The Folium of Descartes

Edgar Valdebenito

abstract

In this note we briefly examine the curve:

$$x^3 + y^3 = \sqrt{2}xy$$

1. Introduction

- The folium of Descartes: A plane curve proposed by Descartes (1638).
- Equation in cartesian coordinates:

$$x^3 + y^3 = 3axy \qquad (1)$$

- Polar equation:

$$r(\theta) = \frac{3a\sin\theta\cos\theta}{\sin^3\theta + \cos^3\theta} \qquad (2)$$

- Parametric equations:

$$x = \frac{3at}{1+t^3} \ , \ y = \frac{3at^2}{1+t^3} \quad , t \neq -1 \qquad (3)$$

- Equation of the asymptote:

$$y + x + a = 0 \qquad (4)$$

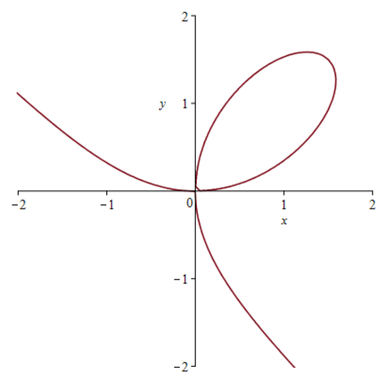

Fig. 1.

2. The folium : $x^3+y^3=\sqrt{2}xy$.

$$F: \quad x^3+y^3=\sqrt{2}xy \tag{5}$$

$$C: \quad x^2+y^2=1 \tag{6}$$

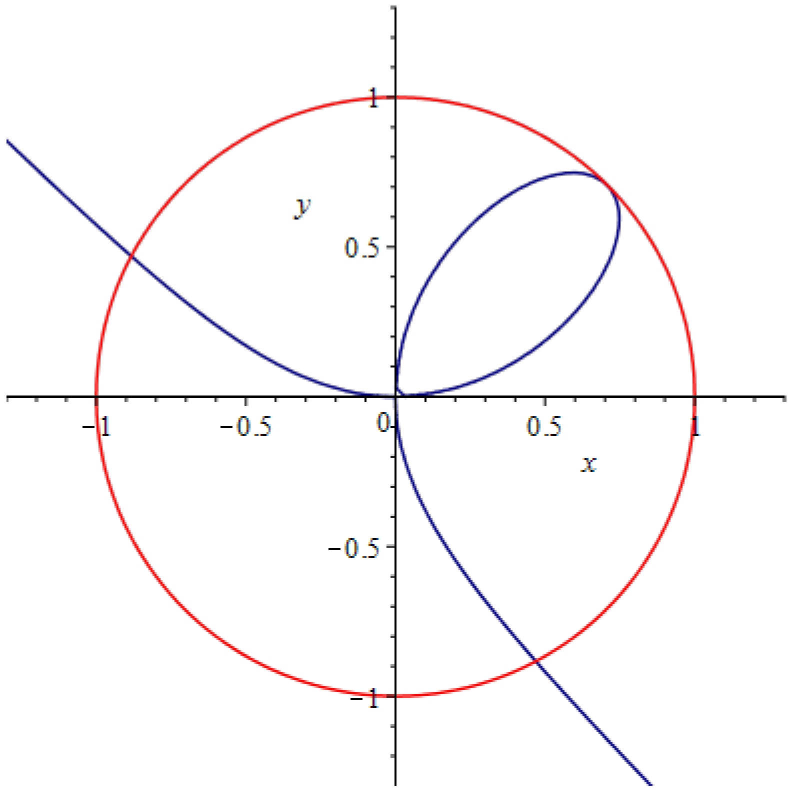

Fig.2. ● $x^2+y^2=1$, ● $x^3+y^3=\sqrt{2}xy$

$$F\cap C=\{P_1,P_2,P_3\} \tag{7}$$

$$P_1=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) \tag{8}$$

$$P_2=\left(\frac{\sqrt{2\sqrt{2}-1}-\sqrt{2}+1}{2},-\frac{\sqrt{2\sqrt{2}-1}+\sqrt{2}-1}{2}\right)=(u,-v) \tag{9}$$

$$P_3=\left(-\frac{\sqrt{2\sqrt{2}-1}+\sqrt{2}-1}{2},\frac{\sqrt{2\sqrt{2}-1}-\sqrt{2}+1}{2}\right)=(-v,u) \quad (10)$$

❖ Some pi formulas related with u,v :

$$\pi=2\sin^{-1}u+2\sin^{-1}v \quad (11)$$

$$\pi=2\sqrt{2\sqrt{2}-1}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-4n}\left(\sqrt{2}-1\right)^{2n}}{2n+1}\sum_{k=0}^{n}\binom{2n+1}{2k+1}\left(4\sqrt{2}+5\right)^{k} \quad (12)$$

$$\pi=2\sqrt{2\sqrt{2}-1}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2n+1}\left(\frac{2\sqrt{2}-1}{16}\right)^{n}\sum_{k=0}^{n+1}\binom{2n+1}{2k}\left(\frac{4\sqrt{2}-5}{7}\right)^{k} \quad (13)$$

$$\pi=4\tan^{-1}\left(v^3\right)-4\tan^{-1}\left(u^3\right)+4\tan^{-1}\left(\frac{2-\sqrt{2}}{2}\right) \quad (14)$$

$$\pi=2\sqrt{2\sqrt{2}-1}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-4n}}{2n+1}c_n \quad (15)$$

$$c_{n+2}=4c_{n+1}-16\left(3-2\sqrt{2}\right)c_n \quad ,c_0=1,c_1=8-4\sqrt{2} \quad (16)$$

Let

$$P_4=\left(\frac{1}{2}-\frac{1}{14}\sqrt{28\sqrt{2}-35},\frac{1}{2}+\frac{1}{14}\sqrt{28\sqrt{2}-35}\right)=(s,t) \quad (17)$$

$$P_5=\left(\frac{1}{2}+\frac{1}{14}\sqrt{28\sqrt{2}-35},\frac{1}{2}-\frac{1}{14}\sqrt{28\sqrt{2}-35}\right)=(t,s) \quad (18)$$

we get $P_4,P_5\in F$ (folium).

❖ Pi formula related with s,t :

$$\pi=2\sin^{-1}\left(\sqrt{\sqrt{2}-1}\sqrt{\frac{s}{t}}\right)+2\sin^{-1}\left(\sqrt{\sqrt{2}-1}\sqrt{\frac{t}{s}}\right) \quad (19)$$

Let

$$P_6=\left(\frac{1}{2},-\frac{1}{2}\sqrt[3]{1+2\sqrt{2}\sqrt[3]{1+2\sqrt{2}\sqrt[3]{1+\dots}}}\right)=\left(\frac{1}{2},-w\right) \quad (20)$$

we get $P_6\in F$ (folium).

❖ Pi formulas related with w :

$$\pi = 2\sin^{-1}\left(\frac{1}{w\sqrt[4]{2}}\right) + 2\sin^{-1}\left(\frac{1}{2w\sqrt{2w}}\right) \tag{21}$$

$$\pi = 3\sin^{-1}\left(\frac{\sqrt{3\sqrt{2}}}{2}w\right) - \sin^{-1}\left(\frac{3\sqrt{6\sqrt{2}}}{16}\right) \tag{22}$$

Let

$$P_7 = \left(\sqrt[3]{\frac{a\left(\sqrt{2}+\sqrt{2-4a}\right)}{2}}, \sqrt[3]{\frac{a\left(\sqrt{2}-\sqrt{2-4a}\right)}{2}}\right) = (x, y) \tag{23}$$

$$a = \frac{1}{6}\left(108+36\sqrt{9+8\sqrt{6}}\right)^{1/3} - 2\sqrt{3}\left(108+36\sqrt{9+8\sqrt{6}}\right)^{-1/3} \tag{24}$$

we get $P_7 \in F$ (folium).

- Pi formula related with x, y :

$$\pi = 6\tan^{-1}\left(x^3\right) + 6\tan^{-1}\left(y^3\right) \tag{25}$$

References

1. Lawrence, J.D.: A Catalog of Special Plane Curves. New York:Dover,pp.106-109,1972.