DERIVATION OF A TRANSPORT FLUX
FROM THE FOKKER-PLANCK EQUATION

By

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The derivation result used will be that of C. Kittel[1] (and his source), in addition to other sources[2]. The derivation derives from the Smoluchowski equation using a conditional probability P(z|y, t) that a particle at z at t = 0 will be at y during the time interval Δt. The result derived is the Fokker-Planck equation without sources:

$$\frac{\partial P}{\partial t}+\frac{\partial\left[A(y)P\right]}{\partial y}-\frac{1}{2}\frac{\partial^2\left[B(y)P\right]}{\partial y^2}=0$$

If A(y) = 0 , then there are equal probabilities of moving either left or right. If B(y) is independent of position or in other words there is an isotropic environment, then this reduces to the usual diffusion equation. Now let's construct the covariant formulation[3] utilizing the symbolism that $_{/\mu}$ is a covariant partial derivative with respect to coordinate μ and $^{/\mu}$ represents the contravariant partial derivative with respect to coordinate μ.

$$\left(A^{\mu}P\right)_{/\mu}=\nabla\bullet\left(\mathbf{A}P\right)+\frac{\partial\left(A^4P\right)}{\partial\left(ct\right)}=\left(A^1P\right)_{/1}+\left(A^2P\right)_{/2}+\left(A^3P\right)_{/3}+\left(A^4P\right)_{/4}$$

This equation then incorporates the first two terms on the left of the Fokker-Planck equation.

$$\nabla^2\left(BP\right)=\nabla\bullet\nabla\left(BP\right)$$

which can be represented covariantly as $\left(BP\right)^{/\mu}{}_{/\mu}$

$\left(A^{\mu}P\right)_{/\mu}+\left(BP\right)^{/\mu}{}_{/\mu}=S=$ source / sink.

This is still not in completely covariant form since we have not used the covariant derivative // which involves the Christoffel symbols in curved coordinate systems. When we do this, we obtain the following:

$$\left(A^{\mu}P\right)_{//\mu}+\left(BP\right)^{/\mu}_{//\mu}=S$$

which is now a covariant equation. We can now factor out the covariant derivative and obtain:

$$\left[\left(A^{\mu}P\right)+\left(BP\right)^{/\mu}\right]_{//\mu}=S=J^{\mu}_{//\mu}$$

$J^{\mu}=\left(A^{\mu}P\right)+\left(BP\right)^{/\mu}=$ transport flux or current density. This is slightly more general than other transport fluxes usually found and represented in irreversible thermodynamics, such as (again from Kittel):

$\mathbf{J}_x K_x = -\nabla W$

Ohms law: x = e and K_e = electrical conductivity, W = electric potential, $\mathbf{J}_e$ = current density
Fourier's law: x = q and K_q = thermal conductivity, W = Temperature, $\mathbf{J}_q$ = heat current density
Fick's law: x = m and K_m = diffusivity, W = particle or mass concentration, $\mathbf{J}_m$ = mass or particle current density.

$J_\mu = \left(A_\mu P\right) + \left(BP\right)_{/\mu}$ has at least two applications.

Number 1:[4]

$$J_s = \frac{e^* |\psi|^2}{m^*}\left(\hbar\nabla\varphi - \frac{e^*}{c}\mathbf{A}\right)$$

Equation 4.32 from reference 4, which is the Ginzburg-Landau equation from super conductivity, where $e^* = 2e$ and $m^* = 2m$ for electron pairs.

Number 2:[5]

$$\frac{d\xi^\mu_{(1)}}{ds} = au^\mu + ib\varphi^{/\mu}$$ = mass velocity vector

This clearly shows that the mass velocity vector of this paper (see reference 5) has the same covariant mathematical structure of a transport flux (or current density) as derived from the 4 dimensional covariant formulation of the Fokker-Planck equation. The author doubts that the mass velocity vector is related to probabilistic considerations, because the author basically obtained the idea from Bernoulli fluid flow mechanics.

In conclusion, it would be interesting to know in how many other venues this structural form for a transport flux arises, and what are, if any, the common characteristics they all share.

REFERENCES

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3. R. Adler, M. Bazin, M. Schiffer, *Introduction to General Relativity*, McGraw-Hill, (1965).

4. http://www.icmr.ucsb.edu/programs/documents/Beasley.pdf

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