Schwarzschild Cosmology

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Abstract

In the current paper, the internal Schwarzschild solution is examined in the context of a cosmological model. The expanding Universe is likened to a collapsing star in this context, where the future spacetime of the Universe is empty (justifying the vacuum solution) and the present is a 3-sphere boundary akin to the 2-sphere surface of a collapsing star. Because the present time of the Universe is the boundary, it sees the vacuum solution just as the surface of a collapsing star sees the vacuum solution. It is shown that the model predicts an accelerated expansion that agrees with current observations of the expansion history of our Universe, namely that the initial expansion is infinitely fast, and then the expansion slows for some time followed by an accelerated expansion. With a simple coordinate change we get a metric resembling the FRW metric for flat space with a time dependent scale factor. It is shown that the singularity at $r = 0$ can be interpreted as a point in time where the geodesics reverse sharply, perhaps causing the expansion and collapse of the Universe to cycle. The consequences for an observer in freefall in a local gravitational field are then examined in the context of this model.

Expansion Along a Timelike Dimension

The current Big Bang model of the Universe says that the Universe expanded from an infinitely dense gravitational singularity at some time in the past. Current cosmological data suggests that this expansion was slowing down for some time, but is now continuing to expand at an accelerated rate. The Cosmological Principle suggests that from any reference frame in the Universe, the mass distribution is spherically symmetric and isotropic. It is proposed that the observed expansion of the Universe is the result of a freefall in the time dimension. To analyze the spherically symmetric Universe freefalling through the time dimension, we need the Schwarzschild solution where the radial coordinate is the timelike coordinate. The interior ($r < 1$) solution of the Schwarzschild field (throughout the paper, we will work in units with Schwarzschild radius equal to 1) gives us precisely that. For $r < 1$, the signature of the Schwarzschild metric flips and the radial coordinate becomes a dimension measuring time while the t coordinate becomes a dimension measuring space.

But the Schwarzschild metric is a vacuum solution to Einstein's equations. Given that the Universe is full of energy, further justification is required to use the vacuum solution. Consider an isotropically collapsing spherical star (such that the surface of the star is in freefall). The external Schwarzschild solution is valid outside the surface of the star and is static as the star collapses. The spacetime inside the star is not governed by the Schwarzschild metric due to the energy inside the star. However, the metric inside the star and the metric outside the star must agree at the surface of the star and therefore the particles on the surface of the star see the Schwarzschild metric on the surface. For the $r < 1$ Schwarzschild metric, it will be shown that at a fixed r, we have a spacelike 3-dimensional slice of the Universe (the space of the Universe at a given cosmological time). Therefore, if we assume that in our frame of reference the future spacetime of the Universe is empty (i.e. the Universe for cosmological times greater than our current cosmological time is a vacuum), we see that the $r < 1$ Schwarzschild metric can be applied to our cosmology for the same reasons that we can apply the external Schwarzschild metric to the surface of a collapsing star. Thus, whereas in the case of the collapsing star, the boundary was a 2-sphere collapsing in space, in the cosmological case the boundary (present time) is a 3-sphere expanding in time (in both cases, the boundary is defined by a given value of r). In fact, if we begin with the assumption that the future spacetime is a vacuum, then according to Birkhoff's theorem, the $r < 1$ Schwarzschild solution is the only possible cosmological solution. Thus, the Schwarzschild solution as a cosmological model is based on 2 assumptions:

1. The Universe is spacetime and its geometry is governed by the equations of General Relativity

2. The 'present' of the Universe is defined as the state of the Universe in which space is uniformly filled with energy at the present time and void of energy at all later times

That energy does not fill all times can also be justified by looking at the momentum 4-vector:

$$P = m_0 \frac{dX}{d\tau} = \begin{bmatrix} m_0 \frac{dt}{d\tau} \\ p^1 \\ p^2 \\ p^3 \end{bmatrix} = \begin{bmatrix} E \\ p^1 \\ p^2 \\ p^3 \end{bmatrix} \tag{2}$$

As can be seen from Equation 2, the rest energy comes from the velocity of the mass through the time dimension. So energy can't just exist statically at some particular place and time, as would be the case if the future were already filled with energy. For the energy to exist, the matter must be flowing through time, as though the present state of energy in the Universe is freefalling through the time dimension into the future vacuum. Equation 2 will also help us understand the extreme points of this freefall.

So let us take the center of our galaxy as the origin of an inertial reference frame. We can draw a line through the center of the reference frame that extends infinitely in both directions radially outward. This line will correspond to fixed angular coordinates (θ, ϕ). There are infinitely many such lines, but since we have an isotropic, spherically symmetric Universe, we only need to analyze this model along one of these lines, and the result will be the same for any line.

The radial distance in this frame is kind of a compound dimension. It is a distance in space as well as a distance in time. The farther away a galaxy is from us, the farther back in time the light we currently receive from it was emitted. Fortunately the $r < 1$ spacetime of the Schwarzschild solution plotted in Kruskal-Szekeres coordinates provides us with a method to understand this radial direction. Figure 1 shows the $r < 1$ solution on a Kruskal-Szekeres coordinate chart where, in this model, the hyperbolas of constant r represent spacelike slices of constant cosmological time and the rays of t represent radial distances (each point on this plot is a 2-sphere and each hyperbola is a 3-sphere).

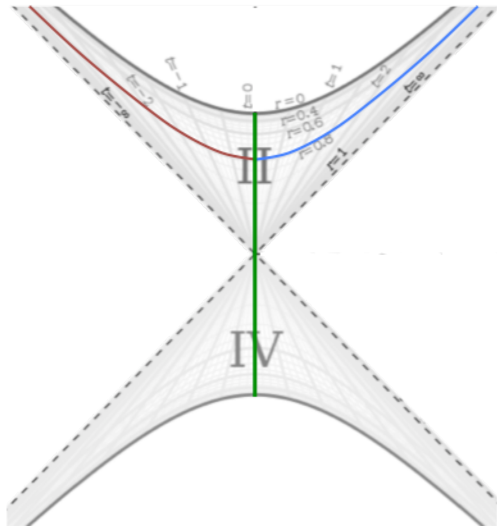

Figure 1 – Freefall Through Cosmological Time[1]

We must first determine the paths of inertial observers in the spacetime. For this we need the internal Schwarzschild metric and the geodesic equations for the internal Schwarzschild metric [1]:

$$d\tau^2 = \frac{r}{1-r} dr^2 - \frac{1-r}{r} dt^2 - r^2 d\Omega^2 \qquad (2)$$

$$\frac{d^2 t}{d\tau^2} = \frac{1}{r(1-r)} \frac{dr}{d\tau} \frac{dt}{d\tau} \qquad (3)$$

$$\frac{d^2 r}{d\tau^2} = \frac{GM}{r^2} \left[\frac{1-r}{r} \left(\frac{dt}{d\tau} \right)^2 - \frac{r}{1-r} \left(\frac{dr}{d\tau} \right)^2 \right] \qquad (4)$$

In Equations 2, 3, and 4, we use units where $c = 2GM = 1$ and equations 3 and 4 assume no angular motion. Looking at points $0 < r < 1$, then by inspection of Equation 3 it is clear that an inertial observer at rest at t will remain at rest at t ($\frac{d^2 t}{d\tau^2} = 0$ if $\frac{dt}{d\tau} = 0$). Also, we see that if an observer is moving inertially with some initial $\frac{dt}{d\tau}$, then if $\frac{dr}{d\tau} < 0$, the coordinate speed of the observer will be reduced over time (the coordinates are

[1] Diagram modified from: "Kruskal diagram of Schwarzschild chart" by Dr Greg. Licensed under CC BY-SA 3.0 via Wikimedia Commons - http://commons.wikimedia.org/wiki/File:Kruskal_diagram_of_Schwarzschild_chart.svg#/media/File:Kruskal_diagram_of_Schwarzschild_chart.svg

expanding beneath her) and if $\frac{dr}{d\tau} > 0$, the coordinate speed will be increased over time (the coordinates are collapsing beneath her).

Let us therefore examine Equation 4 for an inertial observer at rest at *t*. From Equation 1, we see that this corresponds to $\frac{dt}{d\tau} = 0$ and $\frac{dr}{d\tau} = \sqrt{\frac{1-r}{r}}$. Plugging these expressions into Equations 4 gives:

$$\frac{d^2r}{d\tau^2} = -\frac{GM}{r^2} \tag{5}$$

Therefore, the inertial observer's acceleration through cosmological time takes the form of Newton's law of gravity, where *r* (a time coordinate) varies from 1 to 0. So we will use Figure 1 to describe the freefall of the galaxies through the cosmological time dimension where galaxies (or galaxy clusters) follow lines of constant *t* (and any such observer can choose $t = 0$ as their coordinate). The 'Big Bang' will have occurred at the center of Figure 1 at $r = 1$. We know this because the above analysis showed that space expands if $\frac{dr}{d\tau}$ is negative, so for our current cosmological time, our worldlines must be moving toward $r = 0$.

Expressions for the proper time interval along lines of constant *t* and Ω and the proper distance interval along hyperbolas of constant *r* and Ω from Equation 1 are:

$$\frac{dr}{d\tau} = \pm\sqrt{\frac{1-r}{r}} \tag{6}$$

$$\frac{ds}{dt} = \pm\sqrt{\frac{1-r}{r}} \tag{7}$$

First we should notice that neither Equation 6 nor 7 depend on the *t* coordinate. This is good because the *t* coordinate marks the position of other galaxies relative to ours. Since all galaxies are freefalling in time inertially, the particular position of any one galaxy should not matter. The proper velocity and proper distance only depends on the cosmological time *r*. When $r = 1$, Equations 6 and 7 are both 0. At this point (the Big Bang), it is our proper velocity in time that is zero. So at that instant, we are no longer moving through time and therefore all points in space are coincident (the observer can reach every point in space without moving through time). This is analogous to being at rest in space where all points of time essentially converge to a single point in space (the observer can reach every point in time without moving through space). So this why the proper distance goes to zero there and why the lines of *t* in Figure 1 converge at that point; it is an instant where our velocity through cosmological time goes to zero as our speed through cosmological time changes from positive to negative (we can see that if we draw a worldline through the center point, $\frac{dr}{d\tau}$ will change signs as it passes the $r = 1$ point).

At $r = 0$, both equations 6 and 7 are infinite. So when the worldlines enter or exit one of the $r = 0$ hyperbolas, they do so at infinite proper speed *through the time dimension*. If something is travelling through space at the speed to light, the proper distance between points in space is zero. In this case, since we have infinite proper velocity in the time dimension, the proper distance between points in space will be infinite, because you would traverse an infinite amount of time in order to move through an infinitesimal amount of space. What we see then is that at $r = 0$ space will be infinitely expanded.

We can also understand these extreme points using Equation 2. At $r = 1, \frac{dr}{d\tau} = 0$ and according to Equation 2, this would mean that massive objects no longer have rest energy. Thus there are no inertial reference frames in the Universe at that point; all worldlines are lightlike at that instant. Since lightlike objects see space infinitely contracted, we see that at $r = 1$ space will be infinitely contracted in all frames. At $r = 0, \frac{dr}{d\tau} = \infty$, meaning that the rest energies of massive objects is effectively infinite. We can think of this as objects having infinite inertia. With infinite inertia, objects at rest cannot be accelerated and thus space is effectively infinitely expanded since motion is no longer possible. We saw previously that inertial observers with non-zero initial $\frac{dt}{d\tau}$ will be slowed down during expansion and this was explained by saying that the coordinates were expanding beneath her. But we can also think of it as her inertia increasing as a result of her increasing $\frac{dr}{d\tau}$, stopping her motion completely at $r = 0$.

A plot of $\frac{dr}{d\tau}$ vs. r during the expansion $\left(\frac{dr}{d\tau} < 0\right)$ is given in Figure 2 below:

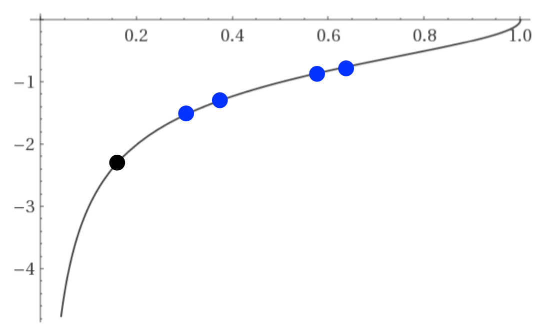

Figure 2 - $\frac{dr}{d\tau}$ vs. r

In Figure 2, time moves forward as we move right to left along the diagram. So the leftmost dot represents us at our current cosmological time in the Universe. The other dots represent galaxies at various distances from us (the farther they are to the right of the diagram, the greater the distance from us) whose signals we are currently receiving. Since the signals we receive now were emitted in the past, they were emitted from galaxies with a lower proper velocity than we have currently. Thus, the signals should all be redshifted since our proper velocity is currently greater than the velocities of the galaxies when they emitted the signals and the magnitude of the redshift should be proportional to the difference in velocity. Now note that in the region where the other dots are, the difference in velocity of the two more distant emitters is less than the difference in velocity of the closer emitters. This difference means that when we get the signals from the galaxies, the difference in redshift for the two closer galaxies will be greater than the difference in redshift from the two more distant galaxies, which looks like accelerated expansion. But on this graph, there is actually an inflection point at $r = 0.75$. That means that from $r = 1$ to $r = 0.75$, it would appear as if the Universe is expanding, but the expansion is slowing down. Then from $r = 0.75$ to $r = 0$, the Universe will look like it is expanding at an accelerated rate. This change from a negatively accelerating expansion to a positively accelerating expansion is consistent with current cosmological data. Note that Equation 7, which describes the proper distance between coordinate points at a given r, has the same form as the function plotted in Figure 2 such that the rate of increase in proper distance decreases from $r = 1$ to $r = 0.75$ and then increases afterwards. Thus, from this equation, we can see that the rate at which wavelengths of light are stretched will first decrease and then increase, just as was discussed above (the exact equation for the redshift will be examined in the last section of this paper). A plot of $\frac{d}{dr}\left(\frac{ds}{dt}\right)$, which is the rate of change of proper distance between coordinates (i.e. expansion rate), is shown in Figure 3 below, demonstrating this expansion profile.

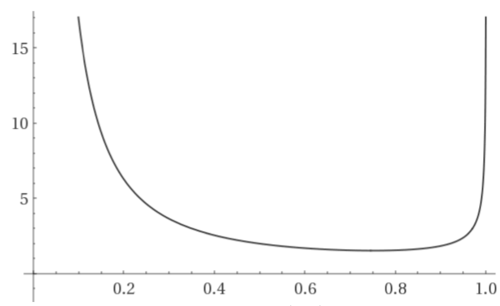

Figure 3 - $\frac{d}{dr}\left(\frac{ds}{dt}\right)$ vs. r

Figure 4 shows the past light cone of an inertial observer at a given time during the expansion:

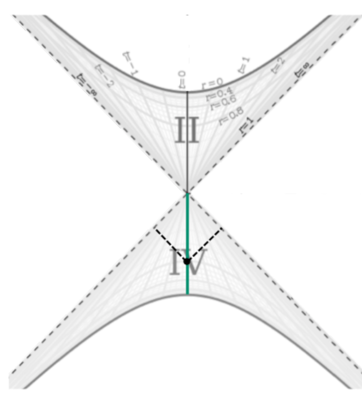

Figure 4 – Past Light Cone of Inertial Observer During the Expansion[2]

[2] Diagram modified from: "Kruskal diagram of Schwarzschild chart" by Dr Greg. Licensed under CC BY-SA 3.0 via Wikimedia Commons - http://commons.wikimedia.org/wiki/File:Kruskal_diagram_of_Schwarzschild_chart.svg#/media/File:Kruskal_diagram_of_Schwarzschild_chart.svg

Notice that at all times during the expansion, the past light cone includes the entire width of the Universe. This means that the observable Universe is the entire Universe. From this we can conclude that the Schwarzschild radius of the entire Universe (r_s, which we have so far set to 1 in the current paper) is the Schwarzschild radius of the observable Universe. We see however from Equation 7 that the proper distance to the edge of the observable Universe when $r < 1$ is infinite (because t ranges from negative infinity to infinity).

We can calculate the duration of the expansion of the Universe in the frame of an inertial observer by integrating Equation 6 from 0 to 1. This integral yields a value of $\frac{\pi}{2}$, but this is in units where the Schwarzschild radius is 1. The total time of expansion is therefore:

$$\tau = \frac{\pi}{2} r_s \tag{8}$$

Where r_s is measured in light-years and τ is in years (or equivalent units where the speed of light is 1).

A plot of τ vs. r from the uppermost to lowermost hyperbola in Figure 1 is given in Figure 5 below. It illustrates well the relationship to typical spatial projectile motion.

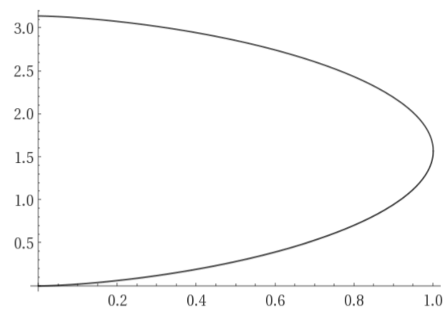

Figure 5 - τ vs. r

Consider a perfectly rigid and elastic ball in simple Newtonian mechanics. If we throw it straight up in the air with initial velocity $\frac{dx}{d\tau}$, the velocity will continuously decrease until at some height $\frac{dx}{d\tau} = 0$, at which point the ball will reverse direction and fall with increasingly negative $\frac{dx}{d\tau}$ until it returns to the ground. When it hits the ground, since it is perfectly rigid and elastic, it will experience an infinite acceleration that will bounce it back toward its maximum height and this cycle will continue ad infinitum. So there are two turnaround points for the ball. One point is maximum height, where the ball does not experience any special acceleration; it just stops moving through space as it turns around. The second point is a hard acceleration that the ball can really feel a (infinite) force changing its direction.

Likewise, we can see that the Schwarzschild cosmology is a similar situation except that the Universe is the ball and the acceleration is through time rather than space. The point $r = 1$ (Big Bang) corresponds to maximum height, where the Universe's velocity through time changes sign. The Hyperbolas are the 'bounce'. When the ball bounced, it experienced an infinite acceleration. In the cosmological case, when $r = 0$ the curvature of the spacetime is infinite [1]. This infinite curvature may be a point in time where the worldlines of the Universe turn back on themselves as if the spacetime is folded there and the worldlines go up one side and down the other (the infinite curvature is at the fold). Equation 5 is the inertial coordinate acceleration through time (proper acceleration is zero). If we express that equation in terms of proper acceleration we get:

$$A^r = \frac{d^2r}{d\tau^2} + \frac{GM}{r^2} \tag{9}$$

Immediately before $r = 0$, the coordinate velocity $\frac{dr}{d\tau}$ for the rest observers will be negatively infinite and immediately after the bounce the coordinate velocity will be infinitely positive. Therefore the inertial coordinate acceleration will need to be instantaneously zero at $r = 0$ and there will be a real proper acceleration given by:

$$A^r = \frac{GM}{r^2} \tag{10}$$

As was the case when the ball bounced, the proper acceleration will be infinite in the direction of increasing r. It is this instantaneous acceleration that would reverse the flow of the geodesics when the Universe is infinitely cold and dilute.

What we see from this paper is that the Schwarzschild solution describes two different scenarios, neither of which is the so-called black hole. The solution for $r > 1$ describes the gravitational field when the gravitational source is a location in space for all time whereas the solution for $r < 1$ describes the gravitational field when the gravitational source is a location in time for all space, where the gravitational center is at $r = 1$ in both cases. This is why the metric signature flips at $r = 1$.

Coordinate Change and the FRW Metric

We can make a coordinate change to make the metric resemble the FRW metric, which is the currently accepted metric for the Universe at-large. Basically, we want a radial coordinate whose interval is equal to the proper time interval of the inertial observer at rest ($t = const$). Thus, we can use Equation 5 to define T such that $\frac{dT}{dr} = \pm\sqrt{\frac{r}{1-r}}$. (+ on one the top half of Figure 1, - on the bottom or vice versa) Substituting this into Equation 2, we get the following:

$$d\tau^2 = dT^2 - \frac{1-r}{r}dt^2 - r^2 d\Omega^2 \tag{11}$$

In these coordinates, the proper time interval of the inertial observer at rest is just dT. The t and Ω intervals are multiplied by time-dependent functions (the r coordinate is a timelike coordinate) that play the role of the scale factors in the FRW metric for flat space. The T coordinate ranges from 0 at $r = 1$ to $\pm\frac{\pi}{2}$ at $r = 0$. As we can see, the scale factor squared in front of the dt^2 is just $\left(\frac{dr}{d\tau}\right)^2$ from Equation 6, which we have found can be interpreted as a proper velocity through time for an inertial observer.

We can use the fact that $\frac{1-r}{r}$ is the square of the scale factor and get the expression for cosmological redshift caused by the expansion [1]:

$$z = \sqrt{\frac{r_{emit}}{(1-r_{emit})}}\sqrt{\frac{(1-r)}{r}} - 1 \tag{12}$$

In Equation 12, r_{emit} is the time at which the signal was emitted and r is the time when the signal was received. Plots of Equation 12 for fixed $r_{emit} = 0.9$ (left) and $r = 0.1$ (right) are shown in Figure 6 below:

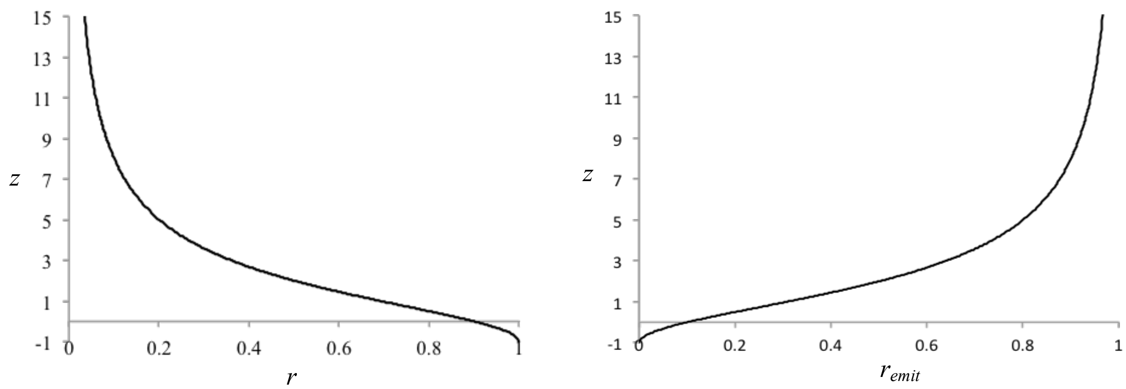

Figure 6 – z vs. r for $r_{emit} = 0.9$ (left) and z vs. r_{emit} for $r = 0.1$ (right)

Interestingly, when looking at the right plot in Figure 6 where we see redshifts seen by an observer at $r = 0.1$ from emitters at various distances, the inflection point occurs at $r = 0.25$ as opposed to Figure 3, where one can see that the spatial expansion changes from decelerating to accelerating at $r = 0.75$. Thus, there is a significant lag between the time the expansion begins to accelerate and the time that redshift measurements would indicate the acceleration.

Returning to Equation 11, a notable issue is the $r^2 d\Omega^2$ term. This is because r is a time coordinate in this context. Normally, the r in that term would be a spacelike Euclidean radius. However, the fact that this metric is describing a cosmological geometry makes the temporal characteristic of the radius more understandable. When we look to distant galaxies, we are seeing them in the past. The more distant the galaxy, the farther into the past we are seeing it. In our current reference frame, farther into the past corresponds to a larger r. Therefore, we can assign a unique r to the shell of galaxies around us at a fixed distance. Furthermore, the more distant the shell, the larger the value of r we can assign. What makes having a time coordinate as the radius make the most sense is that we can't see farther back in time than the Big Bang, so even though the Universe is ever expanding, we will always only have a finitely observable Universe. This is because since we are looking back in time as we look out in space, the past of the Universe is finite and thus the maximum radius must be finite, in this case, $r = 1$.

Freefall In Space

Let us consider a meter stick at rest at the center of a collapsing spherically symmetric collapsing shell. According to Birkhoff's theorem, the spacetime inside the shell, where the meter stick resides, will be flat Minkowski spacetime. The meter stick inside the shell stretches from the center of the shell out to a distance *2GM* (the shell is at a radius greater than *2GM* so the entire stick is in flat spacetime). An observer in freefall on the collapsing shell does so with speed (in natural units measured by her clock) [2]:

$$\frac{dr}{d\tau} = -\sqrt{\frac{2GM}{r}} \tag{13}$$

Therefore, the freefall observer will see observers at rest at *r* moving past her at the speed given in Equation 13. Since the meter stick is also at rest relative to observers at rest at any *r*, Equation 13 will also give the relative velocity between the freefall observer and the meter stick when the shell is at *r*. Since the spacetime between the freefall observer and central observer is flat, they will each see the other's clock dilated by the Special Relativity Relationship:

$$d\tau = dt\sqrt{1 - V^2} = dt\sqrt{1 - \frac{2GM}{r}} \tag{14}$$

Because the meter stick will appear to be moving in the frame of the freefalling observer, its length in her frame would be:

$$L = 2GM\sqrt{1 - \frac{2GM}{r}} \tag{15}$$

We see from Equation 15 that as the freefalling observer approaches $r = 2GM$ the length of the meter stick in her frame will contract to zero length. Thus, in her frame the center of the collapsing shell ($r = 0$) and $r = 2GM$ are coincident. This implies that in her frame, the event horizon lies at $r = 0$ which, cosmologically speaking, is the state of the Universe when it is fully expanded. This is also bolstered by the fact that the freefaller's velocity through time is given by:

$$\frac{dt}{d\tau} = \left[\sqrt{1 - \frac{2GM}{r}}\right]^{-1} \tag{16}$$

Her rest energy/inertia therefore goes to infinity as she falls, just as the rest energy/inertia of the aforementioned cosmological observers at rest goes to infinity. We also know that the clock of the freefalling observer ticks slower than the clock of an observer at infinity, and thus we expect that the shell observer will reach $r = 0$ in less time according to her clock than the observer at infinity.

Let us make a radial coordinate change for the freefalling observer. We choose R such that $\frac{dR}{dr} = \frac{r}{r-2GM}$. This coordinate varies identically to the r coordinate for large r and then diverges from it at the horizon. Note that $R \to \infty$ as $r \to \infty$ and $R \to -\infty$ as $r \to 2GM$. The coordinate velocity of the freefalling observer with this coordinate is given by:

$$\frac{dR}{dt} = -\sqrt{\frac{2GM}{r}} = -\sqrt{\frac{2GM}{W(e^{R-1})+1}} \tag{17}$$

Where W is the product-log function. This coordinate choice is also useful because the speed of light in these coordinates is 1 independent of R and t. The external Schwarzschild metric with the new coordinate becomes:

$$d\tau^2 = \frac{r-2GM}{r}\left[dt^2 - dR^2\right] \tag{18}$$

A plot of the integral of Equation 17 is given in Figure 7 below:

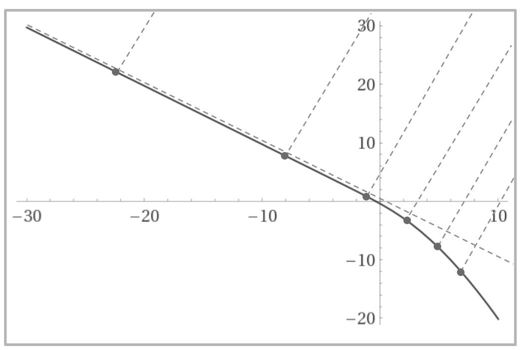

Figure 7 – Light Signals on *t*-*R* Chart

Figure 7 is a *t*-*R* chart that shows a single infalling signal representing the signal to which the freefall worldline is asymptotic. The freefalling observer will never receive this signal or any subsequent signal before the Universe reaches $r = 0$. The dots in Figure 7 represent intervals of equal proper time along the worldline and we can see that rest observers will receive signals from the freefalling observer at longer and longer intervals.

In the frame of the freefalling observer, rest observers will be moving away from her at the speed given in Equation 13. Therefore, she will see the external Universe accelerating away from her at an even faster rate than observers at infinity see other observers at infinity accelerating away from them, their signals increasingly redshifted as time passes. Nonetheless, the freefalling observer will never fall into a 'black hole', she will simply reach the cosmological $r = 0$ more quickly than observers far from any local gravitational fields. In fact, as observers on the shell approach $r = 2GM$, their feefall velocity will be opposed by the spatial acceleration defined by Equation 3 such that they will never reach full collapse.

References

[1] "Lecture Notes on General Relativity": https://arxiv.org/pdf/gr-qc/9712019.pdf. Cited July 17, 2017.

[2] Raine, D., Thomas, E.: Black Holes: A Student Text. Imperial College Press, (2015).