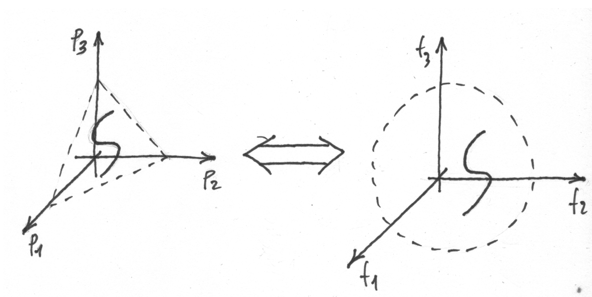
I search to obtain a dynamics for a probabilistic system: a system with a finite number of variables $P_i \geq 0$ such that $\sum_i P_i = 1$.

The dynamics of the probabilistic system is on a a face of a octahedron, but it is complex to require a dynamics on one face (the general solutions $\frac{dP_i}{dt} = F(P_1, \cdots, P_n)$ tend to cover the whole octahedron faces).

I simplify the problem using the probability amplitude f_i such that $P_i = f_i^2$, so that the probabilities are defined as positive.



The dynamics of the system is:

$$\frac{df_i}{dt} = a_i + \sum_i a_{ij} f_j + \sum_i a_{ijk} f_j f_k + \cdots$$

so that it is simple to obtain the normalization:

$$0 = \frac{1}{2}\frac{d}{dt}\sum_i P_i = \frac{1}{2}\frac{d}{dt}\sum_i f_i^2 = \sum_i f_i \frac{df_i}{dt} = \sum_i a_i f_i + \sum_{ij} a_{ij} f_i f_j + \sum_{ijk} a_{ijk} f_i f_j f_k + \cdots$$

for each arbitrary values of the amplitudes this polynomial must be zero (even for points near the octahedron surfaces), so that

$$\begin{aligned}
&a_i = 0 \\
&a_{ij} + a_{ji} = 0 \\
&a_{iij} + a_{iji} + a_{jii} = 0 \\
&a_{ijk} + a_{ikj} + a_{jik} + a_{kij} + a_{jki} + a_{kji} = 0
\end{aligned}$$

so that $\sum_P a_{P(i,j,k,\cdots)} = 0$, so that the sum of the coefficients with the permutation of the indices is null.

The amplitudes dynamics is on a sphere, and if the initial amplitude is on a unitary sphere, then the probability dynamics is normalized to one.

It is possible to use more complex dynamics, for example $P_i = f_i^{2n}$, or $P_i = f_i f_i^*$ (using a quantum mechanics analogy).

The quantum analogy contain the Schrödinger equation (I use the Einstein notation, so that summation is applied when a index appear twice in a single term):

$$\frac{df_i}{dt} = a^i + a^i_j f_j + a^{ij} f^*_j + a^i_{jk} f_j f_k + a^{ij}_k f^*_j f_k + a^{ijk} f^*_j f^*_k + \cdots$$

and

$$\frac{df^*_i}{dt} = (a^i)^* + (a^i_j)^* f^*_j + (a^{ij})^* f_j + (a^i_{jk})^* f^*_j f^*_k + (a^{ij}_k)^* f_j f^*_k + (a^{ijk})^* f_j f_k + \cdots$$

the normalization condition is:

$$0 = \frac{d}{dt} \sum_i P_i = f_i \frac{df^*_i}{dt} + \frac{df_i}{dt} f^*_i = a^i f^*_i + (a^i)^* f_i + a^{ij} f^*_i f^*_j + (a^{ij})^* f_i f_j + \cdots$$

the upper, and the lower, indices behaves as the ususal dynamics:

$$\begin{array}{l} a_i = 0 \\ a^{ij} + a^{ji} = 0 \\ a^i_j + (a^j_i)^* = 0 \\ a^i_{jk} + (a^{kj}_i)^* = 0 \\ a^{ijk} + a^{ikj} + a^{jik} + a^{kij} + a^{jki} + a^{kji} = 0 \end{array}$$

the lower not zero terms seem the elements of the Hamiltonian matrix, when one use the equation $i\frac{df}{dt} = Hf$ instead of $\frac{df}{dt} = Hf$. This is the general differential equation for a quantum system that is normalizable (the f_i can be the f_x amplitude, where x is a discrete coodinate in a three-dimensional space, and a_{ij} can be an interaction betweeen two coordinate points).