Integration Over Manifolds with Variable Coordinate Density

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Abstract

In this paper, the integration of a function over a curved manifold is examined in the case where the curvature of the manifold results in a varying density of coordinates over which the function is being integrated where the upper bound of the of integration is infinity. It is shown that when the coordinate density varies in such a case, the true area under the curve is not correctly calculated by traditional techniques of integration, but must account for the varying coordinate density. This integration technique is then applied to the Schwarzschild metric of General Relativity to examine the proper time taken for a freefalling observer to reach the event horizon of a black hole.

Integration and Coordinate Density

Consider a velocity $\frac{dx}{dt}$ defined by some function that is parameterized by a variable r where r increases (or decreases) as t increases such that $\frac{dx}{dt} = f(r)$. We will begin by integrating this function between t_0 and some finite t. In Figure 1, we see this function plotted in two different cases: on the left, we have a flat manifold where the time coordinate density is constant along the length of the function, and on the right we have a curved manifold which causes the density of the time coordinates to increase as r increases.

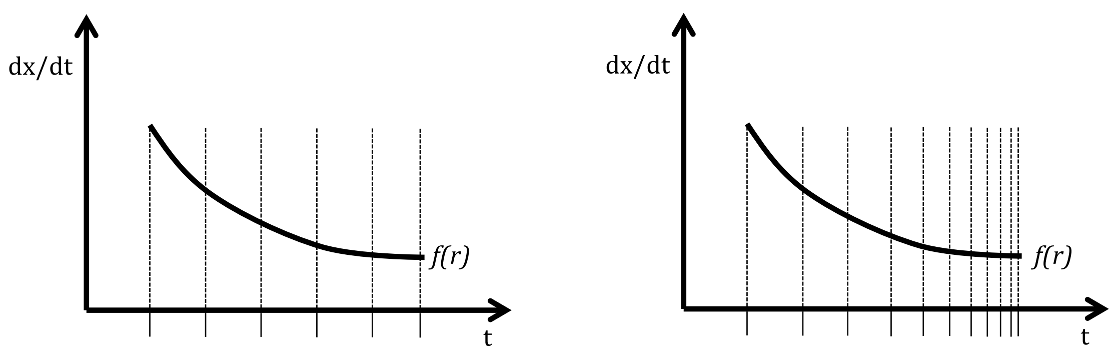

Figure 1 – Velocity vs. Time on Flat (left) and Curved (right) Manifolds (Finite Upper Bound)

On both the left and right sides of Figure 1, the numerical Δt between tick marks on the time axis is the same between any two adjacent tick marks. Now let us examine what happens when we approximate the integrals by summing the areas of the rectangles (where each rectangle has area $A = \left(\frac{dx}{dt}(t)\right)(\Delta t)$). When comparing the left and right images in Figure 1, we see that since the Δt between tick marks in both images is the same, the approximate integral on the right side will give a larger value than the integral on the left as a result of the increasing density of the time coordinate as r increases (we have more rectangles of equal Δt on the right side). Now, it is not entirely surprising that the integral on the right would be larger than that on the left since we are essentially integrating the function over a larger time interval on the right side (we could just stretch the coordinates on the right side such that they have equal density and then integrate normally over a larger time interval). The purpose of Figure 1 is to introduce the effect of increasing coordinate density on an integral to emphasize that when we approximate the integral by summing rectangles when the density increases, we get a larger value for the integral because we are summing more rectangles of equal Δt.

Now consider the same type of velocity function integrated from some finite time to *infinite* time in flat and curved spacetime. Figure 2 shows both of these cases for a velocity function that decreases to zero as t goes to infinity.

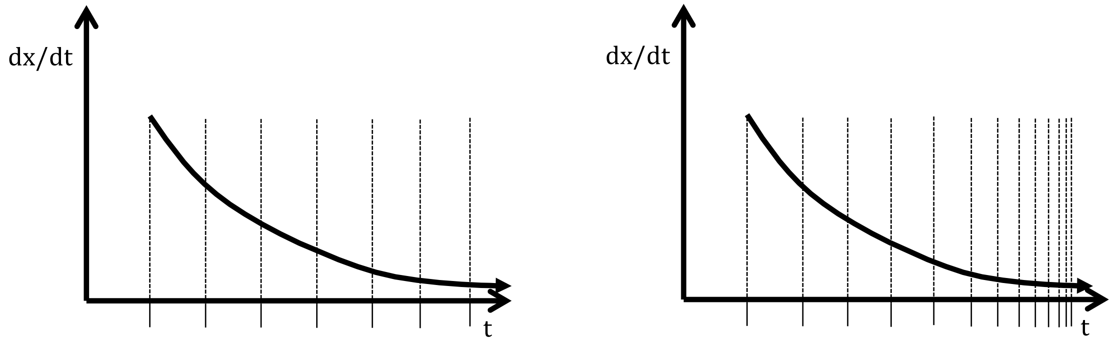

Figure 2 – Velocity vs. Time on Flat (left) and Curved (right) Manifolds (Infinite Upper Bound)

Just as was the case in Figure 1, we can see that when we approximate the integrals in Figure 2 by summing the areas of rectangles, the integral on the right will give a larger than the integral on the left. However, in this case, since the upper bound of t is infinite in both cases, we can't attribute the increase in area to an increase in the time interval. If we suppose that $f(r)$ decreases in such a way that the integral from t_0 to ∞ on the left side (flat manifold) gives a finite value, we can see that the integral on the right side will give a value greater than that and if the coordinate density goes to infinity as $\frac{dx}{dt}$ goes to zero, the integral can even be infinite. This idea of coordinate density can be thought of as being analogous to a dynamic unit change. For instance, in the flat manifold case, suppose x and t were measured in the same units and we multiply the integral by a constant. That would essentially be a change of units (minutes to seconds or mm to meters). But a change of units is really just a rescaling of the axes. So since the coordinate density is describing how the coordinates are scaled over the manifold, it is as if the coordinate units are being changed as you move along the manifold. This is essentially what length contraction and time dilation in General Relativity is, a relative stretching or squeezing of the coordinate axes.

This concept will next be applied to the worldline of a freefalling observer in Schwarzschild spacetime to examine the limit of proper time of the freefaller she approaches the event horizon.

Freefall in the Schwarzschild Field

The well-known Schwarzschild metric is given in (1) below (note we will be using units where the Schwarzschild radius is 1 and we will drop the angular term of the metric since we will only be examining radial freefall):

$$d\tau^2 = \left[1 - \frac{1}{r}\right] dt^2 - \left[1 - \frac{1}{r}\right]^{-1} dr^2 \tag{1}$$

These coordinates are quite useful for describing the spacetime for observers at rest in the gravitational field, particularly the observer at infinity in asymptotically flat spacetime. The *r* coordinate represents some notion of distance from the center of the gravitational source, where the units of *r* are in units of Schwarzschild radius of the source. Thus, this radial coordinate gives circles around the source where, in a top-down view of the source, the circle radii increase linearly as one moves away from the center. Let's now consider the coordinate speed of a freefalling observer (who starts to fall from rest at infinity) in the frame of an observer at infinity in Schwarzschild coordinates [1]:

$$\frac{dr}{dt} = -\sqrt{\frac{1}{r}}\left[1 - \frac{1}{r}\right] \tag{2}$$

Let us now substitute (2) into (1) to examine the proper time of the freefalling observer:

$$d\tau = \left[1 - \frac{1}{r}\right] dt \tag{3}$$

It is conjectured here that Equation 3 is the case described in the first section of this paper where the *t* coordinate density increases as one moves toward the event horizon. If one integrates (3) in the usual way starting from some finite distance from the horizon to the horizon, the integral will yield a finite proper time, but it will be argued that when accounting for the increasing coordinate density near the horizon, the actual time measured by the freefalling observer will be infinite. In order to demonstrate this, we must first make a change of variables for the radial coordinate.

Radial Coordinate Transformation

It is conjectured that the freefalling observer will fall for infinite proper time before reaching the event horizon, and this means that the freefalling observer must traverse an infinite amount of space while falling to the horizon. But the Schwarzschild radial coordinate *r* is defined such that if someone begins falling from some finite distance from the horizon, they will traverse a finite *r*. This is depicted graphically in Figure 3 below.

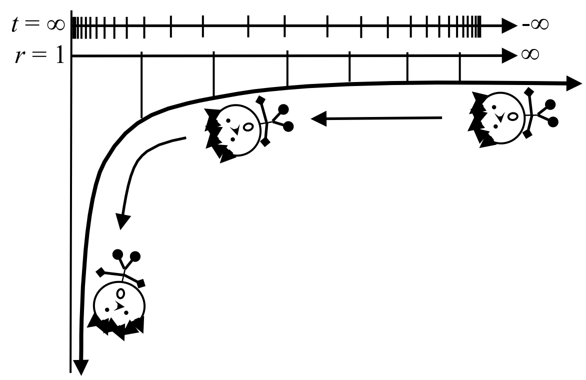

Figure 3- Relationship Between Schwarzschild Coordinates and the Curved Manifold

In Figure 3, we see our intrepid explorer, Scout, freefalling along a radial geodesic in the Schwarzschild gravitational field in the frame of observers at rest in the field. The infinite observer would be off to the right on this diagram where the geodesic (the dark black line) would be horizontal. Since, in this particular depiction, the tangent to the manifold is horizontal at the infinite observer who is inertial in flat space, the acceleration needed for an observer to remain at rest at a given point is proportional to the slope of the tangent at that point.

It is desirable at this point to make a coordinate change for the radial coordinate such that it is better able to capture the curvature near the horizon in the same way the time coordinate does. We will choose coordinate R such that $\frac{dR}{dr} = \frac{r}{r-1}$. This coordinate varies identically to the r coordinate for large r (this is good because r is a good physical coordinate at large r) and then diverges from it at the horizon. Integrating the expression gives:

$$R = r + \ln(r-1),\ r = W(e^{R-1}) + 1 \tag{4}$$

Where W is the product-log function. Note that $R \to \infty$ as $r \to \infty$ and $R \to -\infty$ as $r \to 1$. R is zero in the region of the elbow of the geodesic pictured in Figure 3. Making this coordinate substitution in (2) gives:

$$\frac{dR}{dt} = -\sqrt{\frac{1}{W(e^{R-1})+1}} = -\sqrt{\frac{1}{r}} \tag{5}$$

This coordinate choice is also useful because the speed of light in these coordinates is 1 independent of R and t. The Schwarzschild metric with the new coordinate becomes:

$$d\tau^2 = \frac{W(e^{R-1})}{W(e^{R-1})+1}[dt^2 - dR^2] = \frac{r-1}{r}[dt^2 - dR^2] \tag{6}$$

A portion of the worldline of a freefalling observer plotted on the t-R plane is shown in Figure 4 below:

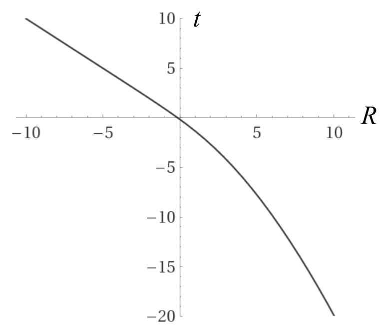

Figure 4 – *t* vs. *R*

The slope of the worldline is close to but less than 1 in the upper right quadrant for all finite *R* and *t*.

If we substitute Equation 5 into Equation 6, we get the expression for proper time of the freefalling observer:

$$d\tau = -\sqrt{\frac{W(e^{R-1})}{W(e^{R-1})+1}\left[W(e^{R-1}) + 1 - 1\right]}\, dR = -\sqrt{(r-1)}\sqrt{\frac{r-1}{r}}\, dR \qquad (7)$$

This function decreases to zero as $R \to -\infty$ and if it is integrated directly as-is, we find that there is a finite proper time to reach the horizon from any finite *R*. But we know that the coordinate density increases as the freefaller approaches the horizon and therefore, as discussed in the first section of this paper, a typical integration will not give the correct quantity of proper time elapsed.

Inertial Motion in General Relativity

Suppose we have two observers in flat spacetime where for some period of time, their times are related by the expression $dt_1 = dt_2$. Then, this relationship changes such that their times are related by $dt_1 = 0.5dt_2$. We might depict this as shown on the graph on the left side of Figure 5 below:

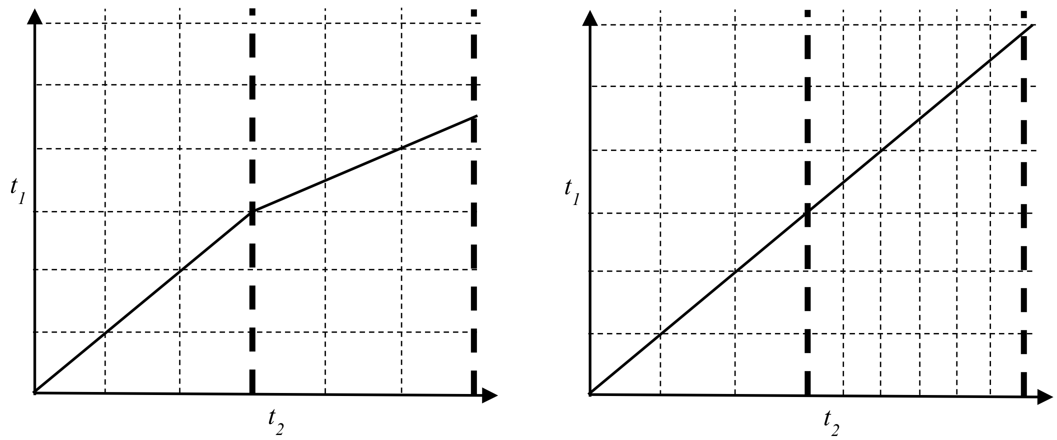

Figure 5 – Acceleration as a Change of Units

The left side of Figure 5 suggests that at some point, observer 1 accelerated relative to observer 2, changing the slope of the worldline. But when we ask both observers, they each say that they never felt any acceleration (i.e. they were at rest relative to one another the entire time). How can this be? After inspecting both clocks, it is found that in the first half of the graph, both clocks were measuring seconds. But in the second half, observer 1's clock ticked off seconds while observer 2's clocks was ticking off half-seconds. The problem with the graph on the left of Figure 5 is that it was constructed using the derivatives given earlier, without accounting for the fact that the change in the derivative from 1 to 0.5 was not caused by acceleration, but was simply the result of a unit change in t_2. Therefore, we can see the correct depiction of the situation on the right side of Figure 5. The coordinate marks represent the actual clock ticks. Since they were both at rest relative to each other the whole time, we know that if both of their clocks had ticked off seconds the whole time, they would have agreed that the same amount of time had passed. Therefore, we should draw the relationship as a 45-degree straight line on the graph. But since observer 2's clock ticked off half-seconds in the second half of the graph, the tick marks must be twice as dense on the second half relative to those on the first half. This allows us to preserve the derivatives relating the times.

This scenario is what we have when considering an observer freefalling in a Schwarzschild field. Both the observer at infinity and the freefalling observer remain inertial as the freefaller falls. The apparent acceleration between them is coming from the curvature of the freefaller's time coordinate relative to the infinite observer's time coordinate. This curvature manifests itself as an increase in time coordinate density for the freefaller as she falls. It is analogous to a continuous unit change of the freefaller's clock relative to the infinite observer's clock. The metric in Equation 6 is useful because it is essentially the Minkowski metric with a variable multiplicative factor. This multiplicative factor is analogous to a function keeping track of the units of the freefaller's clock relative to the infinite observer's clock. Since the metric is quasi-Minkowskian, we can compare the worldline of the freefalling observer, defined by Equation 5, to the same worldline for an observer accelerating in Minkowski space. This comparison is shown in Figure 6 below:

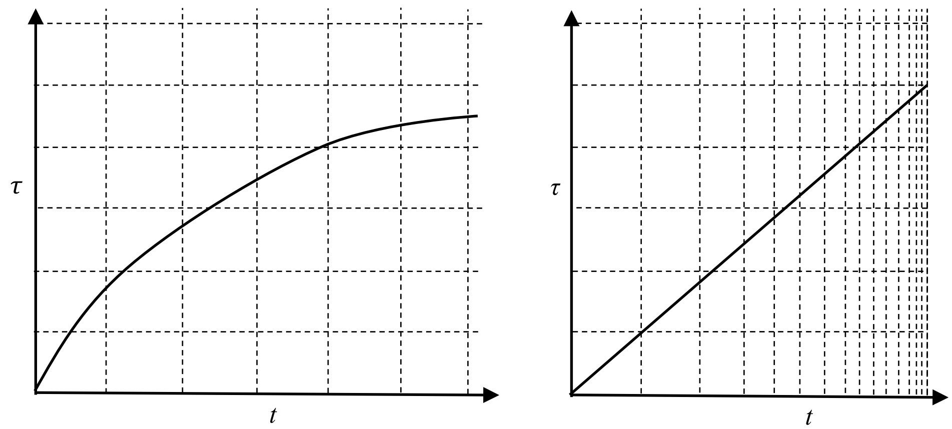

Figure 6 – τ vs. t for the Accelerating Observer in Flat Space (left) and τ vs. t for the Freefalling Observer (right)

In Figure 6, we see the worldline of Equation 5 in both Minkowski space and in the Schwarzschild field. The graph on the right side of Figure 6 can be constructed by laying the flat grid from the left graph on it and then adding tick marks to the *t* axis between the flat ticks. The number of ticks to add between two flat ticks will be proportional to $\frac{r}{r-1}$. As we can see in Figure 6, since the *t* axis of the flat grid goes off to infinity, the *t* axis on the Schwarzschild grid will do the same with an ever-increasing coordinate density. The result of this is that there must be an infinite amount of proper time elapsed when falling toward the horizon from any finite distance. If the freefaller accelerates at any time or starts falling with a non-zero velocity relative to the infinite observer, then the straight worldline on the left of Figure 6 will be curved or will have a different slope depending on that acceleration/initial velocity.

So in the Minkowski case, the observer is in an accelerating reference frame, causing her to approach a maximum τ asymptotically as $t \to \infty$. Figure 6 therefore makes plain the difference between freefall in a Schwarzschild field and acceleration in flat spacetime when both observers have the same $\frac{dR}{dt}$. It shows that for the freefalling observer, the underlying coordinate curvature is the cause of the acceleration whereas in flat space it is a 'real' acceleration. This 'real' acceleration manifests itself as a curvature in the worldline itself as shown in Figure 6. Figure 6 is therefore an effective graphical representation of the Equivalence Principle at work.

As will be shown in the conclusion of this paper, the freefalling observer will see the rest observer's clock slow as she falls. This is just like the symmetry of Special Relativity where inertial observers moving relative to each other each claim that it is the other person who is moving. In order to actually compare their clocks, one of the observers must accelerate toward the other, breaking the symmetry and resulting in less time having passed for the accelerating observer when the two observers meet (so in order to meet and compare clocks, either the freefaller has to accelerate away from the center of gravity or the rest observer must accelerate toward it in order to catch up to the freefaller).

In terms of the R coordinate, the τ-R grid will look very much like Figure 4 because the t and R coordinates are curved by the same factor. However, we see that we can express Equation 7 as $\frac{d\tau}{dR} = \frac{d\tau}{dt}\sqrt{r}$. Therefore, rather than getting a straight line as was the case in Figure 6, the $\sqrt{r}$ factor gives us the worldline shown in Figure 7 (recall that $r \to 1$ as $R \to -\infty$):

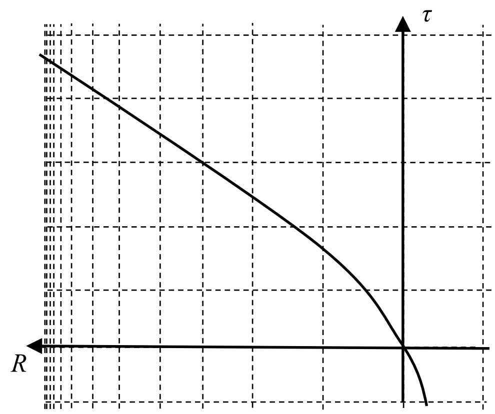

Figure 7 – τ vs. R for the Freefalling Observer

Combining Equations 1 and 2, we see that $\frac{d\tau}{dr} \to 1$ as $r \to 1$. This is because as r goes to 1, both $\frac{dr}{dt}$ and $\frac{d\tau}{dt}$ for the freefalling observer go to zero, where $\frac{d\tau}{dt}$ goes to zero for the reasons discussed above, and $\frac{dr}{dt}$ goes to zero because of the extreme curvature of space near $r = 1$, which will be examined in further detail below.

Given that the freefalling worldline is a geodesic in curved spacetime, we can use the grid on the right side of Figure 6 to show the τ vs. t relationship for observers at rest in the gravitational field. This is depicted in Figure 8:

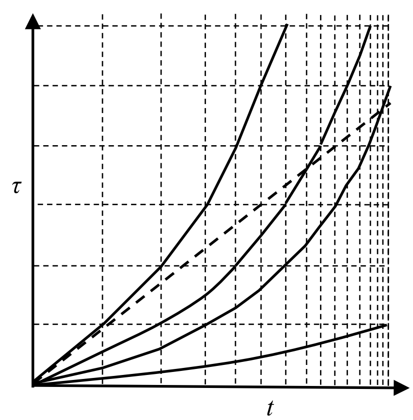

Figure 8 - τ vs. t for Observers at Rest in a Gravitational Field

For observers at rest, $\frac{d\tau}{dt}$ is constant. But the observers at rest are also in an accelerated reference frame, thus we should expect that the worldlines be curved in the curved spacetime. This is what we see in Figure 8, where the curved lines are lines of constant r. The lines of constant r are curved such that the change in τ per change in t is constant, but since the time dimension is curved, the worldline becomes curved relative to the worldline of the inertial observer, which reflects the fact that the rest observer is in an accelerated frame. The dashed line in Figure 8 is the freefalling observer. The rest observer at the event horizon would just be a horizontal line at the bottom of Figure 8. The freefalling worldline will never intersect that line and therefore we see again that it must take an infinite amount of proper time to reach the horizon. Note that if we plot curves of constant *R* coordinate on Figure 8, we would see that curves of equal intervals of *R* would get closer and closer together as *R* goes to negative infinity in the same way that the *t* coordinate spacing decreases. This is what we would expect from Equation 6 where the *R* and *t* coordinates are both curved gravitationally by the same factor.

By manipulating Equation 6, we can see that the straight line over the curved of the time coordinate in Figure 8 is the result of two factors:

$$d\tau = \sqrt{1 - V^2}\sqrt{\frac{r-1}{r}}\,dt \tag{8}$$

Where $V = \sqrt{\frac{1}{r}}$. The factor $\sqrt{\frac{r-1}{r}}$ in Equation 8 is the gravitational time dilation caused by the spacetime curvature (the Minkowski time coordinate is related to the Schwarzschild time coordinate by $dT = \sqrt{\frac{r-1}{r}}dt$). This factor is what governs the time coordinate spacing in Figure 8. The factor $\sqrt{1 - V^2}$ is an additional time dilation caused by the relative velocity between the freefaller and the rest observers. We see that as the worldline moves to increasing *t*, the factor $\sqrt{1 - V^2}$ gets closer to zero, reducing the amount of proper time elapsed for each interval of coordinate time passed. On Figure 8, this reduction is what keeps the worldline straight. Notice that when a rest observer worldline intersects straight line, its apparent slope on Figure 8 is greater than one. These observers have $\frac{d\tau}{dt} = \sqrt{\frac{r-1}{r}}$, the gravitational time dilation factor. For the inertial worldline to remain straight, this slope must be decreased, and that is what the $\sqrt{1 - V^2}$ factor does. This is why the inertial observer accelerates relative to the spacetime coordinates in order to remain inertial.

Next, let's consider the proper distance of a spacelike slice of the Schwarzschild metric. From Equation 1, we see that the relationship between proper distance and the *r* coordinate in this case is given by $\frac{ds}{dr} = \sqrt{\frac{r}{r-1}}$. In Minkowski space, this relationship would be $\frac{ds}{dr} = 1$ and the difference between the two is caused by the manifold curvature. So Figure 9 shows a spacelike *s* vs. *r* graph for the Schwarzschild case:

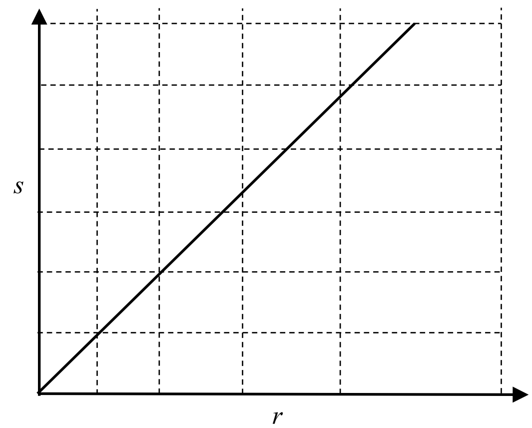

Figure 9 – s vs. r for Spacelike Slice of the Schwarzschild Metric

Since the flat space relationship between s and r is a 45-degree line on the graph, we draw that first. Then we dynamically stretch the axis with the radial coordinate as r goes to 1 to match the derivative given by the Schwarzschild metric (in Figure 9, r starts at some finite distance far from the horizon on the left side and then approaches 1 as we move to the right). So although there will only be a finite number of total r ticks on Figure 9, as we get closer to $r = 1$ the coordinate marks get increasingly stretched until they get infinitely stretched as r goes to 1. Thus, we see that starting from any finite coordinate distance from the horizon, there will be an infinite proper distance to the horizon. The r coordinate curvature in Figure 9 for the worldline of the inertial observer can be deduced from the metric as was done with the time coordinate. Namely, the relationship $d\tau = -\sqrt{r}dr$ for the inertial observer can be expressed as:

$$d\tau = -\sqrt{r-1}\sqrt{\frac{r}{r-1}}dr = -\sqrt{\frac{1}{V^2}-1}\sqrt{\frac{r}{r-1}}dr \tag{9}$$

Equation 9 shows the relationship between the time dilation caused by the inertial observer's velocity and the curvature of the r coordinate, just as Equation 8 did the same with the t coordinate. In this case, we see that the r coordinate becomes increasingly stretched near $r = 1$ as opposed to compressed as is the case with the t and R coordinates. Comparing Equation 9 to Equation 7, we see that the shape of the freefall worldline on a τ vs. r plot should be the same as the one shown in Figure 7, the only difference in the plots being that the r coordinate will be stretched (as in Figure 9) instead of the R coordinate being compressed. It is notable that for the freefalling observer, $\frac{dr}{d\tau} = -V$. Therefore, when the freefalling observer is at some radius r, the rest observers will be moving relative to her with velocity $\frac{dr}{d\tau} = -V$ such that as she approaches $r = 1$, the rest observers will appear to be moving closer and closer to the speed of light relative to her.

Next, imagine an observer at rest at the center of a collapsing spherically symmetric shell. According to Birkhoff's theorem, the space inside the shell, where the central observer is, will be flat. Therefore, according to the clock of an observer at infinity, light within the shell will travel just like it does at infinity. Therefore, as the collapsing shell approaches its Schwarzschild radius (say 1 light-second), the observer at infinity will find that according to her clock, it will take just over 1 second for a signal to travel from the central observer to an observer on the shell. But the clocks of both the central observer and shell observer will slow to a near stop relative to the observer at infinity. Thus in the frames of the central and shell observers, signals exchanged between them will be received almost instantly as the shell approaches its Schwarzschild radius. Thus, in their frame, it will appear as though the space between $r = 0$ and $r = 1$ contracts to zero proper distance as the shell reaches its Schwarzschild radius. In other words, in the collapsing frame, $r = 1$ will correspond to the center of gravity (there will be nowhere else to fall after that in the freefall frame). It is also notable that the clock of the central observer ticks at the same rate as an observer at rest at the location of the shell. Therefore, if the shell were actually able to reach the horizon, the central observer's clock would stop ticking and signals from it would be infinitely blueshifted when received by the collapsing shell (we can also see this from Equation 8, where the time dilation between the freefalling and central clocks is governed by the $\sqrt{1 - V^2}$ factor which goes to zero at $r = 1$). This is yet another example as to why it is nonsensical for the shell to be able to reach $r = 1$ in a finite time.

Figure 10 shows the freefall worldline depicted with all relevant quantities such that all the Schwarzschild differential relationships are captured:

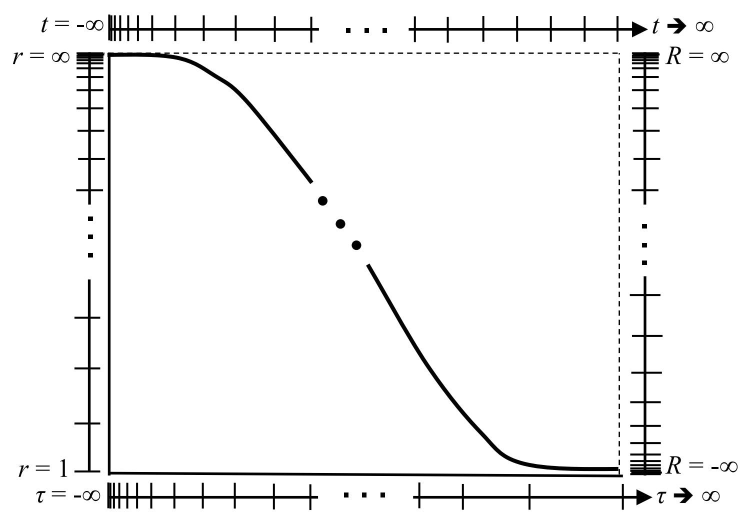

Figure 10 – Freefalling Geodesic Plotted against Multiple Coordinates

Conclusion

It has been shown that when accounting for curved spacetime while integrating the freefall geodesic, the freefaller experiences an infinite amount of proper time before reaching the horizon. We also know that the freefalling worldline approaches a null geodesic asymptotically, as can be deduced from Figure 4. This means that there will be a final light signal receivable by the freefaller from rest observers. Therefore, we must conclude that in the frame of the freefalling observer near the horizon, when she looks out to signals coming from the rest observers, those observers will appear to her to be slowing down since she experiences infinite proper time in her frame while receiving a finite number of light signals from the rest observers. What we find is that the rest observers will see the freefalling observer slow *exponentially* as their times go to infinity, while the freefaller will see the rest observers slow *asymptotically* as her time goes to infinity. This means that in the rest observer frame, the freefaller will have an open future, unfolding at an exponentially slower rate over time, while in the freefalling frame the rest observers will have a closed future, where the rest observers will appear to evolve toward a finite future time at an asymptotically slower rate. These features are shown in Figure 11 below:

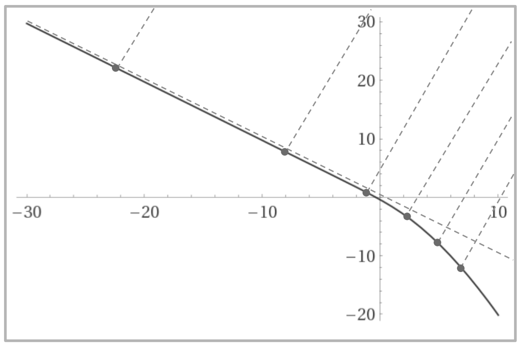

Figure 11 – Light Signals on *t*-*R* Chart

Figure 11 is a *t*-*R* chart that shows a single infalling signal representing the signal to which the freefall worldline is asymptotic. The freefalling observer will receive this signal after an infinite proper time and will receive no signals lying above that one on the chart. If at any time the freefaller accelerates in a direction away from the black hole, he will receive more future signals from the rest observers beyond this asymptotic signal since his worldline will curve upwards on Figure 11 as a result of his acceleration. Then if he stops accelerating and begins freefall again, there will be a new light signal to which his worldline will be asymptotic. The dots in Figure 11 represent intervals of equal proper time along the worldline and we can see that since the worldline is infinite (with tangents always below the speed of light) on this chart, there will be an infinite number of dots on the line spaced increasingly far apart and rest observers will receive an infinite number of signals from the freefalling observer at longer and longer intervals.

Note that in this case we showed that given the curvature of coordinates near the event horizon, a finite-looking integral actually has an infinite result. It should also be true that in different applications, where the coordinate density decreases rather than increases as it does in the present paper, an integral that, when integrated in the traditional manner, gives an infinite answer may in fact give a finite value when the methods demonstrated here are applied.

References

[1] Raine, D., Thomas, E.: Black Holes: A Student Text. Imperial College Press, (2015).