Solution of Poincare's vector-field problem

When a meromorphic vector field is given on the projective plane, a complete holomorphic limit cycle, because it is a closed singular submanifold of projective space, is defined by algebraic equations. Also the meromorphic vector field is an algebraic object. Poincare had asked, is there just an algebraic calculation leading from the vector field to the defining equations of the solution, without the mysterious intermediary of the dynamical system.

The answer is yes, that there is nothing more mysterious or wonderful that happens when a complete holomorphic limit cycle is formed than could have been defined using algebra.

Poincare's problem may be a precursor of the recent efforts to create a type of mathematical production line. In that case what he asked for would have been just an algorithm starting with the vector field to produce defining equations of the curve. Therefore the problem is not solved unless one can bound the discrepancy at each singular point independently of the choice of local analytic germ.

There will be no draft on arXiv due to the history of the arXiv server at Los Alamos.

Rational relations between eigenvalues

At any singluar point p of the foliation, a generating vector field in a neighourhood of that point induces an endomorphism of the vector space m_p/m_p^2, in this case a two-dimensional complex vector space. The set of roots of the characteristic polynomial is an unordered pair (λ_1, λ_2).

If $\lambda_1 \neq \lambda_2$, we can write the system of differential equations in local coordinates at p in the form

$$\frac{dx}{dt} = \lambda_1 x + f(x, y)$$

$$\frac{dy}{dt} = \lambda_2 y + g(x, y)$$

where $f(x, y)$ and $g(x, y)$ are convergent power series with no term of degree zero or one. If we were to look at just one term of $f(x, y)$ of lowest degree r say $ex^i y^j$ with $i + j = r$, where e is just a complex number, then we can consider for a different complex number c the function

$$x + cx^i y^j,$$

and we can try to choose c so that the coefficient of $x^i y^j$ in

$$\frac{d}{dt}(x + cx^i y^j) - \lambda_1(x + cx^i y^j)$$

is zero. The coefficient is

$$e + c[(i\lambda_1 + j\lambda_2) - \lambda_1]$$

If

$$\lambda_1 \neq i\lambda_1 + j\lambda_2$$

then we can set

$$c = -\frac{e}{(i-1)\lambda_1 + j\lambda_2}$$

and in this way change coordinates to remove one of the terms of the power series f.

Textbooks use the term *resonance* to refer to a situation where there exists any natural numbers i, j with $\lambda_1 = i\lambda_1 + j\lambda_2$ or $\lambda_2 = i\lambda_1 + j\lambda_2$. If resonance does not occur, then it is possible in this way to change coordinates formally such that the system of differential equations corresponds to a linear system; moroever this change of coordinates is in fact analytic [2],[3].

A second way that rational relations between λ_1 and λ_2 arise is if we consider the effect of a point blowup on the linear system of differential equations above, with $f(x, y) = g(x, y) = 0$. There results from each such singular point, a pair of singular points, with unordered eigenvalue pairs $(\lambda_1, \lambda_2 - \lambda_1)$ and $(\lambda_1 - \lambda_2, \lambda_2)$. If the λ_i are rationally related we can take them both to be integers (by replacing a local generating vector field of the foliation by an integer multiple), and to be relatively prime if we like. If both are positive and unequal, then of the new pairs one has entries of opposite sign, and the other has entries both positive but with a smaller maximum value.

The case of eigenvalues of opposite sign (a saddle) has only finitely many local analytic germs, and if we by this process chase an infinite set of analytic germs, we may ignore the new singular point created at each step with eigenvalues of opposite sign, and the one which we do not ignore has the eigenvalue pair affected by precisely the Euclid algorithm for natural numbers, until they become equal. At this point is where one sees the infinity of solutions of the system of differential equations $\frac{dx}{dt} = x; \frac{dy}{dt} = y$, and at the next step the singular point disappears, all the leaves through one point resolved into leaves which cross disjoint points of an exceptional projective line.

Finally, a third way that rational relations between the λ_i occur is if we consider what we mean by an analytic germ of a solution. In *Comments about Hilbert's 16'th problem* I mentioned that it suffices to solve the Poincare problem if we could find a bound over algebraic solution germs. Now it turns out that a better class of solution germs to restrict to, which is strictly larger than the algebraic ones, are the analytic solution germs which are non-Zariski dense. If we consider the system of differential equations above with $f(x, y) = g(x, y) = 0$ and λ_1, λ_2 non-rationally related, then the general solution depending on choice of initial conditions is $x = Ae^{\lambda_1 t}, y = Be^{\lambda_2 t}$ for A, B constants, and yet this solution is analytically Zariski dense except for a finite number of values of the ratio $[A : B]$.

The Bendixson, Seidenbert, Dumortier, Camacho, Sad theory

Next let's explain how one encounters a part of the Bendixson, Seidenberg, Dumortier, Camacho, Cano, Sad theory. This had been a difficulty which was difficult to overcome. If one has a singular foliation of the plane, and a smooth plane curve C, one might wonder, is it possible by blowing up iteratively singular points of the foliation which are on the proper transform, to arrange that the proper transform has a smoothly foliated neighbourhood? Here is why the answer is no.

Start with a smooth curve which by analytic choice of coordinates we can assume is the curve y=0 in the (x, y) plane.

Say that δ is a derivation on the plane and $\delta(x)$, $\delta(y)$ are relatively prime and in the maximal ideal of the analytic local ring at the origin.

The total transform of the x axis in the coordinate charts $(x/y, y)$ and $(x, y/x)$, in the first one it is just the exceptional divisor and nothing else.

That is, the exceptional divisor in $(x/y, y)$ coordinates is $(x, y) = (y(\frac{x}{y}), y) = (y)$ and agrees with the total transform so the proper transform is empty here, it does not intersect this chart.

So one only looks at $(x, y/x)$ and looks at what happens to the foliation.

The origin here is the image of the origin in the proper transform of the x axis, and so for the purposes of considering a neighbourhood of the proper transform one only is interested in again the maximal ideal at the origin.

If you write originally $\delta(x) = P, \delta(y) = Q$, then here you get

$$\delta(x) = P$$

$$\delta(\frac{y}{x}) = (xQ - yP)/x^2$$

and the fractional ideal class which controls singularity of the foliation at the point with coordinates $(x, y/x)$ is

$$(xQ - yP, x^2P) = (xQ - x(\frac{y}{x})P, x^2P)$$
$$= (Q - (\frac{y}{x})P, xP) \qquad (1)$$

Choose d as small as possible so P, Q were in $(x, y)^d$.

Each monomial $x^a y^b$ of lowest total degree so $a + b = d$ gets written

$$x^d(\frac{y}{x})^b$$

just as Hironaka often mentioned in his paper.

This means that Q and P here are now, viewed as

$$Q(x, (\frac{y}{x}).x) \text{ and} P(x, (\frac{y}{x}).x)$$

both divisible by x^d, and we can divide through.

Now assume that d does not decrease upon this blowing up.

Since the total degree of P has gone down by d, in the ideal displayed in equation (1), it must be that the only degree d monomials in P had been xy^{d-1} and y^d and none others.

For example if P included the monomial x^2y^{d-2} then the second entry xP of the fractional ideal above would be

$$xP(x, x.(\frac{y}{x}))$$

which would contain the monomial

$$x.x^2.x^{d-2}.(\frac{y}{x})^{d-2}$$

and when we divided by x^d we would have

$$x(\frac{y}{x})^{d-2}$$

which only has degree $d - 1$ in coordinates $(x, \frac{y}{x})$.

So that we have that the original P is of the form

$$axy^{d-1} + by^d + \text{ terms of total degree larger than } d$$

for some complex numbers a, b.

Likewise the first entry of (1)

$$Q - (\frac{y}{x})P$$

if it is to belong to m^d for m the maximal ideal of the point of the blowup, we have to think that in each term, rewriting a monomal $x^a y^b$ of total degree d

$$x^{a+b}(\frac{y}{x})^b = x^d(\frac{y}{x})^b$$

and then dividing by x^d, had the effect of replacing x by 1 and replacing y by $\frac{y}{x}$.

We already saw that the lowest degree monomials in P are xy^{d-1} and y^d. These become $(\frac{y}{x})^{d-1}$ and $(\frac{y}{x})^d$ and when we multiply by $(\frac{y}{x})$ here we get the lowest possible total degree monomal being $(\frac{y}{x})^d$ itself.

This means that the only way the total degree can stay equal to d is if Q itslelf has only one monomial of degree d, that is y^d and nothing else.

So now we can go back to the beginning and put in as extra hypotheses that

$$P(x, y) = ax^{d-1}y + by^d + \text{ higher order}$$

$$Q(x, y) = cy^d + \text{ higher order}$$

Now, of these higher order parts, at least one of them must contain a monomial that is merely a power of x.

Otherwise P and Q would have a common factor and we could have divided out without affecting the foliation.

Now the smallest monomial in the highest order terms that is just a power of x, when we divide by x^d and repeat this process, that degree goes down.

This will eventually be smaller than d and we see that d is decreased *unless* $d = 1$. Then it is possible that x itself occurs as one of the monomials of P, and the process is not guaranteed to decrease d any further.

In fact we see examples of this when we just take

$$\delta(y) = ay, \;\; \delta(x) = bx$$

for a, b constants. We have

$$\delta(\frac{y}{x}) = (b-a)(\frac{y}{x})$$

(the number b-a is a difference of two logarithmic derivatives).

The value of d has not gone down since the minimum degree of $bx, (b-a)(\frac{y}{x})$ in coordinates $(x, \frac{y}{x})$ is still equal to one.

The case of nonzero linear part

Once the subdegree of P, Q is equal to one, if the point p coordinatized by $x = y = 0$ is a fixed point of the vector field, then when we return to our equations

$$\delta(x) = P$$

$$\delta(\frac{y}{x}) = (xQ - yP)/x^2$$

we see that the point on the exceptional divisor where $(x, \frac{y}{x}) = (0, 0)$ is a fixed point if and only if the degree two part of $xQ - yP$ is a polynomial multiple of xy. (We can change the choice of point on the exceptional divisor by replacing y by a linear form $ax + y$). The principal multple of (1) which is relevant is its multiple by x^{-1}, and this is the multiple which occurs if we merely use the lifted vector field as a generating vector field of the foliation.

That is to say, the lifted vector field is not zero on the whole exceptional projective line, and it is not necessary to make a change in the generating vector field of the foliation.

It follows then that once the action fixes a point p and acts by a nonzero linear transformation on m_p/m_p^2, then once the vector field is lifted to the blowup of p, and a fixed point q on the exceptional divisor is chosen which is fixed by the lifted vector field, the linear transformation of m_q/m_q^2 that results depends only on the original linear transformation of m_p/m_p^2. It can be calculated by replacing the vector field with a linear vector field.

Note on the coordinate-free approach

In 'on resolving vector fields' I mentioned (and that this would be true also on a singular variety, and for foliations of any dimension) that we can describe the singularities of the lifted foliation using first principal parts. In this case, starting with p the origin in the plane, we consider two coherent sheaves, both invertible after blowing up $m = m_p$ and both generated by global sections; these are (x, y) and the pullback modulo torsion of the second exterior power of the first principal parts of m which is

$$(x,y)^2(\delta(x),\delta(y)) + (x\delta(y) - y\delta(x))$$
$$= (x^2\delta(x), x^2\delta(y), xy\delta(x), xy\delta(y), y^2\delta(x), y^2\delta(y), x\delta(y) - y\delta(x)).$$

The difference (ratio)

$$(x,y)^{-e}(x^2\delta(x), x^2\delta(y), xy\delta(x), xy\delta(y), y^2\delta(x), y^2\delta(y), x\delta(y) - y\delta(x))$$

with e taken as large as possible so that the sheaf of fractional ideals is an actual ideal sheaf (contained in the structure sheaf) is the defining ideal sheaf of the singular subscheme of the lifted foliation. The difference (ratio) is not in general generated by global sections. This all remains true in the analytic setting.

In the main case under consideration here, where $(\delta(x), \delta(y)) = (x, y)$, we have $e = 2$ and the ideal sheaf is

$$(x, y) + (x, y)^{-2}(x\delta(y) - y\delta(x)).$$

When $x\delta(y) - y\delta(x)$ describes a pair of arcs meeting at the origin before blowing up, the ideal sheaf above describes the two points where the exceptional divisor (defined by the ideal sheaf (x, y)) meets with the union of the proper transform of these arcs.

The Euclid algorithm

Now that we've seen the three ways rational ratios of eigenvalues arise in the subject, and the reduction to the linear case, let's look back at the Euclid algorithm. A previous section shows that when the linear transformation of m_p/m_p^2 is not identically zero, the vector field acts in a neighbourhood of the exceptional divisor associated to p by a flow with isolated fixed points and no pole. This means that the lifted vector field remains a generating vector field of the foliation. That is, the action on m_q/m_q^2 for q a fixed point on the exceptional divisor mapping to p depends only on the action on m_p/m_p^2, and is independent of the terms in any power series expansion of the vector field coefficient besides the linear terms. The action on m_q/m_q^2 is the same as if power series representations of P, Q had been replaced by their linear parts before blowing up. Another way of saying this is that since the coefficients of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ cannot develop a common divisor in passing from coordinates such as (x, y) to coordinates such as $(x, y/x)$, the effect on nonlinear monomials in the coefficients of a generating vector field of the foliation a substitution to increase the degree, and nonlinear monomials never become linear. If the eigenvalues are rationally related and the generating vector field is chosen to make them integers, the ideal in the ring of integers generated by the eigenvalues is unaffected by passage from p to q.

Once we arrive at λ_1, λ_2 not both zero for every singular point, by point blow-ups, if $[\lambda_1 : \lambda_2]$ is not rational, it is not resonant and we can linearize; of the infinity of analytic solution germs only finitely many are non-Zariski-dense.

In the case when the Jordan form of the action on m_p/m_p^2 is not the semisimple type, there can only be finitely many solutions.

The remaining three cases are those described in Ilyashenko and Yakovenko's book [2] as resonant saddle, resonant node, and resonant saddle-node. For the resonant saddle, it is still true, as for any saddle, that it does not have infinitely many analytic solution germs.

Although we cannot always linearize the remaining two cases, Seidenberg's argument [1] in both cases amounts to observing, as we've observed above the independence of the linear terms from the nonlinear terms. That is, the action on m_q/m_q^2 for q a fixed point on the exceptional divisor mapping to p depends only on the action on m_p/m_p^2, and is independent of the terms in any power series expansion of the vector field coefficient besides the linear terms.

In both the remaining two cases of λ_1, λ_2 both positive integers (which includes the resonant node case), and the case of $(1, 0)$, the saddle-node case the eigenvalue pair is affected by Euclid's algorithm just the same as happens as for the linear case; point blowups reduce to a single case with infinitely many analytic germs, the case of $(1, 1)$.

Note that one further point blow-up desingularizes the vector field and each leaf through the origin becomes a leaf meeting the exceptional divisor at one point.

Preliminary idea of the proof

Let me first give a preliminary idea of the proof, which is not quite what we will use. It is to consider a pretend case when we can use a transverse arc to 'capture' analytic arcs.

What I would mean by 'capturing' arcs is to construct using a an analytic arc B (isomorphic to $\mathbb{C}$) an analytic parametrization of all but finitely many of the analytic solution germs in a neighbourhood U of a singular point. Assuming that each leaf in U meets B at no more than one point, one can consider the hypersurface

$$H \subset B \times U$$

which is the set of pairs consisting of a point b of the arc B together with a point $x \in U$ which belongs to the leaf in U which meets B.

One can embed H in the algebraic variety which results by blowing up a suitable ideal sheaf supported on the singular point of the foliation. It is a singular, in general non-normal surface, but one in which the lifted leaves are all disjoint, and the singular point has been blown up to a compactification of the parameter space which is an irreducible component of the one-dimensional singular locus of the non-normal surface.

Instead of performing an equivariant divisorial simplification of each analytic solution germ separately one can begin with by blowing up $B \times U$ along the singular locus of the non-normal surface H, that is, the closure of the parameter space B, and then perform a divisorial simplification by blowing up curves where proper transform of H is non-normal-crossing to arrive at a proper map $\widetilde{B \times U} \to B \times U$ such that the inverse image of H is a divisor with a discrete and therefore finite non-normal-crossing locus. For each curve which is blown up, letting $\pi : C \to B$ be the map to the parameter space, the support on B of the cokernel of $\pi^*\Omega_B \to \Omega_C$ is also a finite set. By deleting from B both finite sets, the remaining points of B parametrize arcs for which the discrepancy coefficient of each exceptional component of $\widetilde{B \times U} \to B$ agrees with the corresponding discrepancy coefficient of the induced divisorial simplification of the arc itself. Therefore by calculating these, and also directly the ones for the finitely many omitted leaves, one finds the upper bound for the number m_i in *Comments about Hilbert's 16'th problem* for the i'th singular point, and if d is the degree of poles minus zeroes of a chosen vector field which induces the foliation, then all complete holomorphic limit cycles of degree larger than

$$\frac{1}{2}\sqrt{9+4\sum_{i=1}^{s}(m_i-1)(m_i-2)}+d+\frac{3}{2}$$

are found by rational integration.

We see that if this is to work, the compactification of B can only be a Riemann sphere. For, the non-normal hypersurface H admits a locally projective map to the smooth surface U, and therefore must be dominated by a result of a sequence of point blowups starting from U. Then the exceptional locus of $H \to U$ is dominated by a disjoint union of projective lines and must itself be a projective line by Luroth's theorem.

The actual proof.

What we will do is almost the same as the idea above, but we actually will arrange that H is normal, and the transverse arc which we'll use is not actually going to be an arc in $\mathbb{P}^2$, but rather in a partial resolution. We can assume that we begin with either a resonant node or a resonant saddle-node. Then the partial resolution of Seidenberg's paper in either case (either theorem 8b (bis) or 10b (bis)) gives an exceptional divisor including one copy of $\mathbb{P}^1$ which meets all but finitely many of the arc germs which originally pass through the singular point, now consisting of arcs.

One of the arcs meeting this $\mathbb{P}^1$ may be exceptional, and not actually correspond to an arc in the projective plane. If we delete the intersection point, the remaining portion of the Riemann sphere is a copy of $\mathbb{C}$ in the partial resolution. The whole of this copy of $\mathbb{C}$ maps to the one singular point we are considering in $\mathbb{P}^2$, all the arcs meeting this copy of $\mathbb{C}$ map to embedded arcs in $\mathbb{P}^2$ and correspond to all but finitely many of the solution germs through the singular point.

We choose a small neighbourhood U of our singular point, and define H to be the variety consisting of a point of U together with a point of $\mathbb{P}^1$. This is taken to be the intersection point with $\mathbb{C} \subset \mathbb{P}^1$ if the lifted leaf meets a point of $\mathbb{C}$ and otherwise the point at infinity. Thus H is a hypersurface in $\mathbb{P}^1 \times U$ and we perform the divisorial simplification as we described earlier, blowing up first within $\mathbb{P}^1 \times U$ the reduced structure of $\mathbb{P}^1 \times p$ for p our singuar point, and then successively blowing up any any choice of curve along which the total transform of H in $\mathbb{P}^1 \times U$ is generically non-normal-crossing, until no such curves remain.

The infinity of solution germs described by the divisorial simplification share a finite number of discrepancy coefficients (those which arise during the divisorial simplification), and the finite number of omitted non-Zariski dense analytic solution germs give rise to a further finite set of discrepancy coefficients. All entire holomorphic limit cycles whose degree then exceeds the bound above can be found by rational integration and according to Poincare may be ignored. The remainder belong to an algebraic family of curves of known bounded degree and the condition of which are solutions of the system of differential equations is again also then algebraic.

References

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