Results on the Problem of Abraham-Minkowski and the Abraham Force

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Abstract. In this article the final results on the Abraham-Minkowski problem are presented. It is shown that the Minkowski momentum is related to the dielectric characteristics of the medium, and the Abraham momentum is related to the magnetic characteristics of the medium. The total electromagnetic momentum in the medium is equal to the half-sum of the Minkowski and Abraham momentums. Therefore, the Minkowski momentum and the Abraham momentum have equal importance in electrodynamics, and the discussion of which of these forms of electromagnetic momentum is better does not make sense. It is shown that only an asymmetric energy-momentum tensor describes the total electromagnetic momentum in the medium. The Abraham force in a dielectric and conducting medium is described. It is shown that the Abraham force does not transmit a mechanical impulse to the medium, that means it is a fictitious force that really does not exist and it is impossible to detect it experimentally.

Keywords: The Abraham–Minkowski problem, the Minkowski momentum, the Abraham momentum, the Abraham force.

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The contents

1. Introduction

The problem of the interaction of the electromagnetic field with medium has been discussed for dozens of years and is known as the Abraham-Minkowski problem (the Abraham-Minkowski controversy). It contains two questions. The main reason for the debate is the form of representation of the electromagnetic momentum in the medium. The second issue is the Abraham electromagnetic force and its physical existence. For the dielectric medium, two forms of the electromagnetic momentum are known: the Minkowski form and the Abraham form. Theoretical arguments in favor of this or that form of momentum are given in articles [1 - 42]. In order to determine which of these momentums is more appropriate descriptions of experiments, an analysis of their results, and methods for new experiments are given in articles [43 - 62].

The electromagnetic momentum is inextricably linked with the energy-momentum tensor (EMT) and is an integral part of it. The equations of conservation of momentum follow from the EMT. In connection with this, in addition to the Minkowski EMT, various forms of EMT were constructed to describe the electromagnetic momentum in the medium, of which the Abraham symmetrical EMT is best known [63]. There are also other EMTs, such as Herz-Heaviside EMT, Helmholtz-Abraham, Abragam-Brillouin-Pitaevsky, Polevoi-Rytov, Belinfante-Rosenfeld [64-71]. However, the methods of constructing of these tensors were phenomenological and mathematical correctness of their derivation is doubtful. In work [72] from electromagnetic field tensors and electromagnetic induction, the EMT of interaction of an electromagnetic field with a dielectric medium is mathematically rigorously obtained, from which follow the conservation equations of an electromagnetic momentum and energy. Another important issue in the Abraham-Minkowski controversy is the Abraham force, which is defined as the difference in the time derivatives of the Minkowski and Abraham momentum. In many of the listed works, the question of its physical existence is discussed. Experiments have been conducted on its detection, the classic of which is the Lahoz-Walker experiment [44, 45], and new experiments are suggested. The last review of experimental works is given in [40]. The solution to the problem of detecting and measuring the Abraham force was hampered by the lack of its correct equation In [62] this equation was obtained. Its analysis in [42] showed that the Abraham force in a dielectric medium with losses is a reactive vortex force and does not depend on losses in the medium. However, a complete analysis of the electromagnetic momentum in the medium and the Abraham force, taking into account the latest results, has not been done so far.

The purpose of this article is to consider the electromagnetic momentum and the Abraham force in a dielectric and conducting medium and summarize the facts on the Abraham-Minkowski controversy.

2. The electromagnetic momentum in a dielectric medium

The electromagnetic momentum density is a component of the EMT. We can write the canonical EMT in a general form [7]:

$$T_{\mu\nu} = \begin{bmatrix} W & i\frac{1}{c}\mathbf{S} \\ ic\cdot\mathbf{g} & t_{ik} \end{bmatrix} \qquad (\mu, \nu = 0, 1, 2, 3;\ \ i, k = 1, 2, 3) \qquad (1)$$

where: W - energy density

S - energy flux density (Poynting vector);

g - momentum density

$\boldsymbol{t}_{ik}$ - momentum flux density tensor (stress tensor)

For a dielectric medium, the components of the EMT obtained in [72] have the form:

$$W = \mathbf{E} \cdot \mathbf{D} \qquad \mathbf{S} = \mathbf{E} \times \mathbf{H}$$

$$\mathbf{g} = \mathbf{D} \times \mathbf{B} \qquad \boldsymbol{t}_{ik} = E_i D_k + B_i H_k - \delta_{ik}(\mathbf{B} \cdot \mathbf{H}) .$$

Here, **E** and **D** are the electric field strength and electric induction respectively; **H** and **B** are the magnetic field strength and magnetic induction respectively. The electromagnetic momentum density in the Minkowski form has the form $\mathbf{g}^M = \mathbf{D} \times \mathbf{B}$. The electromagnetic momentum density in the form of Abraham has the form $\mathbf{g}^A = \mathbf{S}/c^2 = \mathbf{E} \times \mathbf{H}/c^2$. The purpose of the discussion on the form of the electromagnetic momentum was to find out which one of them more correctly describes the electromagnetic momentum in the medium.

Let’s consider the differences between these two forms of electromagnetic momentum. Sommerfeld A. [74] has divided the electromagnetic values into power and quantitative. He attributed the electromagnetic field strength **E** and the induction of the magnetic field **B** to the power quantities. He related the electric induction **D** and the magnetic field strength **H** to quantitative values. The relationship between **E** and **D**, **B** and **H** is determined by the material equations. For a weak electromagnetic field in an isotropic non-ferromagnetic dielectric medium without dispersion, the material equations are usually taken in the form: $\mathbf{D} = \varepsilon \cdot \varepsilon_0 \cdot \mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu \cdot \mu_0$, where ε and μ, are the relative dielectric and magnetic permeability of the medium respectively. The electric induction **D** and the magnetic field strength **H**, respectively, depend on the electric and magnetic characteristics of the medium. Then the Minkowski electromagnetic momentum density $\mathbf{g}^M = \mathbf{D} \times \mathbf{B}$, which includes the electric induction **D**, describes a part of the electromagnetic momentum associated with the electric characteristics of the medium. The Abraham electromagnetic momentum density $\mathbf{g}^A = \mathbf{E} \times \mathbf{H}/c^2$, which includes the magnetic induction **H**, describes the part of the electromagnetic momentum associated with the magnetic characteristics of the medium. It follows that each of these forms describes only a part of the total electromagnetic momentum, and the discussion of which of these parts is more correct does not make sense, since they are both correct.

This conclusion leads us to another question: which of the forms of the EMT is correct? All known forms of EMT can be divided into asymmetric and symmetric. From asymmetric EMT follows two different conservation equations for the momentum density, since its divergences for each of the indices are different. A symmetric EMT is followed by a single conservation equation for the momentum density, since its divergences for each of the indices are the same.

Let us consider the conservation equations for the electromagnetic momentum density in the Minkowski form. We write them for a dielectric medium with losses by expanding the equations [42] $\partial^{\mu}T^{E}_{\mu\nu} = \partial^{\mu}T^{M}_{\mu\nu}$ and $\partial^{\nu}T^{E}_{\mu\nu} = \partial^{\nu}T^{M}_{\mu\nu}$, where $T^{E}_{\mu\nu}$ - is the electromagnetic energy-momentum tensor, $T^{E}_{\mu\nu}$ - is the mechanical energy-momentum tensor:

$$\partial_t \mathbf{g}^M - \partial_k t_{ik} = \partial_t \mathbf{p} + \nabla(\mathbf{p}\cdot\mathbf{V}) \tag{2}$$

$$\frac{1}{c^2}\partial_t \mathbf{S} - \partial_i t_{ik} = \partial_t \mathbf{p} + \nabla(\mathbf{p}\cdot\mathbf{V}) \text{ or } \partial_t \mathbf{g}^A - \partial_i t_{ik} = \partial_t \mathbf{p} + \nabla(\mathbf{p}\cdot\mathbf{V}) \tag{3}$$

where $\mathbf{p}$ - is the mechanical momentum density; $\mathbf{V}$ - is the velocity of the unit volume of the medium. From these equations follows the conclusion that an asymmetric EMT of Minkowski describes an electromagnetic momentum in the form of Minkowski Eq. (2) and in the form of Abraham Eq. (3). Thus, an asymmetric EMT describes both parts of an electromagnetic momentum in a medium.

Let us consider the conservation equations for the electromagnetic momentum density in the form of Abraham:

$$\partial_t \mathbf{g}^A - \partial_k t_{ik} = \partial_t \mathbf{p} + \nabla(\mathbf{p}\cdot\mathbf{V})$$

$$\frac{1}{c^2}\partial_t \mathbf{S} - \partial_i t_{ik} = \partial_t \mathbf{p} + \nabla(\mathbf{p}\cdot\mathbf{V}) \text{ or } \partial_t \mathbf{g}^A - \partial_i t_{ik} = \partial_t \mathbf{p} + \nabla(\mathbf{p}\cdot\mathbf{V})$$

It follows from these equations that a symmetric EMT of Abraham describes an electromagnetic momentum only in the form of an Abraham. That means that the description of the electromagnetic momentum is incomplete, since there is no part of the electromagnetic momentum associated with the dielectric characteristics of the medium. Thus, in the general case, the correct EMT is an asymmetric EMT, which completely describes the electromagnetic momentum in the medium.

Since the momentum conservation Eqs. (2) and (3) for a single volume of the medium are fulfilled simultaneously, then adding them constructively, we obtain the conservation equation for the total momentum density in this volume of the medium:

$$\partial_t(\mathbf{g}^M + \mathbf{g}^A)/2 - (\partial_k t_{ik} + \partial_i t_{ik})/2 = \partial_t \mathbf{p} + \nabla(\mathbf{p}\cdot\mathbf{V}) \tag{4}$$

It follows from this equation that the density of the total electromagnetic momentum in a dielectric medium is equal to the half-sum of the momentum densities of Minkowski and Abraham.

3. The electromagnetic Momentum in a conducting medium

Let us consider the electromagnetic momentum in a conducting medium. We shall consider the medium for a microfield as the density of free electric charges in a vacuum. The electromagnetic field will be described with the help of the electromagnetic potential $\mathbf{A}_{\mu}(\varphi/c, i\cdot\mathbf{A})$, where φ and $\mathbf{A}$ are the scalar and vector potentials of the electromagnetic field in Euclidean space, and the charges and currents will be described in the form of a four-dimensional current density vector $\mathbf{J}_\nu(c\cdot\rho, i\cdot\mathbf{J})$, where ρ and **J** are the charge density and the conduction current density vector.

The EMT of the interaction of an electromagnetic field with charges and currents $T^E_{\mu\nu}$ is obtained as the tensor product of the four-dimensional vector potential of the electromagnetic field $\mathbf{A}_\mu$ by the four-dimensional current density $\mathbf{J}_\nu$:

$$T^E_{\mu\nu} = \mathbf{A}_\mu \otimes \mathbf{J}_\nu = \begin{pmatrix} \rho\cdot\varphi & \frac{1}{c}i\cdot\varphi\cdot J_x & \frac{1}{c}i\cdot\varphi\cdot J_y & \frac{1}{c}i\cdot\varphi\cdot J_z \\ i\cdot c\cdot\rho\cdot A_x & -A_x\cdot J_x & -A_x\cdot J_y & -A_x\cdot J_z \\ i\cdot c\cdot\rho\cdot A_y & -A_y\cdot J_x & -A_y\cdot J_y & -A_y\cdot J_z \\ i\cdot c\cdot\rho\cdot A_z & -A_z\cdot J_x & -A_z\cdot J_y & -A_z\cdot J_z \end{pmatrix} = \begin{bmatrix} W & \frac{1}{c}i\cdot\mathbf{S} \\ i\cdot c\cdot\mathbf{p} & t_{ik} \end{bmatrix} \tag{5}$$

where $W = \rho\cdot\varphi$ is the density of the total electromagnetic energy, $\mathbf{g} = \rho\cdot\mathbf{A}$ - is the density of the electromagnetic momentum; $\mathbf{S} = \varphi\cdot\mathbf{J}$ - density of electromagnetic energy flow; $t_{ik} = -\mathbf{A}_i \otimes \mathbf{J}_k$ - three-dimensional tensor of the flux density of an electromagnetic momentum or stress tensor. The EMT (5) is asymmetric and corresponds to the canonical energy-momentum tensor (1). We will write the conservation equations in the form of the equality of four-dimensional divergences of the energy-momentum tensors $T^E_{\mu\nu}$ and $T^M_{\mu\nu}$:

$$\partial_\mu T^E_{\mu\nu} = \partial_\mu T^M_{\mu\nu} \quad \text{и} \quad \partial_\nu T^E_{\mu\nu} = \partial_\nu T^M_{\mu\nu}$$

Let us write down two conservation equations for the electromagnetic momentum density in the expanded form:

$$\partial_t(\rho\cdot\mathbf{A}) + \partial_k(\mathbf{A}_i \otimes \mathbf{J}_k) = \partial_t\mathbf{p} + \nabla(\mathbf{p}\cdot\mathbf{V}) \tag{6}$$

$$\frac{1}{c^2}\partial_t(\varphi\cdot\mathbf{J}) + \partial_i(\mathbf{A}_i \otimes \mathbf{J}_k) = \partial_t\mathbf{p} + \nabla(\mathbf{p}\cdot\mathbf{V}) \tag{7}$$

Eqs. (6) and (7) are analogous to Eqs. (2) and (3). They also have two types of electromagnetic momentum, $\mathbf{g} = \rho\cdot\mathbf{A}$ and $\mathbf{g} = \varphi\cdot\mathbf{J}/c^2$. Let us find the density of the total electromagnetic momentum in the form of a sum of Eqs. (6) and (7):

$$\partial_t(\rho\cdot\mathbf{A} + \frac{1}{c^2}\varphi\cdot\mathbf{J})/2 + (\partial_k(\mathbf{A}_i \otimes \mathbf{J}_k) + \partial_i(\mathbf{A}_i \otimes \mathbf{J}_k))/2 = \partial_t\mathbf{p} + \nabla(\mathbf{p}\cdot\mathbf{V}) \tag{8}$$

Here, the density of the total electromagnetic momentum, like in Eq. (4), is equal to the half-sum of the density of the two types of electromagnetic momentum.

4. The Abraham force in a dielectric and conducting medium

The equations for the conservation of the momentum density (2), (3), (6), and (7) can be regarded as equations for the balance of the electromagnetic and mechanical forces density in a medium.

Let us consider the Abraham force in a dielectric medium with losses. It is expressed as a difference in the time derivatives of the Minkowski and Abraham momentum. To obtain it, we find the difference between Eqs. (2) and (3):

$$\partial_t \mathbf{g}^M - \partial_t \mathbf{g}^A - \partial_k t_{ik} + \partial_i t_{ik} = 0 \text{ или } \mathbf{F}_A = \partial_t \mathbf{g}^M - \partial_t \mathbf{g}^A = \partial_k t_{ik} - \partial_i t_{ik} \quad (9)$$

Expanding the right-hand side of the equation, we will obtain

$$\mathbf{F}_A = \partial_t (\mathbf{D}\times\mathbf{B}) - \partial_t (\mathbf{E}\times\mathbf{H})/c^2 = \nabla\times(\mathbf{E}\times\mathbf{D}+\mathbf{B}\times\mathbf{H}) \quad (10)$$

From Eqs. (2), (3) and (4) it follows that the time derivatives of individual parts of the electromagnetic momentum density and the density of the total electromagnetic momentum change the density of the mechanical momentum of the medium (in the presence of losses in the medium). From Eqs. (9) and (10) it follows that the difference in the time derivatives of individual parts of the electromagnetic momentum density does not change the density of its mechanical momentum. Thus, the Abraham force described by Eq. (9) does not have a mechanical effect on the medium, i.e. it is a fictitious force and it is impossible to detect it experimentally.

Now let us consider the Abraham force for a conducting medium. For this case, it is also expressed as the difference in the time derivatives of the two forms of electromagnetic momentums $\mathbf{g} = \rho \cdot \mathbf{A}$ and $\mathbf{g} = \varphi \cdot \mathbf{J} / c^2$. To get it, we will find the difference between the Eqs. (6) and (7):

$$\partial_t (\rho \cdot \mathbf{A}) - \frac{1}{c^2} \partial_t (\varphi \cdot \mathbf{J}) + \partial_k (\mathbf{A}_i \otimes \mathbf{J}_k) - \partial_i (\mathbf{A}_i \otimes \mathbf{J}_k) = 0 \quad (11)$$

$$\mathbf{F}_A = \partial_t (\rho \cdot \mathbf{A}) - \partial_t (\varphi \cdot \mathbf{J})/c^2 = \partial_i (\mathbf{A}_i \otimes \mathbf{J}_k) - \partial_k (\mathbf{A}_i \otimes \mathbf{J}_k) = \nabla\times(\mathbf{A}\times\mathbf{J})$$

From Eqs. (6), (7) and (8) it follows that the time derivatives of individual parts of the electromagnetic momentum density and the density of the total electromagnetic momentum change the density of the mechanical momentum of the medium. It follows from Eq. (11) that the difference in the time derivatives of individual parts of the electromagnetic momentum density does not change the density of its mechanical momentum. Thus, even for a conducting medium, the Abraham force described by Eq. (11) does not have a mechanical effect on it, i.e. it is a fictitious force and it is impossible to detect it experimentally.

Let us consider the cause of the fictitiousness of the Abraham force using the example of a conducting medium (the result for a dielectric medium is the same). To do this, we will find the divergences of the three-dimensional stress tensors in Eqs. (6) and (7) and will write them in expanded forms:

$$\partial_m (\mathbf{A}_m \otimes \mathbf{J}_n) = \mathbf{J}(\nabla\cdot\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{J} \quad \text{и} \quad \partial_n (\mathbf{A}_m \otimes \mathbf{J}_n) = \mathbf{A}(\nabla\cdot\mathbf{J}) + (\mathbf{J}\cdot\nabla)\mathbf{A} \quad (12)$$

Let us apply the known vector identity to the last terms of these expressions:

$$(\mathbf{a}\cdot\nabla)\mathbf{b} = \frac{1}{2}[\nabla(\mathbf{a}\cdot\mathbf{b}) - \nabla\times(\mathbf{a}\times\mathbf{b}) - \mathbf{b}\times\nabla\times\mathbf{a} - \mathbf{a}\times\nabla\times\mathbf{b} - \mathbf{b}\cdot(\nabla\cdot\mathbf{a}) + \mathbf{a}\cdot(\nabla\cdot\mathbf{b})$$

We will write them in expanded form and obtain the expressions:

$$\partial_m(\mathbf{A}_m \otimes \mathbf{J}_n) = \frac{1}{2}[\nabla(\mathbf{A}\cdot\mathbf{J}) - \nabla\times(\mathbf{A}\times\mathbf{J}) - \mathbf{J}\times\nabla\times\mathbf{A} - \mathbf{A}\times\nabla\times\mathbf{J} + \mathbf{J}\cdot(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla\cdot\mathbf{J})] \quad (13)$$

$$\partial_n(\mathbf{A}_m \otimes \mathbf{J}_n) = \frac{1}{2}[\nabla(\mathbf{A}\cdot\mathbf{J}) + \nabla\times(\mathbf{A}\times\mathbf{J}) - \mathbf{A}\times\nabla\times\mathbf{J} - \mathbf{J}\times\nabla\times\mathbf{A} + \mathbf{J}\cdot(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla\cdot\mathbf{J})] \quad (14)$$

Each of these expressions contains the Abraham force $\nabla\times(\mathbf{A}\times\mathbf{J})$, but with opposite signs. Therefore, in the equation for the density of the total electromagnetic momentum (8), obtained by adding the Eqs. (6) and (7), the Abraham force is reduced:

$$\partial_t(\rho\cdot\mathbf{A} + \frac{1}{c^2}\varphi\cdot\mathbf{J})/2 + \nabla(\mathbf{A}\cdot\mathbf{J}) - \mathbf{J}\times\nabla\times\mathbf{A} - \mathbf{A}\times\nabla\times\mathbf{J} + \mathbf{J}\cdot(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla\cdot\mathbf{J}) = \partial_t\mathbf{p} + \nabla(\mathbf{p}\cdot\mathbf{V})$$

Thus, the Abraham force does not really exist in nature and attempts at its experimental detection can be stopped.

5 Conclusion

According to the results of recent works on the Abraham-Minkowski controversy, it can be concluded that the Minkowski momentum and the Abraham momentum are integral parts of the total electromagnetic momentum in the medium. The Minkowski momentum is related to the dielectric characteristics of the medium, and the Abraham momentum is related to the magnetic characteristics of the medium. The total electromagnetic momentum in a dielectric medium is equal to the half-sum of the Minkowski and Abraham momentums. Each of them describes its part of the total electromagnetic momentum. In this regard, there is no point to continue the discussion on which of them describes the electromagnetic momentum in the medium more correctly.

The conservation equation of the total electromagnetic momentum follows only from an asymmetric EMT.

The Abraham force does not change the mechanical momentum of the medium, so it is a fictitious force and cannot be detected experimentally. Thus, the Abraham-Minkowski problem is solved.

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