},
\end{aligned}$$

$$\begin{aligned}
F = &-Zk_3k_6k_9 + Zk_2k_7k_9 + Zk_3k_5k_{10} - Zk_1k_7k_{10} - Zk_2k_5k_{11} + Zk_1k_6k_{11} + Yk_3k_6k_{13} \\
&- Yk_2k_7k_{13} - Xk_3k_{10}k_{13} + Tk_7k_{10}k_{13} + Xk_2k_{11}k_{13} - Tk_6k_{11}k_{13} - Yk_3k_5k_{14} + Yk_1k_7k_{14} \\
&+ Xk_3k_9k_{14} - Tk_7k_9k_{14} - Xk_1k_{11}k_{14} + Tk_5k_{11}k_{14} + Yk_2k_5k_{15} - Yk_1k_6k_{15} - Xk_2k_9k_{15} \\
&+ Tk_6k_9k_{15} + Xk_1k_{10}k_{15} - Tk_5k_{10}k_{15},
\end{aligned}$$

$$\begin{aligned}
H = &Zk_4k_7k_{10} - Zk_3k_8k_{10} - Zk_4k_6k_{11} + Zk_2k_8k_{11} + Zk_3k_6k_{12} - Zk_2k_7k_{12} - Yk_4k_7k_{14} \\
&+ Yk_3k_8k_{14} + Xk_4k_{11}k_{14} - Tk_8k_{11}k_{14} - Xk_3k_{12}k_{14} + Tk_7k_{12}k_{14} + Yk_4k_6k_{15} - Yk_2k_8k_{15} \\
&- Xk_4k_{10}k_{15} + Tk_8k_{10}k_{15} + Xk_2k_{12}k_{15} - Tk_6k_{12}k_{15} - Yk_3k_6k_{16} + Yk_2k_7k_{16} \\
&+ Xk_3k_{10}k_{16} - Tk_7k_{10}k_{16} - Xk_2k_{11}k_{16} + Tk_6k_{11}k_{16}.
\end{aligned}$$

Then

$$\begin{aligned}
&u_t - a^2\left(u_{xx}+u_{yy}+u_{zz}\right)\\
&=\left(g_t-a^2g_{xx}-a^2g_{yy}-a^2g_{zz}\right)h+\left(k_1g-2a^2k_2g_x-2a^2k_3g_y-2a^2k_4g_z\right)h_T\\
&+\left(k_5g-2a^2k_6g_x-2a^2k_7g_y-2a^2k_8g_z\right)h_X+\left(k_9g-2a^2k_{10}g_x-2a^2k_{11}g_y-2a^2k_{12}g_z\right)h_Y\\
&+\left(k_{13}g-2a^2k_{14}g_x-2a^2k_{15}g_y-2a^2k_{16}g_z\right)h_Z-a^2\left(k_2^2+k_3^2+k_4^2\right)gh_{TT}\\
&-a^2\left(k_6^2+k_7^2+k_8^2\right)gh_{XX}-a^2\left(k_{10}^2+k_{11}^2+k_{12}^2\right)gh_{YY}-a^2\left(k_{14}^2+k_{15}^2+k_{16}^2\right)gh_{ZZ}\\
&-2a^2\left(k_2k_6+k_3k_7+k_4k_8\right)gh_{TX}-2a^2\left(k_2k_{10}+k_3k_{11}+k_4k_{12}\right)gh_{TY}\\
&-2a^2\left(k_2k_{14}+k_3k_{15}+k_4k_{16}\right)gh_{TZ}-2a^2\left(k_6k_{10}+k_7k_{11}+k_8k_{12}\right)gh_{XY}\\
&-2a^2\left(k_6k_{14}+k_7k_{15}+k_8k_{16}\right)gh_{XZ}-2a^2\left(k_{10}k_{14}+k_{11}k_{15}+k_{12}k_{16}\right)gh_{YZ}=A\left(X,Y,Z,T\right).
\end{aligned} \tag{217}$$

According to Eq. (217), set

$$g_t-a^2g_{xx}-a^2g_{yy}-a^2g_{zz}=0, \tag{218}$$

$$k_1g-2a^2k_2g_x-2a^2k_3g_y-2a^2k_4g_z=0, \tag{219}$$

$$k_5g-2a^2k_6g_x-2a^2k_7g_y-2a^2k_8g_z=0, \tag{220}$$

$$k_9g-2a^2k_{10}g_x-2a^2k_{11}g_y-2a^2k_{12}g_z=0, \tag{221}$$

$$k_{13}g-2a^2k_{14}g_x-2a^2k_{15}g_y-2a^2k_{16}g_z=0. \tag{222}$$

By (191), the particular solution of Eq. (218) is

$$g\left(t,x,y,z\right)=c_5e^{\frac{c_0\left(c_0t+c_1x+c_2y+c_3z\right)}{a^2\left(c_1^2+c_2^2+c_3^2\right)}}+c_6, \tag{223}$$

where c_1-c_6 are arbitrary constants. Setting $c_6=0$ and substituting Eq. (223) into (219-222) respectively, we get

$$\begin{aligned}
&a_0k_1g+2a_1k_2g_x+2a_2k_3g_y+2a_3k_4g_z\\
&=a_0k_1c_5e^{\frac{-a_0c_0\left(c_0t+c_1x+c_2y+c_3z\right)}{a_1c_1^2+a_2c_2^2+a_3c_3^2}}-2a_0c_0c_5e^{\frac{-a_0c_0\left(c_0t+c_1x+c_2y+c_3z\right)}{a_1c_1^2+a_2c_2^2+a_3c_3^2}}\frac{a_1k_2c_1+a_2k_3c_2+a_3k_4c_3}{a_1c_1^2+a_2c_2^2+a_3c_3^2}\\
&=0.
\end{aligned}$$

Namely

$$k_1=2c_0\frac{k_2c_1+k_3c_2+k_4c_3}{c_1^2+c_2^2+c_3^2}. \tag{224}$$

$$\begin{aligned}
&a_0k_5g+2a_1k_6g_x+2a_2k_7g_y+2a_3k_8g_z\\
&=a_0k_5c_5e^{\frac{-a_0c_0\left(c_0t+c_1x+c_2y+c_3z\right)}{a_1c_1^2+a_2c_2^2+a_3c_3^2}}-2a_0c_0c_5e^{\frac{-a_0c_0\left(c_0t+c_1x+c_2y+c_3z\right)}{a_1c_1^2+a_2c_2^2+a_3c_3^2}}\frac{a_1k_6c_1+a_2k_7c_2+a_3k_8c_3}{a_1c_1^2+a_2c_2^2+a_3c_3^2}\\
&=0.
\end{aligned}$$

That is

$$k_5=2c_0\frac{k_6c_1+k_7c_2+k_8c_3}{c_1^2+c_2^2+c_3^2}. \tag{225}$$

$$\begin{aligned}
&a_0k_9g+2a_1k_{10}g_x+2a_2k_{11}g_y+2a_3k_{12}g_z\\
&=a_0k_9c_5e^{\frac{-a_0c_0\left(c_0t+c_1x+c_2y+c_3z\right)}{a_1c_1^2+a_2c_2^2+a_3c_3^2}}-2a_0c_0c_5e^{\frac{-a_0c_0\left(c_0t+c_1x+c_2y+c_3z\right)}{a_1c_1^2+a_2c_2^2+a_3c_3^2}}\frac{a_1k_{10}c_1+a_2k_{11}c_2+a_3k_{12}c_3}{a_1c_1^2+a_2c_2^2+a_3c_3^2}\\
&=0.
\end{aligned}$$

Namely

$$k_9 = 2c_0 \frac{k_{10}c_1 + k_{11}c_2 + k_{12}c_3}{c_1^2 + c_2^2 + c_3^2}. \tag{226}$$

$$\begin{aligned}
& a_0 k_{13} g + 2a_1 k_{14} g_x + 2a_2 k_{15} g_y + 2a_3 k_{16} g_z \\
& = a_0 k_{13} c_5 e^{\frac{-a_0 c_0 (c_0 t + c_1 x + c_2 y + c_3 z)}{a_1 c_1^2 + a_2 c_2^2 + a_3 c_3^2}} - 2a_0 c_0 c_5 e^{\frac{-a_0 c_0 (c_0 t + c_1 x + c_2 y + c_3 z)}{a_1 c_1^2 + a_2 c_2^2 + a_3 c_3^2}} \frac{a_1 k_{14} c_1 + a_2 k_{15} c_2 + a_3 k_{16} c_3}{a_1 c_1^2 + a_2 c_2^2 + a_3 c_3^2} \\
& = 0.
\end{aligned}$$

So

$$k_{13} = 2c_0 \frac{k_{14}c_1 + k_{15}c_2 + k_{16}c_3}{c_1^2 + c_2^2 + c_3^2}. \tag{227}$$

So (217) is simplified as

$$\begin{aligned}
& u_t - a^2 (u_{xx} + u_{yy} + u_{zz}) \\
& = -a^2 \left(k_2^2 + k_3^2 + k_4^2\right) g h_{TT} - a^2 \left(k_6^2 + k_7^2 + k_8^2\right) g h_{XX} - a^2 \left(k_{10}^2 + k_{11}^2 + k_{12}^2\right) g h_{YY} \\
& - a^2 \left(k_{14}^2 + k_{15}^2 + k_{16}^2\right) g h_{ZZ} - 2a^2 (k_2 k_6 + k_3 k_7 + k_4 k_8) g h_{TX} \\
& - 2a^2 (k_2 k_{10} + k_3 k_{11} + k_4 k_{12}) g h_{TY} - 2a^2 (k_2 k_{14} + k_3 k_{15} + k_4 k_{16}) g h_{TZ} \\
& - 2a^2 (k_6 k_{10} + k_7 k_{11} + k_8 k_{12}) g h_{XY} - 2a^2 (k_6 k_{14} + k_7 k_{15} + k_8 k_{16}) g h_{XZ} \\
& - 2a^2 (k_{10} k_{14} + k_{11} k_{15} + k_{12} k_{16}) g h_{YZ} = A(X, Y, Z, T).
\end{aligned} \tag{228}$$

Set

$$k_6^2 + k_7^2 + k_8^2 = k_{10}^2 + k_{11}^2 + k_{12}^2 = k_{14}^2 + k_{15}^2 + k_{16}^2 = 0. \tag{229}$$

We have

$$k_6 = \pm\sqrt{-k_7^2 - k_8^2}, k_{10} = \pm\sqrt{-k_{11}^2 - k_{12}^2}, k_{14} = \pm\sqrt{-k_{15}^2 - k_{16}^2}\,. \tag{230}$$

Further set

$$\begin{aligned}
a_1 k_2 k_6 + a_2 k_3 k_7 + a_3 k_4 k_8 &= a_1 k_2 k_{10} + a_2 k_3 k_{11} + a_3 k_4 k_{12} = a_1 k_2 k_{14} + a_2 k_3 k_{15} + a_3 k_4 k_{16} \\
&= a_1 k_6 k_{10} + a_2 k_7 k_{11} + a_3 k_8 k_{12} = a_1 k_6 k_{14} + a_2 k_7 k_{15} + a_3 k_8 k_{16} \\
&= a_1 k_{10} k_{14} + a_2 k_{11} k_{15} + a_3 k_{12} k_{16} = 0.
\end{aligned} \tag{231}$$

Similar to the solving method of (63), we obtain

$$k_7 k_{12} = k_8 k_{11}, k_7 k_{16} = k_8 k_{15}, k_{11} k_{16} = k_{12} k_{15}, \tag{232}$$

and

$$u_t - a^2 (u_{xx} + u_{yy} + u_{zz}) = -a^2 \left(k_2^2 + k_3^2 + k_4^2\right) g h_{TT} = A(X, Y, Z, T). \tag{233}$$

The special solution of Eq. (233) is

$$h = -\iint \frac{A(X, Y, Z, T)}{a^2 \left(k_2^2 + k_3^2 + k_4^2\right) g} dT dT.$$

Similar to Eq. (67), it can be verified that the excrescent general solutions of Eq. (223) is

$$h = h_1(X, Y, Z) + T h_2(X, Y, Z) - \iint \frac{A(X, Y, Z, T)}{a^2 \left(k_2^2 + k_3^2 + k_4^2\right) g} dT dT.$$

We do not carry out specific analysis.

So the special solution of Eq. (209) is

$$u = -g \iint \frac{A(X,Y,Z,T)}{a^2\left(k_2^2+k_3^2+k_4^2\right)g} dTdT. \tag{234}$$

Combining (201), we can have the basic general solution of Eq. (209).

For the 2D nonhomogeneous heat equation

$$u_t - a^2\left(u_{xx}+u_{yy}\right) = A(x,y,t).$$

By the similar method we can get its particular solution and general solution.

4.3. Schrödinger Equation

Linear [29-31] and nonlinear [32, 33] stationary state Schrödinger equation are the focus of current research. Consider the following linear equation

$$\frac{\hbar^2}{2m}\triangle u - \left(V(x,y,z)-E\right)u = 0, \tag{235}$$

where m is the mass of the described particle and $\hbar$ is the reduced Plank constant, by Z_2 Transformation, set

$$u(x,y,z) = f(v) = f\left(k_1x+k_2y+k_3z+k_4\right), \tag{236}$$

$$V(x,y,z)-E = a(v) = a\left(k_1x+k_2y+k_3z+k_4\right), \tag{237}$$

where $v(x,y,z) = k_1x+k_2y+k_3z+k_4, k_1-k_4$ are known parameters, $V(x,y,z)-E = a(v)$ is a known function, f is an undetermined second differentiable function, then

$$\frac{\hbar^2}{2m}\triangle u - \left(V(x,y,z)-E\right)u = \frac{\hbar^2}{2m}\left(k_1^2+k_2^2+k_3^2\right)f_v'' - a(v)f = 0.$$

Namely

$$f_v'' + b(v)f = 0. \tag{238}$$

where

$$b(v) = \frac{-2m\left(V(x,y,z)-E\right)}{\hbar^2\left(k_1^2+k_2^2+k_3^2\right)}. \tag{239}$$

If $b(v)$ is some special function [34], Eq. (238) has a particular solution and its general solution may be obtained by the law of second-order linear ODEs (LODEs), such as

$$b(v) = -c\left(cv^{2n}+nv^{n-1}\right), \tag{240}$$

$$V(x,y,z) = a(v)+E = \frac{c\hbar^2\left(k_1^2+k_2^2+k_3^2\right)}{2m}\left(cv^{2n}+nv^{n-1}\right)+E, \tag{241}$$

where c is an arbitrary constant, the particular solution of Eq. (238) under the condition of (241) is

$$f(v) = \exp\left(\frac{cv^{n+1}}{n+1}\right) = \exp\left(\frac{c(k_1x+k_2y+k_3z+k_4)^{n+1}}{n+1}\right).$$

So the particular solution of Eq. (235) under the condition of (241) is

$$u\left(x,y,z\right)=\exp\left(\frac{c(k_1x+k_2y+k_3z+k_4)^{n+1}}{n+1}\right). \tag{242}$$

For getting the general solution of Eq. (235) under the condition of (241), according to Z_3 Transformation, we set

$$u\left(x,y,z\right)=g\left(x,y,z\right)h\left(w\right)=g\left(x,y,z\right)h\left(l_1x+l_2y+l_3z+l_4\right), \tag{243}$$

where $w(x,y,z)=l_1x+l_2y+l_3z+l_4, l_1-l_4$ are parameters to be determined, $h(w)$ and $g(x,y,z)$ are undetermined second differentiable function, so

$$u_{xx}=hg_{xx}+2l_1g_xh_w^{'}+l_1^2gh_w^{''}, \tag{244}$$

$$u_{yy}=hg_{yy}+2l_2g_yh_w^{'}+l_2^2gh_w^{''}, \tag{245}$$

$$u_{zz}=hg_{zz}+2l_3g_zh_w^{'}+l_3^2gh_w^{''}. \tag{246}$$

Then

$$\begin{aligned}&\frac{\hbar^2}{2m}\triangle u-V((x,y,z)-E)u\\&=\frac{\hbar^2}{2m}\left(l_1^2+l_2^2+l_3^2\right)gh_w^{''}+\frac{\hbar^2}{m}\left(l_1g_x+l_2g_y+l_3g_z\right)h_w^{'}\\&+\left(\frac{\hbar^2}{2m}g_{xx}+\frac{\hbar^2}{2m}g_{yy}+\frac{\hbar^2}{2m}g_{zz}+\left(V\left(x,y,z\right)-E\right)g\right)h=0.\end{aligned} \tag{247}$$

Set $h(w)$ an arbitrary second differentiable function, by (247) we get

$$l_1^2+l_2^2+l_3^2=0\Longrightarrow l_1=\pm\sqrt{-l_2^2-l_3^2}, \tag{248}$$

$$l_1g_x+l_2g_y+l_3g_z=0, \tag{249}$$

$$\frac{\hbar^2}{2m}g_{xx}+\frac{\hbar^2}{2m}g_{yy}+\frac{\hbar^2}{2m}g_{zz}+\left(V\left(x,y,z\right)-E\right)g=0. \tag{250}$$

By (242), the particular solution of Eq. (250) on the condition of (241) is

$$g\left(x,y,z\right)=\exp\left(\frac{c(k_1x+k_2y+k_3z+k_4)^{n+1}}{n+1}\right). \tag{251}$$

Substituting from (251) into (249) we get

$$\begin{aligned}&l_1ck_1(k_1x+k_2y+k_3z+k_4)^n\exp\left(\frac{c(k_1x+k_2y+k_3z+k_4)^{n+1}}{n+1}\right)\\&+l_2ck_2(k_1x+k_2y+k_3z+k_4)^n\exp\left(\frac{c(k_1x+k_2y+k_3z+k_4)^{n+1}}{n+1}\right)\\&+l_3ck_3(k_1x+k_2y+k_3z+k_4)^n\exp\left(\frac{c(k_1x+k_2y+k_3z+k_4)^{n+1}}{n+1}\right)=0\\&\Longrightarrow l_1=\frac{-k_2l_2-k_3l_3}{k_1}=\pm\sqrt{-l_2^2-l_3^2}\end{aligned}$$

.

Namely

$$l_2 = \frac{-k_1 k_2 l_3 \pm \sqrt{-k_1^4 l_3^2 - k_1^2 k_3^2 l_3^2 - k_2^2 k_3^2 l_3^2}}{k_1^2 + k_2^2}. \tag{252}$$

Then

$$u(x,y,z) = g(x,y,z)\,h(w) = \exp\left(\frac{c(k_1 x + k_2 y + k_3 z + k_4)^{n+1}}{n+1}\right) h(l_1 x + l_2 y + l_3 z + l_4).$$

So the general solution of Eq. (235) on the condition of (241) is

$$\begin{aligned} u = {} & \exp\left(\frac{c(k_1 x + k_2 y + k_3 z + k_4)^{n+1}}{n+1}\right) \\ & \left(h_1\left(\sqrt{-l_2^2 - l_3^2}x + l_2 y + l_3 z + l_4\right) + h_2\left(-\sqrt{-l_{12}^2 - l_{13}^2}x + l_{12} y + l_{13} z + l_{14}\right)\right), \end{aligned} \tag{253}$$

where h_1 and h_2 are arbitrary second differentiable unary function, $k_1 - k_4$ and c are determinate parameters, l_3, l_4, l_{13} and l_{14} are arbitrary constants, $l_{12} = \frac{-k_1 k_2 l_{13} \pm l_{13}\sqrt{-k_1^4 - k_1^2 k_3^2 - k_2^2 k_3^2}}{k_1^2 + k_2^2}$.

Time dependent Schrödinger equation is always the focus of research [35-40], in addition, the related nonlinear equation [41, 42] and the time fractional Schrödinger equations (TFSEs) [43, 44] are the deeply researched field. Consider the following linear equation

$$i\hbar u_t + \frac{\hbar^2}{2m} \triangle u - V(x,y,z,t)\,u = 0. \tag{254}$$

According to Z_2 Transformation, set

$$u(x,y,z,t) = f(v) = f(k_1 x + k_2 y + k_3 z + k_4 t + k_5), \tag{255}$$

$$V(x,y,z,t) = a(v) = a(k_1 x + k_2 y + k_3 z + k_4 t + k_5), \tag{256}$$

where $v = k_1 x + k_2 y + k_3 z + k_4 t + k_5, k_1 - k_5$ are known parameters, $V(x,y,z,t) = a(v)$ is a known function, f is an undetermined second differentiable function, then

$$i\hbar u_t + \frac{\hbar^2}{2m} \triangle u - V(x,y,z,t)\,u = i\hbar k_4 f_v' + \frac{\hbar^2}{2m}\left(k_1^2 + k_2^2 + k_3^2\right) f_v'' - a(v)\,f = 0.$$

Namely

$$f_v'' + k f_v' + b(v)\,f = 0, \tag{257}$$

where

$$k = \frac{i2mk_4}{\hbar\left(k_1^2 + k_2^2 + k_3^2\right)}\ ,\ b(v) = \frac{-2ma(v)}{\hbar^2\left(k_1^2 + k_2^2 + k_3^2\right)}. \tag{258}$$

If $b(v)$ is some special function [34], Eq. (257) has a particular solution, such as

$$b(v) = c\left(-cv^{2n} + kv^n + nv^{n-1}\right).$$

The particular solution of Eq. (257) is

$$f(v) = \exp\left(-\frac{cv^{n+1}}{n+1}\right).$$

Namely

$$V(x,y,z,t) = a(v) = \frac{-\,c\hbar^2\left(k_1^2 + k_2^2 + k_3^2\right)}{2m}\left(-cv^{2n} + kv^n + nv^{n-1}\right). \tag{259}$$

The particular solution of Eq. (254) on the condition of (259) is

$$u\left(x,y,z,t\right)=\exp\left(-\frac{c(k_1x+k_2y+k_3z+k_4t+k_5)^{n+1}}{n+1}\right). \tag{260}$$

By

$$f_v^{''}+\left(g\left(v\right)+h\left(v\right)\right)f_v^{'}+\left(g\left(v\right)h\left(v\right)+g_v^{'}\right)f=0. \tag{261}$$

The particular solution of Eq. (261) is

$$f\left(v\right)=\exp\left(-\int g\left(v\right)dv\right). \tag{262}$$

Set $h(v)=-g(v)+k$, where $g(v)$ is an arbitrary unary first differentiable function, then

$$b\left(v\right)=-g^2\left(v\right)+kg\left(v\right)+g_v^{'}. \tag{263}$$

Namely

$$V\left(x,y,z,t\right)=a\left(v\right)=\frac{\hbar^2\left(k_1^2+k_2^2+k_3^2\right)}{2m}\left(g^2\left(v\right)-kg\left(v\right)-g_v^{'}\right). \tag{264}$$

The particular solution of Eq. (254) on the condition of (264) is

$$u\left(x,y,z,t\right)=f\left(v\right)=\exp\left(-\int g\left(v\right)dv\right). \tag{265}$$

For getting the general solution of Eq. (254) on the condition of (259), according to Z_3 Transformation, we set

$$u\left(x,y,z,t\right)=g\left(x,y,z,t\right)h\left(w\right)=g\left(x,y,z,t\right)h\left(l_1x+l_2y+l_3z+l_4t+l_5\right), \tag{266}$$

where $w=l_1x+l_2y+l_3z+l_4t+l_5, l_1-l_5$ are constants to be determined, h and g are undetermined second differentiable functions, by (266), we have

$$\begin{aligned}&i\hbar u_t+\frac{\hbar^2}{2m}\triangle u-V\left(x,y,z,t\right)u\\&=\frac{\hbar^2}{2m}\left(l_1^2+l_2^2+l_3^2\right)gh_w^{''}+\hbar\left(il_4g+\frac{\hbar}{m}l_1g_x+\frac{\hbar}{m}l_2g_y+\frac{\hbar}{m}l_3g_z\right)h_w^{'}\\&+\left(i\hbar g_t+\frac{\hbar^2}{2m}g_{xx}+\frac{\hbar^2}{2m}g_{yy}+\frac{\hbar^2}{2m}g_{zz}-Vg\right)h=0.\end{aligned} \tag{267}$$

Set $h(w)$ an arbitrary second differentiable function, by (267) we get

$$l_1^2+l_2^2+l_3^2=0\Longrightarrow l_1=\pm\sqrt{-l_2^2-l_3^2}, \tag{268}$$

$$il_4g+\frac{\hbar}{m}l_1g_x+\frac{\hbar}{m}l_2g_y+\frac{\hbar}{m}l_3g_z=0, \tag{269}$$

$$i\hbar g_t+\frac{\hbar^2}{2m}g_{xx}+\frac{\hbar^2}{2m}g_{yy}+\frac{\hbar^2}{2m}g_{zz}-Vg=0. \tag{270}$$

By (260) the particular solution of Eq. (270) on the condition of (259) is

$$g\left(x,y,z,t\right)=\exp\left(-\frac{c(k_1x+k_2y+k_3z+k_4t+k_5)^{n+1}}{n+1}\right). \tag{271}$$

Substituting from (271) into (269) we get

$$
\begin{aligned}
& il_4\exp\left(-\frac{c(k_1x+k_2y+k_3z+k_4t+k_5)^{n+1}}{n+1}\right) \\
& -\frac{\hbar}{m}l_1ck_1(k_1x+k_2y+k_3z+k_4t+k_5)^n\exp\left(-\frac{c(k_1x+k_2y+k_3z+k_4t+k_5)^{n+1}}{n+1}\right) \\
& -\frac{\hbar}{m}l_2ck_2(k_1x+k_2y+k_3z+k_4t+k_5)^n\exp\left(-\frac{c(k_1x+k_2y+k_3z+k_4t+k_5)^{n+1}}{n+1}\right) \\
& -\frac{\hbar}{m}l_3ck_3(k_1x+k_2y+k_3z+k_4t+k_5)^n\exp\left(-\frac{c(k_1x+k_2y+k_3z+k_4t+k_5)^{n+1}}{n+1}\right)=0.
\end{aligned}
$$

Namely

$$
l_4=-\frac{\hbar c}{m}\left(l_1k_1+l_2k_2+l_3k_3\right)\left(k_1x+k_2y+k_3z+k_4t+k_5\right)^n. \tag{272}
$$

Since l_4 is a constant and is not a function of x,y,z and t, if (271) is the particular solution of Eq. (270), by (272) n must equal 0, then

$$
V=\frac{-\,c\hbar^2\left(k_1^2+k_2^2+k_3^2\right)}{2m}\left(-cv^{2n}+kv^n+nv^{n-1}\right)=\frac{-\,c\hbar^2\left(k_1^2+k_2^2+k_3^2\right)}{2m}\left(-c+k\right). \tag{273}
$$

Since k_1-k_5,k and c are determinate constants, so $V(x,y,z,t)$ is an determinate constants too, namely

$$
l_4=-\frac{\hbar c}{m}\left(l_1k_1+l_2k_2+l_3k_3\right). \tag{274}
$$

Then

$$
u=gh=\exp\left(-\frac{c(k_1x+k_2y+k_3z+k_4t+k_5)^{n+1}}{n+1}\right)h\left(l_1x+l_2y+l_3z+l_4t+l_5\right).
$$

So the general solution of Eq. (254) on the condition of (273) is

$$
\begin{aligned}
& u=e^{-c(k_1x+k_2y+k_3z+k_4t+k_5)} \\
& \left(h_1\left(\sqrt{-l_2^2-l_3^2}x+l_2y+l_3z+l_4t+l_5\right)+h_2\left(-\sqrt{-l_{12}^2-l_{13}^2}x+l_{12}y+l_{13}z+l_{14}t+l_{15}\right)\right),
\end{aligned} \tag{275}
$$

where h_1 and h_2 are arbitrary second differentiable functions, $l_{14}=-\frac{\hbar c}{m}\left(l_{11}k_1+l_{12}k_2+l_{13}k_3\right)$, $l_2,l_3,l_5,l_{12},l_{13}$ and l_{15} are arbitrary constants.

According to Z_3 Transformation, if set

$$
u\left(x,y,z,t\right)=g\left(x,y,z,t\right)h\left(X,Y,Z,T\right),
$$

where

$$
\begin{aligned}
T&=k_1t+k_2x+k_3y+k_4z,\\
X&=k_5t+k_6x+k_7y+k_8z,\\
Y&=k_9t+k_{10}x+k_{11}y+k_{12}z,\\
Z&=k_{13}t+k_{14}x+k_{15}y+k_{16}z,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial\left(X,Y,Z,T\right)}{\partial\left(x,y,z,t\right)}\\
&=k_4k_7k_{10}k_{13}-k_3k_8k_{10}k_{13}-k_4k_6k_{11}k_{13}+k_2k_8k_{11}k_{13}+k_3k_6k_{12}k_{13}-k_2k_7k_{12}k_{13}\\
&-k_4k_7k_9k_{14}+k_3k_8k_9k_{14}+k_4k_5k_{11}k_{14}-k_1k_8k_{11}k_{14}-k_3k_5k_{12}k_{14}+k_1k_7k_{12}k_{14}\\
&+k_4k_6k_9k_{15}-k_2k_8k_9k_{15}-k_4k_5k_{10}k_{15}+k_1k_8k_{10}k_{15}+k_2k_5k_{12}k_{15}-k_1k_6k_{12}k_{15}\\
&-k_3k_6k_9k_{16}+k_2k_7k_9k_{16}+k_3k_5k_{10}k_{16}-k_1k_7k_{10}k_{16}-k_2k_5k_{11}k_{16}+k_1k_6k_{11}k_{16}\neq 0.
\end{aligned}
$$

It can be verified that we only obtain excrescent general solutions of Eq. (254) on the condition of (273), which is similar to Eq. (235), and do not carry out specific analysis.

Consider the following initial value problem of Eq. (254) on the condition of (273)

$$
u\left(x,y,z,0\right)=e^{x+y+z}\left(\varphi_1\left(\sqrt{-2}x+y+z\right)+\varphi_2\left(-\sqrt{-2}x+y+z\right)\right), \tag{276}
$$

$$
\begin{aligned}
&u_t\left(x,y,z,0\right)\\
&=e^{x+y+z}\left(\varphi_1\left(\sqrt{-2}x+y+z\right)+\varphi_2\left(-\sqrt{-2}x+y+z\right)\right)\\
&+\frac{\hbar}{m}e^{x+y+z}\Big(\left(2+\sqrt{-2}\right)\varphi_1'\left(\sqrt{-2}x+y+z\right)+\left(2-\sqrt{-2}\right)\varphi_2'\left(-\sqrt{-2}x+y+z\right)\Big).
\end{aligned} \tag{277}
$$

Comparing (275) with (276) we have

$$
k_1=k_2=k_3=-\frac{1}{c}\;,l_2=l_3=l_{12}=l_{13}=1\;,k_5=l_5=l_{15}=0.
$$

By further calculation which is in Appendix D, the exact solutions of the initial value problem is

$$
\begin{aligned}
u=&e^{x+y+z+t}(\varphi_1(\sqrt{-2}x+y+z+\frac{\hbar}{m}(2+\sqrt{-2})t)\\
&+\varphi_2(-\sqrt{-2}x+y+z+\frac{\hbar}{m}(2-\sqrt{-2})t)).
\end{aligned} \tag{278}
$$

When $V(x,y,z,t)$ is a constant, Eq. (254) is also an important case of the diffusion equation with a source [45], its general solution and the exact solutions of the Cauchy problem are applicable to the diffusion equation.

5. Conclusion

In this paper, we propose a verification axiom and a conjecture that the general solution of PDEs is related to the spatial dimension, and put forward a concept and the relevant law of the equivalent function.

In order to effectively solve the linear and nonlinear PDEs, we propose three kinds of Z transformations. According to the actual case, we find that the arbitrary constants in solutions of PDEs have two types which are absolutely arbitrary constants and relatively arbitrary constants. We find the limitation of the characteristic equation method to solve some first order PDEs.

Since mathematical physics equations (MPEs) are very important in PDEs, their progress is always been noticed especially [46]. In this paper, we have obtained the general solutions and exact solutions of the problems of definite solutions of various typical MPEs, and found that in the more universal case PDEs have basic general solution, series general solution, transformational general solution, generalized series general solution and so on.

Appendix

Appendix A

In (37) it can be proved that if $k_1, l_1 \neq 0$ and $k_1, l_1 \nrightarrow 0$, $c_1 v$ can be described by f_1 and f_2, set

$$k_i = l_i = C_i, (i = 2, 3, \dots n+1).$$

Then

$$f_1 = f_1 \left(kx_1 + C_2 x_2 + \ldots + C_n x_n + C_{n+1} \right),$$
$$f_2 = f_2 \left(-kx_1 + C_2 x_2 + \ldots + C_n x_n + C_{n+1} \right),$$

where

$$k = \left(-\frac{a_2 C_2^2 + a_3 C_3^2 + \ldots + a_n C_n^2 + a_{n+1} C_2 C_3}{a_1} \right)^{\frac{1}{2}}.$$

Set

$$Ac_1(kx_1 + C_2x_2 + \ldots + C_nx_n + C_{n+1}) + Bc_1(-kx_1 + C_2x_2 + \ldots + C_nx_n$$
$$+C_{n+1}) = (A+B)c_1(C_1x_1 + C_2x_2 + \ldots + C_nx_n + C_{n+1}) \Longrightarrow C_1 = \frac{A-B}{A+B}k.$$

If $A = B \neq 0$, then $\frac{A-B}{A+B} = 0$. If $B = -A + 1$, then

$$\lim_{A\to\infty} \frac{A-B}{A+B} = \lim_{A\to\infty} (2A-1) \to \infty, \lim_{A\to\infty} \frac{A-B}{A+B} = \lim_{A\to\infty} (2A-1) \to -\infty.$$

Namely $\frac{A-B}{A+B} \in (-\infty, \infty)$, if $k \neq 0$ and $k \nrightarrow 0$, selecting A, B felicitously, C_1 may equal to arbitrary real number, so $c_1 v$ can be described by f_1, f_2, and

$$\begin{aligned} c_1 v &= c_1 \left(C_1 x_1 + C_2 x_2 + \ldots + C_n x_n + C_{n+1} \right) \\ &= \frac{Ac_1}{A+B} \left(kx_1 + C_2 x_2 + \ldots + C_n x_n + C_{n+1} \right) \\ &+ \frac{Bc_1}{A+B} \left(-kx_1 + C_2 x_2 + \ldots + C_n x_n + C_{n+1} \right), \end{aligned}$$

where $C_1 = \frac{A-B}{A+B}k$.

Appendix B

The calculation of (43) as follows.
In (40), set $c_1 = 0, k_{i_j} = l_{i_j}, (i = 1, 2, \cdots s, j = 2, 3, \cdots n+1)$

$$k_{i_1} = \left(-\left(a_2 k_{i_2}^2 + \ldots + a_n k_{i_n}^2 + a_{n+1} k_{i_2} k_{i_3} \right) / a_1 \right)^{\frac{1}{2}}. \tag{44}$$

According to (40)-(42)

$$\begin{aligned} u\left(0, x_2, \ldots x_n\right) &= \sum_{i=1}^{s} (f_{1_i}(k_{i_2}x_2 + \ldots + k_{i_n}x_n + k_{i_{n+1}}) + f_{2_i}(k_{i_2}x_2 + \ldots + k_{i_n}x_n + k_{i_{n+1}})) \\ &= \sum_{i=1}^{s} \varphi_i(k_{i_2}x_2 + \ldots + k_{i_n}x_n + k_{i_{n+1}}), \end{aligned}$$

$$\begin{aligned} u_{x_1}\left(0, x_2, \ldots x_n\right) &= \sum_{i=1}^{s} (k_{i_1} f'_{1_i}(k_{i_2}x_2 + \ldots + k_{i_n}x_n + k_{i_{n+1}}) - k_{i_1} f'_{2_i}(k_{i_2}x_2 + \ldots + k_{i_n}x_n + k_{i_{n+1}})) \\ &= \sum_{i=1}^{s} \psi_i(k_{i_2}x_2 + \ldots + k_{i_n}x_n + k_{i_{n+1}}). \end{aligned}$$

We have

$$\begin{aligned}&f_{1_i}\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right)+f_{2_i}\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right)\\&=\varphi_i\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right),\end{aligned}\tag{279}$$

$$\begin{aligned}&k_{i_1}f_{1_i}^{'}\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right)-k_{i_1}f_{2_i}^{'}\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right)\\&=\psi_i\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right).\end{aligned}\tag{280}$$

According to (280) we get

$$\begin{aligned}&f_{1_i}(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}})-f_{2_i}(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}})\\&=\frac{1}{k_{i_1}}\int_{k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}}^{k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}}\psi(\xi_i)d\xi_i+f_{1_i}(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}})\\&-f_{2_i}(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}).\end{aligned}\tag{281}$$

Combining (279) and (281), then

$$\begin{aligned}&f_{1_i}\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right)\\&=\tfrac{1}{2}\varphi_i\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right)+\tfrac{1}{2k_{i_1}}\int_{k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}}^{k_{i_2}x_2+\ldots k_{i_n}x_n+k_{i_{n+1}}}\psi\left(\xi_i\right)d\xi_i\\&+\tfrac{1}{2}f_{1_i}\left(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}\right)-\tfrac{1}{2}f_{2_i}\left(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}\right)\\&\Longrightarrow f_{1_i}\left(k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right)\\&=\tfrac{1}{2}\varphi_i\left(k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right)+\tfrac{1}{2k_{i_1}}\int_{k_{i_2}x_{2_0}+k_{i_3}x_{3_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}}^{k_{i_1}x_1+k_{i_2}x_2+\ldots k_{i_n}x_n+k_{i_{n+1}}}\psi\left(\xi_i\right)d\xi_i\\&+\tfrac{1}{2}f_{1_i}\left(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}\right)-\tfrac{1}{2}f_{2_i}\left(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}\right),\end{aligned}$$

$$\begin{aligned}&f_{2_i}\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right)\\&=\tfrac{1}{2}\varphi_i\left(k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right)-\tfrac{1}{2k_{i_1}}\int_{k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}}^{k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}}\psi\left(\xi_i\right)d\xi_i-\\&\tfrac{1}{2}f_{1_i}\left(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}\right)+\tfrac{1}{2}f_{2_i}\left(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}\right)\\&\Longrightarrow f_{2_i}\left(-k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right)\\&=\tfrac{1}{2}\varphi_i\left(-k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right)-\tfrac{1}{2k_{i_1}}\int_{k_{i_2}x_{2_0}+k_{i_3}x_{3_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}}^{-k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}}\psi\left(\xi_i\right)d\xi_i\\&-\tfrac{1}{2}f_{1_i}\left(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}\right)+\tfrac{1}{2}f_{2_i}\left(k_{i_2}x_{2_0}+\ldots+k_{i_n}x_{n_0}+k_{i_{n+1}}\right).\end{aligned}$$

In the conditions of (41) and (42), the exact solution of Eq. (38) is

$$\begin{aligned}u=&\frac{1}{2}\sum_{i=1}^{s}(\varphi_i(k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}})\\&+\varphi_i(-k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}})\\&+\frac{1}{k_{i_1}}\int_{-k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}}^{k_{i_1}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}}\psi(\xi_i)d\xi_i).\end{aligned}\tag{43}$$

Appendix C

In $\mathbb{R}^n$, using Z_1 Transformation to get the general solution of

$$a_1u_{x_1x_1}+a_2u_{x_2x_2}=A\left(x_1,x_2,\ldots x_n\right),\tag{282}$$

where a_1 and a_2 are arbitrary known constants, $A(x_1,x_2,\cdots x_n)$ is any known function.

According to Z_1 Transformation, set

$$y_1 = k_1 x_1 + k_2 x_2, y_2 = k_3 x_1 + k_4 x_2, \tag{283}$$

where $k_1 - k_4$ are undetermined constants, and set

$$\frac{\partial (y_1, y_2, x_3, x_4, \dots x_n)}{\partial (x_1, x_2, x_3, x_4, \dots x_n)} = k_1 k_4 - k_2 k_3 \neq 0. \tag{284}$$

By Eq. (283), we get

$$x_1 = \frac{k_4 y_1 - k_2 y_2}{k_1 k_4 - k_2 k_3}, x_2 = \frac{-k_3 y_1 + k_1 y_2}{k_1 k_4 - k_2 k_3}. \tag{285}$$

Then

$$\begin{aligned} & a_1 u_{x_1 x_1} + a_2 u_{x_2 x_2} \\ & = \left(a_1 k_1^2 + a_2 k_2^2\right) u_{y_1 y_1} + \left(a_1 k_3^2 + a_2 k_4^2\right) u_{y_2 y_2} + 2 \left(a_1 k_1 k_3 + a_2 k_2 k_4\right) u_{y_1 y_2} \\ & = A (x_1, x_2, \dots x_n). \end{aligned} \tag{286}$$

Set

$$a_1 k_1^2 + a_2 k_2^2 = a_1 k_3^2 + a_2 k_4^2 = 0 \Longrightarrow k_1 = \pm \sqrt{-\frac{a_2}{a_1}} k_2, k_3 = \pm \sqrt{-\frac{a_2}{a_1}} k_4.$$

And set

$$k_1 = \sqrt{-\frac{a_2}{a_1}} k_2, k_3 = -\sqrt{-\frac{a_2}{a_1}} k_4, (k_2, k_4 \neq 0). \tag{287}$$

So

$$k_1 k_4 - k_2 k_3 = 2 \sqrt{-\frac{a_2}{a_1}} k_2 k_4 \neq 0.$$

Thus

$$\begin{aligned} a_1 u_{x_1 x_1} + a_2 u_{x_2 x_2} & = 2 \left(a_1 k_1 k_3 + a_2 k_2 k_4\right) u_{y_1 y_2} \\ & = 2 \left(-a_1 \sqrt{-\frac{a_2}{a_1}} k_2 \sqrt{-\frac{a_2}{a_1}} k_4 + a_2 k_2 k_4\right) u_{y_1 y_2} \\ & = 4 a_2 k_2 k_4 u_{y_1 y_2} = A (x_1, x_2, \dots x_n). \end{aligned} \tag{288}$$

So the general solution of Eq. (282) is

$$u = f_1 (y_1, x_3, x_4, \dots x_n) + f_2 (y_2, x_3, x_4, \dots x_n) + \frac{\iint A (x_1, x_2, \dots x_n) d y_1 d y_2}{4 a_2 k_2 k_4}, \tag{289}$$

where

$$y_1 = \sqrt{-\frac{a_2}{a_1}} k_2 x_1 + k_2 x_2, y_2 = -\sqrt{-\frac{a_2}{a_1}} k_4 x_1 + k_4 x_2,$$

f_1 and f_2 are arbitrary second differentiable functions, k_2 and k_4 are relatively arbitrary constants which cannot equal to zero.

If set

$$k_2 = k_4 = 1.$$

Then

$$x_1 = \frac{k_4 y_1 - k_2 y_2}{k_1 k_4 - k_2 k_3} = \frac{y_1 - y_2}{2\sqrt{-\frac{a_2}{a_1}}}, x_2 = \frac{-k_3 y_1 + k_1 y_2}{k_1 k_4 - k_2 k_3} = \frac{y_1 + y_2}{2}.$$

The general solution of Eq. (282) may be written as

$$\begin{aligned} u = & f_1 \left(\sqrt{-\frac{a_2}{a_1}} x_1 + x_2, x_3, x_4, \dots x_n\right) + f_2 \left(-\sqrt{-\frac{a_2}{a_1}} x_1 + x_2, x_3, x_4, \dots x_n\right) \\ & + \frac{1}{4 a_2} \iint A \left(\sqrt{-\frac{a_1}{a_2}} \frac{y_1 - y_2}{2}, \frac{y_1 + y_2}{2}, x_3, x_4, \dots x_n\right) d y_1 d y_2, \end{aligned} \tag{290}$$

where

$$y_1 = \sqrt{-\frac{a_2}{a_1}}x_1 + x_2, \tag{291}$$

$$y_2 = -\sqrt{-\frac{a_2}{a_1}}x_1 + x_2. \tag{292}$$

In $\mathbb{R}^2$,the form of the nonhomogeneous 1D wave equation in Cartesian coordinate system is

$$u_{tt} - a^2 u_{xx} = A(x,t). \tag{293}$$

According to (290-292), its basic general solution is

$$u = f_1(x+at) + f_2(x-at) - \frac{1}{4a^2}\iint A\left(\frac{y_1 - y_2}{2a}, \frac{y_1 + y_2}{2}\right) dy_1 dy_2, \tag{294}$$

where

$$y_1 = x + at, y_2 = x - at. \tag{295}$$

Consider the following initial value problem of Eq. (293)

$$u(0,x) = \varphi(x), \tag{296}$$

$$u_t(0,x) = \psi(x). \tag{297}$$

Set

$$B(t,x) = -\frac{1}{4a^2}\iint A\left(\frac{y_1 - y_2}{2a}, \frac{y_1 + y_2}{2}\right) dy_1 dy_2. \tag{298}$$

Then

$$u(0,x) = f_1(x) + f_2(x) + B(0,x) = \varphi(x), \tag{299}$$

$$u_t(0,x) = af_1'(x) - af_2'(x) + B_t(0,x) = \psi(x). \tag{300}$$

So

$$f_1(x) + f_2(x) = \varphi(x) - B(0,x),$$

$$af_1'(x) - af_2'(x) = \psi(x) - B_t(0,x) \Longrightarrow f_1(x) - f_2(x) = \frac{1}{a}\int_{x_0}^{x} (\psi(\xi) - B_t(0,\xi))\, d\xi.$$

Namely

$$\begin{aligned} f_1(x) &= \frac{1}{2}\left(\varphi(x) - B(0,x) + \frac{1}{a}\int_{x_0}^{x} (\psi(\xi) - B_t(0,\xi))\, d\xi\right) \\ \Longrightarrow f_1(x+at) &= \frac{1}{2}\left(\varphi(x+at) - B(0,x+at) + \frac{1}{a}\int_{x_0}^{x+at} (\psi(\xi) - B_t(0,\xi))\, d\xi\right), \\ f_2(x) &= \frac{1}{2}\left(\varphi(x) - B(0,x) - \frac{1}{a}\int_{x_0}^{x} (\psi(\xi) - B_t(0,\xi))\, d\xi\right) \\ \Longrightarrow f_2(x-at) &= \frac{1}{2}\left(\varphi(x-at) - B(0,x-at) - \frac{1}{a}\int_{x_0}^{x-at} (\psi(\xi) - B_t(0,\xi))\, d\xi\right). \end{aligned}$$

Thus

$$\begin{aligned} u &= f_1(x+at) + f_2(x-at) + B(t,x) \\ &= \frac{1}{2}\left(\varphi(x+at) - B(0,x+at) + \frac{1}{a}\int_{x_0}^{x+at} (\psi(\xi) - B_t(0,\xi))\, d\xi\right) \\ &+ \frac{1}{2}\left(\varphi(x-at) - B(0,x-at) - \frac{1}{a}\int_{x_0}^{x-at} (\psi(\xi) - B_t(0,\xi))\, d\xi\right) + B(t,x). \end{aligned}$$

So the exact solution of Eq. (293) on the conditions of (296) and (297) is

$$u=\frac{1}{2}\left(\varphi\left(x+at\right)+\varphi\left(x-at\right)-B\left(0,x+at\right)-B\left(0,x-at\right)+\frac{1}{a}\int_{x-at}^{x+at}\left(\psi\left(\xi\right)-B_t\left(0,\xi\right)\right)d\xi\right)$$
$$+B\left(t,x\right). \tag{301}$$

Appendix D

Consider the following initial value problem of Eq. (254) on the condition of (273)

$$u\left(x,y,z,0\right)=e^{x+y+z}\left(\varphi_1\left(\sqrt{-2}x+y+z\right)+\varphi_2\left(-\sqrt{-2}x+y+z\right)\right), \tag{276}$$

$$\begin{aligned}&u_t\left(x,y,z,0\right)\\&=e^{x+y+z}\left(\varphi_1\left(\sqrt{-2}x+y+z\right)+\varphi_2\left(-\sqrt{-2}x+y+z\right)\right)\\&+\frac{\hbar}{m}e^{x+y+z}\left(\left(2+\sqrt{-2}\right)\varphi_1'\left(\sqrt{-2}x+y+z\right)+\left(2-\sqrt{-2}\right)\varphi_2'\left(-\sqrt{-2}x+y+z\right)\right).\end{aligned} \tag{277}$$

Comparing (275) with (276) we have

$$k_1=k_2=k_3=-\frac{1}{c}\ ,l_2=l_3=l_{12}=l_{13}=1\ ,k_5=l_5=l_{15}=0.$$

Then

$$\begin{aligned}u\left(x,y,z,t\right)=&e^{x+y+z-ck_4t}(h_1(\sqrt{-2}x+y+z+\frac{\hbar}{m}(2+\sqrt{-2})t)\\&+h_2(-\sqrt{-2}x+y+z+\frac{\hbar}{m}(2-\sqrt{-2})t)),\\(x,y,z,t)=&-ck_4e^{x+y+z-ck_4t}(h_1(\sqrt{-2}x+y+z+\frac{\hbar}{m}(2+\sqrt{-2})t)\\&+h_2(-\sqrt{-2}x+y+z+\frac{\hbar}{m}(2-\sqrt{-2})t))\\&+e^{x+y+z-ck_4t}(\frac{\hbar}{m}(2+\sqrt{-2})h_1'(\sqrt{-2}x+y+z+\frac{\hbar}{m}(2+\sqrt{-2})t)\\&+\frac{\hbar}{m}(2-\sqrt{-2})h_2'(-\sqrt{-2}x+y+z+\frac{\hbar}{m}(2-\sqrt{-2})t)).\end{aligned}$$

Therefore

$$\begin{aligned}&u\left(x,y,z,0\right)=e^{x+y+z}\left(\varphi_1\left(\sqrt{-2}x+y+z\right)+\varphi_2\left(-\sqrt{-2}x+y+z\right)\right)\\&=e^{x+y+z}\left(h_1\left(\sqrt{-2}x+y+z\right)+h_2\left(-\sqrt{-2}x+y+z\right)\right)\\&\Longrightarrow h_1\left(\sqrt{-2}x+y+z\right)=\varphi_1\left(\sqrt{-2}x+y+z\right)\\&\Longrightarrow h_1\left(\sqrt{-2}x+y+z+\frac{\hbar}{m}\left(2+\sqrt{-2}\right)t\right)=\varphi_1\left(\sqrt{-2}x+y+z+\frac{\hbar}{m}\left(2+\sqrt{-2}\right)t\right).\end{aligned}$$

Namely

$$h_1\left(\sqrt{-2}x+y+z+\frac{\hbar}{m}\left(2+\sqrt{-2}\right)t\right)=\varphi_1\left(\sqrt{-2}x+y+z+\frac{\hbar}{m}\left(2+\sqrt{-2}\right)t\right), \tag{302}$$

$$h_2\left(-\sqrt{-2}x+y+z+\frac{\hbar}{m}\left(2-\sqrt{-2}\right)t\right)=\varphi_2\left(-\sqrt{-2}x+y+z+\frac{\hbar}{m}\left(2-\sqrt{-2}\right)t\right). \tag{303}$$

Thus

$$
\begin{aligned}
&u_t\left(x,y,z,0\right) = e^{x+y+z}\left(\varphi_1\left(\sqrt{-2}x+y+z\right)+\varphi_2\left(-\sqrt{-2}x+y+z\right)\right)\\
&+\frac{\hbar}{m}e^{x+y+z}\left(\left(2+\sqrt{-2}\right)\varphi_1^{'}\left(\sqrt{-2}x+y+z\right)+\left(2-\sqrt{-2}\right)\varphi_2^{'}\left(-\sqrt{-2}x+y+z\right)\right)\\
&=-ck_4e^{x+y+z}\left(h_1\left(\sqrt{-2}x+y+z\right)+h_2\left(-\sqrt{-2}x+y+z\right)\right)\\
&+e^{x+y+z}\left(\frac{\hbar}{m}\left(2+\sqrt{-2}\right)h_1^{'}\left(\sqrt{-2}x+y+z\right)+\frac{\hbar}{m}\left(2-\sqrt{-2}\right)h_2^{'}\left(-\sqrt{-2}x+y+z\right)\right)\\
&\Longrightarrow k_4=-\frac{1}{c}.
\end{aligned}
$$

So the exact solutions of the initial value problem is

$$
\begin{aligned}
u =&e^{x+y+z+t}(\varphi_1(\sqrt{-2}x+y+z+\frac{\hbar}{m}(2+\sqrt{-2})t)\\
&+\varphi_2(-\sqrt{-2}x+y+z+\frac{\hbar}{m}(2-\sqrt{-2})t)).
\end{aligned}
\tag{278}
$$

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