Dynamics of Statistical Fermionic and Boson-Fermionic Quantum System in Terms of Occupation Numbers

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Abstract

The ergodic second-order approach of entropy gradient maximization, applied on the problem of a quantum bosonic system, does not provide dynamic equations for pure fermionic system. The first-order dynamic equation results for a system of bosonic and fermionic degrees of freedom interacting by a conservation of a common sum of quantum occupation numbers.

1 Introduction

The entropy gradient maximization was proposed as an alternative formalism generating the dynamic equations for a closed system with arbitrary degrees of freedom [8]. It is based on the fundamental principle of the *entropy maximization of closed system* (2nd law of thermodynamics).

It has been demonstrated that this principle applied locally to the infinitesimal variation of entropy produces first- and second-order generalized dynamic equations and leads independently to basic principles of causality and special relativity. Further developments of this approach are aimed at recovering all fundamental relationships in physics (including conservation laws and interaction phenomena), assuming a minimal number of primary statements. Ideally this should be the *only one, equivalent to the second law of thermodynamics*, and the only one basic object governed thereby - the entropy (or equivalently, the statistical weight).

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Also the formalism has been demonstrated to be a successful and promising approach for number of other applications.

The primary first-order approach firstly proposed in [8] provides the first-order dynamics by maximizing the first order entropy variation with one or more additional relationships - *ergodicity conditions*. These conditions can be interpreted as pre-assumed conservation laws [8, 9]. At the same time they imply a *correlation* between degrees of freedom , construed in the framework of the formalism as an *interaction.* The *ergodicity conditions* were firstly introduced on the example of the energy conservation as a relationship for conserved energy value for a given scalar function on the space of degrees of freedom q: $h(q) = \mathcal{E} = \text{const}$

The extension of the entropy variation to include second-order contributions also implies a corresponding extended formalism based on this approach. Especially it allows for generating dynamic equations by a conditionless maximization procedure.

This approach has been shown to produce the first order dynamic equations without demanding the energy conservation and is therefore determined as a *non-ergodic* formalism. Some most significant results of this extension are outlined in the recent report [10], which was devoted to the pure *non-ergodic second-order approach.*

The second-order approach for the local entropy gradient maximization for systems with many degrees of freedom without additional ergodicity conditions, e.g.like conservation laws, provides the first order dynamic equation and *light-like* (massless) second order dynamics.

Especially, it has been shown, that the resulting dynamics correspond to a system of non-interacting degrees of freedom . This appears as long as the generating functional of entropy variation does not contain entropy derivatives of order higher than two (the formalism is restricted on the second order).

Additionally it was demonstrated in [9, 10], that the application of this approach for the non-ergodic bosonic system leads to physically meaningless unrestricted growth of all occupation numbers independently of each other.

For the entropy construction of a kind like that considered in [9, 10] all derivatives of entropy of order higher than two disappear identically, even if no restrictions on the entropy variation are demanded. Hence, in order to obtain physically meaningful interactive dynamics, it is necessary to couple the maximized entropy variation with any ergodicity condition, for instance a conservation of total bosonic occupation number [9]. This *ergodic second-*

order approach was readily demonstrated to generate collective first-order dynamics, as applied for a bosonic quantum system in terms of occupation numbers. There both the second order entropy variation and the bosonic number conservation as the ergodicity have been assumed. The latter is revealed to be the issue of collective interaction between bosonic degrees of freedom .

In the present note the same approach has been applied to two problems:

I. System of identical fermionic degrees of freedom with the conservation of the total sum of fermionic occupation numbers;

II. System consisting of one-type bosonic and one-type fermionic degrees of freedom with the constant total occupation number (conservation of the total fermion+boson common number).

It will be shown below, that a straightforward application of this approach for a system of identical fermions in terms of occupation numbers does not provide any dynamics of a pure fermionic system.

A meaningful dynamics first appear for a system consisting of both fermionic and bosonic degrees of freedom with an ergodicity condition, which constrains both occupation numbers together. A simplest possibility would be demanding a conservation of a common sum of occupation numbers.

The content of the article is ordered conventionally.

The Section 2 contains the generalized formulation of the second order approach, providing the local first-order dynamics for bosonic degrees of freedom and the global first-order dynamics for bosonic and fermionic ensembles.

The derivation of the dynamical equations in the Sec.2 is followed by Sec.3 illustrating the dynamics by solutions for several special cases. Some conceivable physical interpretations are suggested.

The final Section 4 summarizes the observed results and contains a short subsequent discussion.

2 Dynamical equations

2.1 System of fermionic degrees of freedom

We start with the unrestricted entropy variation on the finite-dimensional space of fermionic occupation numbers f_α as degrees of freedom with dimension $F, \alpha = 1, 2, ..., F$ with the *discoupled time reference* τ [11]

$$\delta S^f = \sum_\alpha S_{f_\alpha} df_\alpha \quad + \quad \frac{1}{2}\sum_{\alpha,\beta} S_{f_\alpha f_\beta} df_\alpha df_\beta + \frac{1}{3!}\sum_{\alpha,\beta,\gamma} S_{f_\alpha f_\beta f_\gamma} df_\alpha df_\beta df_\gamma + ...$$
$$+ \quad S_\tau d\tau + \frac{1}{2} S_{\tau\tau} + ... \tag{1}$$

which presumably contains contributions from all-order-derivatives of entropy. However, since the second-order and all higher-order derivatives of entropy disappear (Appendix A.1), the formalism remains restricted to the first order approach.

Consequently, it confirms the fact, that fermionic degrees of freedom are governed by first-order dynamic equations (e.g. of Dirac-type for massive or Weyl type for massless fermionic fields respectively)

The interaction between f_α is provided by the *ergodicity condition* imposed in the form of a global:

$$\mathcal{F} := \sum_{\alpha=1}^{F} f_\alpha = \text{ const} \tag{2}$$

or a local:

$$\sum_{\alpha=1}^{F} df_\alpha = 0 \tag{3}$$

conservation law. The latter together with the surviving first-order entropy variation

$$\delta S^f = \sum_\alpha S_{f_\alpha} df_\alpha + S_\tau d\tau \tag{4}$$

leads finally to the conclusion that the direct exchange between fermionic degrees of freedom in form of a conservation law (3, 2) does not provide any dynamics.

Again it is in accordance with the fact, that there are no observable interactions between fermionic fields immediately.

2.2 Boson-fermionic system

As a next step we consider the extended system including both bosonic b_i and fermionic f_α occupation numbers as quantum degrees of freedom , with

$$\mathcal{F} := \mathcal{F}(f_1, f_2, ..., f_F) = \sum_{\alpha=1}^{F} f_\alpha \quad \text{and} \quad \mathcal{B} := \mathcal{B}(b_1, b_2, ..., b_B) = \sum_{i=1}^{B} b_i$$

total fermionic and bosonic occupation numbers in the present state $\{f_1, f_2, ..., f_F; b_1, b_2, ..., b_B\}$ respectively, where the positive value $\mathcal{B}$ can be arbitrary large,

$$0 \leq \mathcal{B} < +\infty,$$

while the $\mathcal{F}$ is restricted by

$$0 \leq \mathcal{F} \leq F.$$

We can now proceed with the *entropy gradient maximization* procedure using the same scenario as for the pure fermionic system above. The formalism starts with the second order entropy variation to be maximized:

$$\begin{aligned} \delta S[b_i, f_\alpha, \tau] &= \sum_i S_{b_i} db_i + \frac{1}{2} \sum_{i,k} S_{b_i b_k} db_i db_k \\ &+ \sum_\alpha S_{f_\alpha} df_\alpha + \frac{1}{2} \sum_{\alpha,\beta} S_{f_\alpha f_\beta} df_\alpha df_\beta \\ &+ S_\tau d\tau + \frac{1}{2} S_{\tau\tau} d\tau^2 \end{aligned} \tag{5}$$

An assumption of a conservation of the common total occupation number provides the additional (ergodicity) condition in a global

$$\mathcal{F} + \mathcal{B} = \sum_{\alpha=1}^{F} f_\alpha + \sum_{i=1}^{B} b_i = \mathcal{T} = \text{ const} \tag{6}$$

and a local form

$$\sum_{\alpha=1}^{F} df_\alpha + \sum_{i=1}^{B} db_\alpha = 0. \tag{7}$$

The partial derivatives entering in (6) are (Appendix A.1)

$$S_{b_j} = \sum_{i=0}^{B} b_i - b_j = \mathcal{B} - b_j; \quad S_{b_j b_k} = \left\{ \begin{array}{l} 0,\ j = k, \\ 1,\ j \neq k \end{array} \right\} = 1 - \delta_{jk}, \tag{8}$$

for bosonic and

$$S_{f_\beta} = \sum_{\alpha=0}^{F} = \mathcal{F}; \quad S_{f_\alpha f_\beta} = 1 \tag{9}$$

for fermionic degrees of freedom .

Using these results, a straightforward application of the maximization procedure (Appendix A.2) produces evolution equations for the dynamics of a single bosonic DoF's b_j:

$$\dot{b}_j := \frac{db_j}{d\tau} = -\frac{S_{\tau\tau}}{S_\tau}\left(\frac{\mathcal{B}+\mathcal{F}}{2B-1} - b_j\right) = \frac{1}{\Omega}\left(\frac{\mathcal{T}}{2B-1} - b_j\right), \tag{10}$$

for the entire bosonic ensemble $\mathcal{B}$

$$\dot{\mathcal{B}} := \frac{d\mathcal{B}}{d\tau} = \frac{1}{\Omega}\left(\frac{B}{2B-1}\mathcal{T} - \mathcal{B}\right) \tag{11}$$

and for the fermionic ensemble $\mathcal{F}$

$$\dot{\mathcal{F}} := \frac{d\mathcal{F}}{d\tau} = \frac{1}{\Omega}\left(-\frac{B}{2B-1}\mathcal{F} + \frac{B-1}{2B-1}\mathcal{B}\right) \tag{12}$$

as well.

A generalization of this result for high number $j \to \infty$ of bosonic states provides the continuous form of this system:

$$\begin{aligned}
\Omega(\tau)\frac{d}{d\tau}b(i) &= \bar{b} - b(i) \\
\text{where the effective bosonic number } \bar{b}: &= \frac{1}{2}\lim_{B\to\infty}\frac{1}{B}\int_1^B b(j) + f(j) \quad dj; \\
\Omega(\tau)\frac{d}{d\tau}\int b(j)dj &= \frac{1}{2}\int b(j) + f(j) \quad dj; \\
\Omega(\tau)\frac{d}{d\tau}\int f(j)dj &= \frac{1}{2}\int b(j) - f(j) \quad dj;
\end{aligned} \tag{13}$$

The former equation describes the dynamics of single bosonic modes, like a shape expansion (heat equation) or a propagation (wave equation), that was shown in [9] to be proper for bosonic systems. The latter two equations determine the global behaviors only of entire ensembles.

Some possible application scenarios of the discrete model for special cases can be considered in this context.

3 Examples

The dynamical equations

$$\frac{d\mathcal{B}}{d\tau} = \frac{1}{\Omega}\left(\frac{B}{2B-1}\mathcal{T} - \mathcal{B}\right) \tag{14}$$

$$\frac{d\mathcal{F}}{d\tau} = \frac{1}{\Omega}\left(-\frac{B}{2B-1}\mathcal{F} + \frac{B-1}{2B-1}\mathcal{B}\right) \tag{15}$$

can be rewritten in a standard matrix form for a simple linear dynamical system with two DoF's $\{\mathcal{B},\ \mathcal{F}\}$:

$$\frac{d}{d\tau}\left[\begin{array}{c} \mathcal{B} \\ \mathcal{F} \end{array}\right] = \frac{1}{\Omega(2B-1)}\left[\begin{array}{rr} -(B-1) & B \\ B-1 & -B \end{array}\right]\left[\begin{array}{c} \mathcal{B} \\ \mathcal{F} \end{array}\right], \tag{16}$$

with the singular matrix. This means, the system is effectively 1-dimensional (since $\mathcal{B} + \mathcal{F} = \text{const}$)

Some special cases of the ensemble dynamics defined by the equation(16) are considered below. They also could allow for an appropriate physical interpretation.

3.1 Vacuum state

For the trivial case of a pure fermionic world with no bosonic states,

$$B = 0,\ \text{that also implies } \mathcal{B} = 0$$

the equations (37 - 39) result merely in

$$\frac{d\mathcal{B}}{d\tau} = \frac{d\mathcal{F}}{d\tau} = 0$$

with a solution

$$\mathcal{F}(\tau) = \mathcal{F}(0) = \text{ const.}$$

But since $\mathcal{B} = 0$ such states are physically non-observable.

3.2 Single bosonic state

An extension of the model by one bosonic state

$$B = 1,$$

which can be occupied by an arbitrary positive number $\mathcal{B}$, produces a simple dynamical system

$$\frac{d\mathcal{B}}{d\tau} = \frac{1}{\Omega}\mathcal{F} \tag{17}$$

$$\frac{d\mathcal{F}}{d\tau} = -\frac{1}{\Omega}\mathcal{F} \tag{18}$$

with the solutions

$$\mathcal{F}(\tau) = \mathcal{F}(0)e^{-\frac{1}{\Omega}\tau}$$

$$\mathcal{B}(\tau) = \mathcal{B}(0) + \mathcal{F}(0)\Omega\left(1 - e^{-\frac{1}{\Omega}\tau}\right)$$

describing a natural exponential decay of the fermionic ensemble into the single bosonic mode.

3.3 Symmetric ensemble

The next trivial case is the system with equal numbers of bosonic and fermionic modes,

$$\mathcal{F}(\tau) = \mathcal{B}(\tau) = \frac{\mathcal{T}}{2} \tag{19}$$

The equations (38, 39) then become

$$\frac{d\mathcal{T}}{d\tau} = \frac{1}{2B-1}\mathcal{T}, \quad \frac{d\mathcal{T}}{d\tau} = -\frac{1}{2B-1}\mathcal{T}$$

with the trivial constant solution,

$$\frac{d\mathcal{T}}{d\tau} = 0, \quad \mathcal{T} = \text{ const.}$$

3.4 Large bosonic ensemble

An increasing number of bosonic states

$$B \to \infty,$$

which corresponds to

$$\begin{aligned} \frac{d\mathcal{B}}{d\tau} &= \frac{1}{2\Omega}\left(\mathcal{F}-\mathcal{B}\right) \\ \frac{d\mathcal{F}}{d\tau} &= -\frac{1}{2\Omega}\left(\mathcal{F}-\mathcal{B}\right). \end{aligned} \tag{20}$$

Surprisingly, this arranges for stabilizing the fermionic occupation number, since the solution of (20)

$$\mathcal{F}(\tau)-\mathcal{B}(\tau) = \left[\mathcal{F}(0)-\mathcal{B}(0)\right] e^{-\frac{1}{2\Omega}\tau}$$

exponentially approaches the equilibrium symmetric ("supersymmetric") state with

$$\mathcal{F}(\tau) = \mathcal{B}(\tau), \tau \to \infty$$

Thus, the boson-fermionic dynamics possesses an asymptotic limit corresponding to the stabilizing fermionic ensemble in a bosonic environment. This phenomena could for example be considered as stable leptons, which are stabilized through the interaction with an infinite photonic background.

3.5 Discrete bosonic & fermionic ensembles

The question of a special interest is to investigate the case where all degrees of freedom are integers, since the model is aimed to reproduce a structure of quantum state space.

To this end we recall, that both numbers of states B, F and all total occupation numbers of ensembles $\mathcal{F}, \mathcal{B}$ are generally considered to be integers,

$$B,\ F,\ \mathcal{B},\ \mathcal{F}\ (0 \le \mathcal{F} \le F) \in \mathbb{N},$$

as well as change steps of all degrees of freedom , the time reference τ including,

$$d\mathcal{F}, d\mathcal{B}, d\tau \in \mathbb{Z}.$$

Then the equations (38-39) for an integer value N of bosonic states, $B = N$ and a minimal change step

$$-d\mathcal{F} = d\mathcal{B} = 1 \tag{21}$$

transform to two identical relations

$$\begin{aligned} 2N - 1 &= N\mathcal{F} - (N-1)\mathcal{B} & (22) \\ -(2N-1) &= -N\mathcal{F} + (N-1)\mathcal{B} & (23) \end{aligned}$$

($\Omega = 1$ for simplicity) with a strongly restricted discrete spectrum of admissible solutions. E.g. for $N = 8$ the corresponding equation

$$8\mathcal{F} - 7\mathcal{B} = 15$$

is first satisfied by the couple of numbers

$$\mathcal{F} = 8; \ \mathcal{B} = 7.$$

The next solutions are

$$\begin{aligned} \mathcal{F} &= \mathcal{B} = 15, \\ \mathcal{F} &= 22; \ \mathcal{B} = 23, \\ \mathcal{F} &= 29; \ \mathcal{B} = 31, \end{aligned}$$

and so on.

It means, the discrete topology of the state space (21) forms an another discrete structure of the "space-time" represented by $\{\mathcal{F}, \ \mathcal{B}; \ \tau\}$.

In framework of this statement, an existence of stable confined systems could be explained simply by the only existence of discrete states with integer-numbered bosonic and fermionic modes with a fixed number of bosonic states N. Among the non-observable case

$$N = 1 : \mathcal{F} = 1; \ \mathcal{B} = 0,$$

we have further:

$$N = 2 : \ \mathcal{F} = 2; \ \mathcal{B} = 1,$$

$$N = 3 : \ \mathcal{F} = 3; \ \mathcal{B} = 2.$$

These discrete sets of states with a small fermionic number could be interpreted, for example, as a two-quark meson confined by one gluon or a three-quark baryon confined by two gluons.

4 Discussions and Conclusions

In the present note, the basic principles of the entropy gradient maximization (EGM) approach have been applied to derive a dynamics and dynamical equations for statistical system on the quantum level like [13, 9]The EGM-procedure was elaborated in previous applications [8] - [12] for systems of continuous statistical degrees of freedom .

For a quantum system of identical units the occupation number was considered as the the appropriate degree of freedom.

Since the pure *non-ergodic* approach in the second order EGM formalism describes only systems of non-interacting degrees of freedom , the additional (ergodiciy) condition is needed to generate an interaction.

In a framework of the second order formalism, the interaction can be induced simply by adding a common conservation law as a constraint.

The conservation of the total occupation number has been chosen to be a conservation law generating the interaction in a pure bosonic system [9] and in a fermionic and a mixed statistical systems as well. Some obvious physical facts has been reproduced consistently:

The trivial result is, firstly, that a pure fermionic system cannot have a proper dynamics. A dynamics appears due to interaction with an adjoined bosonic system. Nevertheless the simple model with only one there conservation law still does not determine a local dynamics for single fermionic modes but rather an evolution of an entire fermionic ensemble interacting with a bosonic one. The dynamics of single bosonic degrees of freedom still remains similar to that of pure bosonic system.

The next important conclusion is that the mixed boson-fermionic system possesses an asymptotic stability and is therefore non- controversal to describe the physical world. In particular it follows, that the statistically stablest asymptotic state is the "supersymmetric" state - the boson-fermionic equilibrium.

Some further appropriate results of the model are among others:

- The scenario of an infinite "bosonic bath" offers the option to consider bounded fermionic systems (e.g. multi-electron state like an electron-shell of an atom) as some stable states of multi-fermionic ensembles, interacting with an infinite bosonic background (photons).

- The existence of stable discrete states for small fermionic numbers bounded by small bosonic numbers allows for an interpretation of elementary confined structures as algebraically discrete states for generically integervalued degrees of freedom .

Finally, it is worth mentioning the remarkable issue of the resulting dynamics: only bosonic degrees of freedom reveal the local dynamics which is able to produce expanding and propagating modes. This fact is in accordance with the conventional interpretation of bosons as mediators of interactions between fermionic states.

5 Outlooks

The further aims of the approach outlined above are to reproduce the structure of fundamental dynamical equations of matter: of Maxwell type for massless vector bosonic, of Dirac type for massive fermionic and of Schrödinger / Klein-Gordon type for massive scalar fields.

This would mean, that the physics of quantum world can be reformulated taking the arbitrary high-dimensional space of statistical degrees of freedom for a basis. It's worth remarking, in this regard, there is still no geometric (configuration) space in this approach.

It would be expected, that the structure of the physically observable space arises as a low-dimensional reduction of higher level on the primary high- dimensional space of statistical degrees of freedom .

A Mathematical Supplement

A.1 Partial derivatives of entropy

The statistical weight of the state vector of occupation numbers n_i is given combinatorially by

$$W_{n_i} = \frac{(n_1 + n_2 + ... + n_N)!}{n_1!n_2!...n_N!} = \frac{\left(\sum\limits_{i=-N}^{N} n_i\right)!}{\prod\limits_{i=-N}^{N} n_i!} \tag{24}$$

and the related additive entropy is the logarithm of W - the *discrete bosonic entropy* :

$$S[n_i] = \ln W_{n_i} = \ln \left(\sum_{i=-N}^{N} n_i\right)! - \sum_{i=-N}^{N} \ln n_i! \tag{25}$$

In the continuum limit at very large numbers n_i factorials are replaced by Gamma-functions:

$$S[n(i)] = \ln W_{n(i)} = \ln\ \Gamma\left[\sum_{i=-N}^{N} n(i)\right] - \sum_{i=-N}^{N} \ln\ \Gamma[n(i)] \tag{26}$$

For the both cases (25 - 26) one obtains for partial derivatives (Sec. Appendix):

$$S_{n_i} = \sum_i n_i - n_i;\ S_{n_i n_k} = 1; S_{n_i n_i} = 0$$

By using the continuation of

$$\frac{d}{dn} n! = \lim_{n\to\infty} \frac{(n+1)! - n!}{1} = n! n$$

we can generalize this result for Gamma-function as

$$\frac{d}{dn}\Gamma(n) = n\Gamma(n),$$

and

$$\frac{d}{dn}\ln n! = n, \quad \frac{d}{dn}\ln\Gamma(n) = n.$$

Then we obtain for a *bosonic case* $n_i = b_i$:

$$\frac{\partial}{\partial b_k} S[b(i)] = \frac{\partial}{\partial b_k}\left\{\ln\ \Gamma\left[\sum_{i=0}^{B} b(i)\right] - \sum_{i=}^{B} \ln\ \Gamma[n(i)]\right\} = \sum_{i=0}^{B} b(i) - b(k) \tag{27}$$

and

$$S[b(i)]_{b_k b_k} = \frac{\partial^2}{\partial b_k \partial b_k}\left[\sum_{k=0}^{B} b(k) - b(i) = 0\right]; \tag{28}$$

$$S[b(i)]_{b_k b_l} = \frac{\partial^2}{\partial b_k \partial b_l}\left[\sum_{k=0}^{B} b(k) - b(i) = 1\right],\ k \neq l, \tag{29}$$

finally rewritten for further usage as:

$$S_{b_j} = \sum_{i=0}^{B} b_i - b_j = \mathcal{B} - b_j; \quad S_{b_j b_k} = \left\{\begin{array}{l} 0,\ j = k, \\ 1,\ j \neq k \end{array}\right\} = 1 - \delta_{jk}. \tag{30}$$

In a *fermionic case* $n_i = f_\alpha$, obeying

$$f_\alpha = \{0 \text{ or } 1\}; \quad f_\alpha! = 1, \tag{31}$$

the same formulas (24 25 26) are used. The denominator in (24) disappears due to (31),

$$W_{f_\alpha} = (f_1 + f_2 + ... + f_F)! = \left(\sum_{\alpha=0}^{F} f_\alpha\right)!, \tag{32}$$

and the corresponding entropy reads

$$S[f_\alpha] = \ln\left[\left(\sum_{\alpha=0}^{F} f_\alpha\right)!\right]. \tag{33}$$

Applying the formulas for partial derivatives of entropy (27,29) provides:

$$S_{f_\beta} = \sum_{\alpha=0}^{F} = \mathcal{F}; \quad S_{f_\alpha f_\beta} = 1. \tag{34}$$

A.2 Maximization of the entropy variation

The maximization of (6) with the additional condition (7), adjoined by the Lagrange multiplier λ

$$\delta_\lambda S[b_i, f_\alpha, \tau] = \delta S[b_i, f_\alpha, \tau] + \lambda\left(\sum_{\alpha=1}^{F} df_\alpha + \sum_{i=1}^{B} db_\alpha\right)$$

results in two conditions of extrema

$$\sum_{i=1}^{F} df_i S_{b_i b_k} db_i + S_{b_k} + \lambda = 0,$$

$$\sum_{\alpha=1}^{F} S_{f_\alpha f_\beta} df_\alpha + S_{f_\beta} + \lambda = 0.$$

A summation over the total number of degrees of freedom with partial derivatives of S substituted by (30 - 34) provides after the primer performance:

$$\mathcal{B} - b_j + \sum_k db_k - db_j + \lambda = 0$$

$$\mathcal{F} + \sum_\alpha df_\alpha + \lambda = 0. \tag{35}$$

The latter can be rewritten in terms of b_k by means of (7) as

$$\mathcal{F} - \sum_k db_k + \lambda = 0$$

The elimination of λ (by subtraction of equations from each other),

$$\mathcal{B} - \mathcal{F} - b_j + 2\sum_k db_k - db_j = 0$$

and the repeated summation over bosonic degrees of freedom j

$$B(\mathcal{B} - \mathcal{F}) - \mathcal{B} + 2B\sum_k db_k - \sum_j db_j$$

results after a suitable simplification in

$$\sum_k db_k = \frac{B\mathcal{F} - \mathcal{B}(B-1)}{2B-1},$$

$$db_j = \mathcal{B} - \mathcal{F} - b_j + 2\sum_k db_k = \mathcal{B} - \mathcal{F} - b_j + 2\frac{B\mathcal{F} - \mathcal{B}(B-1)}{2B-1}.$$

Together with

$$d\tau = -\frac{S_\tau}{S_{\tau\tau}} := \Omega(\tau) \tag{36}$$

for the discoupled time reference, we arrive at the dynamic equation for single bosonic DoF's b_j:

$$\dot{b}_j = \frac{db_j}{d\tau} = -\frac{S_{\tau\tau}}{S_\tau}\left(\frac{\mathcal{B}+\mathcal{F}}{2B-1} - b_j\right) = \frac{1}{\Omega}\left(\frac{\mathcal{T}}{2B-1} - b_j\right) \tag{37}$$

Furthermore, with summing over all bosonic modes b_j again, we obtain the equation for the entire bosonic ensemble $\mathcal{B}$

$$\dot{\mathcal{B}} := \frac{d\mathcal{B}}{d\tau} = \frac{1}{\Omega}\left(\frac{B}{2B-1}\mathcal{T} - \mathcal{B}\right) \tag{38}$$

For fermionic degrees of freedom the formalism provides no dynamics for single fermionic modes df_α, since differentials df_α in (35) appear only as a sum over the ensemble. The only dynamics of the entire fermionic ensemble can be obtained. Starting again with

$$\sum_{\alpha} df_{\alpha} + \mathcal{F} + \lambda = 0$$

$$\mathcal{B} - b_j - \sum_{\alpha} df_{\alpha} - db_j + \lambda = 0$$

and proceeding in the same way as above for bosonic modes, we arrive finally at

$$d\mathcal{F} + \frac{B}{2B-1}\mathcal{F} + \frac{1-B}{2B-1}\mathcal{B} = 0$$

which produces together with (36) the dynamic equationfor the fermionic ensemble $\mathcal{F}$

$$\dot{\mathcal{F}} := \frac{d\mathcal{F}}{d\tau} = \frac{1}{\Omega}\left(-\frac{B}{2B-1}\mathcal{F} + \frac{B-1}{2B-1}\mathcal{B}\right) \tag{39}$$

The total balance of the ensemble

$$\dot{\mathcal{T}} = \frac{d}{d\tau}(\mathcal{F} + \mathcal{B}) = 0$$

remains in accordance with (7).

References

[1] Joel Keizer
Statistical Thermodynamics of Nonequilibrium Processes
Springer 1987
ISBN 978-1-4612-1054-2

[2] V. Zubarev, D. Morozov, G. Röpke
Statistical Mechanics of Nonequilibrium Processes
Akademie Verlag, Berlin (1996)
ISBN 3-05-501708-0

[3] L.D. Landau, E.M. Lifshitz
Course of Theoretical Physics, Vol V:
Statistical Physics
Elsevier(1980)

[4] H.Bateman, A, Erdelyi
Higher Transcendental Functions, Vol.1
McGraw Hill Book Company,Inc. (1953)

[5] I.S. Gradsheyn, I.M. Ryzhik,
Tables of Integrals, Sums, Series and Products
Academic Press Inc. (1980)

[6] Richard P. Stanley
Enumerative combinatorics Vol. 1
Wadsworth & Brooks/Cole (1986)

[7] N.I. Akhiezer
The Calculus of Variations
Harwood Academic Publishers GmbH (1988)

[8] I.V. Drozdov
On a local formalism for time evolution of dynamical systems
arxiv: physics.class-ph 1501.02945 (2015)

[9] I.V. Drozdov
Dynamics of Statistical Quantum System in Terms of Bosonic Occupation Numbers
viXra:1510.0109

[10] I.V. Drozdov
The second order local formalism for time evolution of dynamical systems
viXra: 1605.0240 (2016)

[11] I.V. Drozdov
Entropy Gradient Maximization for Systems with Discoupled Time
vixra: 1508.0080 (2015)

[12] I.V. Drozdov
Entropy Gradient Maximization for Local-Ergodic Systems
vixra: 1508.0084 (2015)

[13] I.V. Drozdov
Problem of Thermally Driven Diffusion in Terms of Occupation Numbers
viXra: 1510.0131 (2015)