Multifaceted approaches to a Berkeley problem: part 1

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Abstract

We solve a Berkeley problem in several ways.

1 A Berkeley problem

Problem 3.4.9 (Sp91) *Let $x(t)$ be a nontrivial solution to the system*

$$\frac{dx}{dt} = Ax,$$

where

$$A = \begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix}.$$

Prove that $\|x(t)\|$ is an increasing function of t. (Here, $\|\cdot\|$ denotes the Euclidean norm.)

N.B. For some reason, problem number is **3.4.6** in [1].

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2 Answers

2.1 Not-so-explicit version

First, by *not-so-explicit*, we mean that we don't try to obtain $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ =, *e.g.*, $(t, \sqrt{t}, \sin 5t)^{\mathrm{T}}$, $(4t, t^2, 3)^{\mathrm{T}}$, and so forth. Next, we differentiate $\|x(t)\|$ to get

$$\|x(t)\|'^{\,1} = \left[\sqrt{\{x_1(t)\}^2 + \{x_2(t)\}^2 + \{x_3(t)\}^2}\,\right]' \overset{2}{=} \frac{[\{x_1(t)\}^2+\{x_2(t)\}^2+\{x_3(t)\}^2]'}{2\sqrt{\{x_1(t)\}^2+\{x_2(t)\}^2+\{x_3(t)\}^2}}$$
$$= \frac{2\{x_1'(t)x_1(t)+x_2'(t)x_2(t)+x_3'(t)x_3(t)\}}{2\sqrt{\{x_1(t)\}^2+\{x_2(t)\}^2+\{x_3(t)\}^2}} = \frac{x_1(t)x_1'(t)+x_2(t)x_2'(t)+x_3(t)x_3'(t)}{\sqrt{\{x_1(t)\}^2+\{x_2(t)\}^2+\{x_3(t)\}^2}}.$$

Using REDUCE and wxMaxima 13.04.2, we verify the above as follows: [3], [4], [5]

```
$ reduce
```

```
0.00+0.07 secs          reduce
File  Edit  Font  Break  Load Package  Switch
Reduce (Free CSL version), 04-Aug-11 ...

1: df((x1(t)^2+x2(t)^2+x3(t)^2)^(1/2),t);

Declare x1 operator ?  (Y or N)

?Y

Declare x2 operator ?  (Y or N)

?Y

Declare x3 operator ?  (Y or N)

?Y
```

$$\frac{\frac{\partial x_1(t)}{\partial t}x_1(t) + \frac{\partial x_2(t)}{\partial t}x_2(t) + \frac{\partial x_3(t)}{\partial t}x_3(t)}{\sqrt{x_1(t)^2 + x_2(t)^2 + x_3(t)^2}}$$

---

[1] $'$ stands for differentiation with respect to t.

[2] We have used chain rule.

[3] Throughout this paper, we employ Debian GNU/Linux 8.8 (jessie). Central processing units are the same as those indicated in footnote 3 of [2].

[4] When we verify our computations, we use two kinds of softwares.

[5] We sometimes edit verbatim outputs of softwares to make them look neat. For instance, the font size of the function $A(a)$ in Fig. 1 has been slightly enlarged by using GIMP ver. 2.8.14.

```
$ wxmaxima
(%i1)  ratsimp(diff((x1(t)^2+x2(t)^2+x3(t)^2)^(1/2),t));
```

(%o1) $$\frac{\mathrm{x3}(t)\left(\frac{d}{dt}\mathrm{x3}(t)\right)+\mathrm{x2}(t)\left(\frac{d}{dt}\mathrm{x2}(t)\right)+\mathrm{x1}(t)\left(\frac{d}{dt}\mathrm{x1}(t)\right)}{\sqrt{\mathrm{x3}(t)^2+\mathrm{x2}(t)^2+\mathrm{x1}(t)^2}}$$

Having verified differentiation, we rewrite the numerators of these outputs as shown in the following.

$$x_1(t)x_1'(t)+x_2(t)x_2'(t)+x_3(t)x_3'(t)=\begin{pmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\end{pmatrix}^{\mathrm{T}}\cdot\begin{pmatrix} x_1'(t)\\ x_2'(t)\\ x_3'(t)\end{pmatrix}=\begin{pmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\end{pmatrix}^{\mathrm{T}}\cdot\left(A\begin{pmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\end{pmatrix}\right)$$

$$=\begin{pmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\end{pmatrix}^{\mathrm{T}}\cdot\begin{pmatrix} x_1(t)+6x_2(t)+x_3(t)\\ -4x_1(t)+4x_2(t)+11x_3(t)\\ -3x_1(t)-9x_2(t)+8x_3(t)\end{pmatrix}$$

$$=x_1(t)\{x_1(t)+6x_2(t)+x_3(t)\}+x_2(t)\{-4x_1(t)+4x_2(t)+11x_3(t)\}$$
$$+x_3(t)\{-3x_1(t)-9x_2(t)+8x_3(t)\}.$$

To make notations a bit simpler, we replace $x_1(t)$, $x_2(t)$, and $x_3(t)$ by X, Y, and Z, respectively. Then, we get

$$\|x(t)\|'=\frac{X(X+6Y+Z)+Y(-4X+4Y+11Z)+Z(-3X-9Y+8Z)}{\sqrt{X^2+Y^2+Z^2}}=\frac{X^2+2XY-2ZX+4Y^2+2YZ+8Z^2}{\sqrt{X^2+Y^2+Z^2}}$$

$$=\frac{X^2+2(Y-Z)X+4Y^2+2YZ+8Z^2}{\sqrt{X^2+Y^2+Z^2}}=\frac{\{X+(Y-Z)\}^2-(Y-Z)^2+4Y^2+2YZ+8Z^2}{\sqrt{X^2+Y^2+Z^2}}$$

$$=\frac{(X+Y-Z)^2-Y^2+2YZ-Z^2+4Y^2+2YZ+8Z^2}{\sqrt{X^2+Y^2+Z^2}}=\frac{(X+Y-Z)^2+3Y^2+4YZ+7Z^2}{\sqrt{X^2+Y^2+Z^2}}$$

$$=\frac{(X+Y-Z)^2+3(Y+\frac{2Z}{3})^2-3\cdot(\frac{2Z}{3})^2+7Z^2}{\sqrt{X^2+Y^2+Z^2}}=\frac{(X+Y-Z)^2+3(Y+\frac{2Z}{3})^2+\frac{17Z^2}{3}}{\sqrt{X^2+Y^2+Z^2}}$$

$$>^{6} 0.$$

Hence, $\|x(t)\|$ is an increasing function of t.

[6] If $(X, Y, Z) = (0, 0, 0)$, then $(x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ amounts to $(0, 0, 0)^{\mathrm{T}}$, which is trivial. So we have taken it for granted that $(X, Y, Z) \neq (0, 0, 0)$, and thus both $(X+Y-Z)^2+3(Y+\frac{2Z}{3})^2+\frac{17Z^2}{3}$ and $\sqrt{X^2+Y^2+Z^2}$ are greater than 0.

2.2 Rather intuitive version

In this subsection, we would like to emphasize the role of our intuition in problem-solving and perform slightly explicit computations. We write out the system we have been considering as follows:

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}. \tag{1}$$

Drawing a (very) intuitive parallel between $\frac{dx}{dt} = Ax$ and $\frac{de^{at}}{dt} = ae^{at}$, where $a \in \mathbb{R}$, we immediately get

$$(x_1(t),\ x_2(t),\ x_3(t))^{\mathrm{T}} = (A_1e^{at},\ A_2e^{at},\ A_3e^{at})^{\mathrm{T}}, \tag{2}$$

where $A_i \in \mathbb{R}$, $i = 1, 2, 3$ [3]. Substituting (2) into each side of (1), we obtain

$$\frac{d}{dt}\begin{pmatrix} A_1e^{at} \\ A_2e^{at} \\ A_3e^{at} \end{pmatrix} = \begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix} \begin{pmatrix} A_1e^{at} \\ A_2e^{at} \\ A_3e^{at} \end{pmatrix}.$$

Performing the differentiation in the left-hand side (LHS) of the above yields

$$\begin{pmatrix} A_1ae^{at} \\ A_2ae^{at} \\ A_3ae^{at} \end{pmatrix} = \begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix} \begin{pmatrix} A_1e^{at} \\ A_2e^{at} \\ A_3e^{at} \end{pmatrix}.$$

Then, we divide each side of the above by e^{at}. [7] After some rearrangements, we have

$$\begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = a \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}. \tag{3}$$

[7] This is possible, because $e^{ax} \neq 0$, with $a, x \in \mathbb{R}$. See **Appendix**.

Interpreting $(A_1, A_2, A_3)^{\mathrm{T}}$ as an eigenvector with eigenvalue a, we would like to say that (2) satisfies (1).[8] In order to know the value of a, we perform an expansion about the third column of the matrix

$$\begin{pmatrix} 1-a & 6 & 1 \\ -4 & 4-a & 11 \\ -3 & -9 & 8-a \end{pmatrix}$$

and get the characteristic polynomial $A(a) = 1 \cdot \{-4 \cdot (-9) - (4-a) \cdot (-3)\}$
$-11 \cdot \{(1-a) \cdot (-9) - 6 \cdot (-3)\} + (8-a)\{(1-a) \cdot (4-a) - 6 \cdot (-4)\} = 1 \cdot (48-3a)$
$-11(9a+9) + (8-a)(a^2-5a+28) = 48-3a-99a-99-a^3+13a^2-68a+224 =$

$$-a^3 + 13a^2 - 170a + 173. \tag{4}$$

Now we check this expansion using OpenAxiom and wxMaxima 13.04.2 as follows.

```
$ open-axiom
             OpenAxiom: The Open Scientific Computation Platform
                     Version: OpenAxiom 1.5.0-2013-06-21
               Built on Sunday December 15, 2013 at 18:59:05
-----------------------------------------------------------------------------
   Issue )copyright to view copyright notices.
   Issue )summary for a summary of useful system commands.
   Issue )quit to leave OpenAxiom and return to shell.
-----------------------------------------------------------------------------
   Re-reading interp.daase
   Re-reading operation.daase
   Re-reading category.daase
   Re-reading browse.daase
```

[8] We assume that $(A_1, A_2, A_3)^{\mathrm{T}} \neq (0, 0, 0)^{\mathrm{T}}$. Otherwise we get the trivial solution $(x_1(t), x_2(t), x_3(t))^{\mathrm{T}} = (0, 0, 0)^{\mathrm{T}}$. See footnote 6 and (2).

```
(1) -> A_a:=characteristicPolynomial([[1,6,1],[-4,4,11],
            [-3,-9,8]],a)

             3      2
   (1)  - a   + 13a   - 170a + 173

                                    Type: Polynomial Integer
```

```
$ wxmaxima
```

```
(%i1)  expand(charpoly(A:matrix([1,6,1],[-4,4,11],[-3,-9,8]),
              a));
```

(%o1) $-a^3 + 13a^2 - 170a + 173$

In this way, we have checked (4). Since all polynomial functions are continuous , $A(a) = -a^3 + 13a^2 - 170a + 173$ is continuous. [9] $A(1) \cdot A(2)$ being $15 \cdot (-123) < 0$, the equation

$$A(a) = 0 \tag{5}$$

has at least one root in [1, 2] . [10] Differentiating $A(a)$, we get

$$\frac{dA(a)}{da} = -3a^2 + 26a - 170. \tag{6}$$

Completing the square for the right-hand side (RHS) of (6) yields $-3(a - \frac{13}{3})^2$ $-\frac{341}{3}$, which is less than 0 for all $a \in \mathbb{R}$. So $A(a)$ decreases monotonously in $\mathbb{R}^2$. Taken together, (5) has just one root α in [1, 2] . Using Scilab and wxMaxima 13.04.2, we visualize $A(a)$ as shown below.

[9] Strictly speaking, we need to turn to the ϵ - δ method .

[10] This is due to the intermediate value theorem . See also here .

```
$ scilab
                              scilab-5.5.1
-->a=[1:0.1:2]';
-->A=[-a^3+13*a^2-170*a+173];
-->plot(a,A);xgrid(1);xtitle('A(a)=-a^3+13*a^2-170*a+173');
```

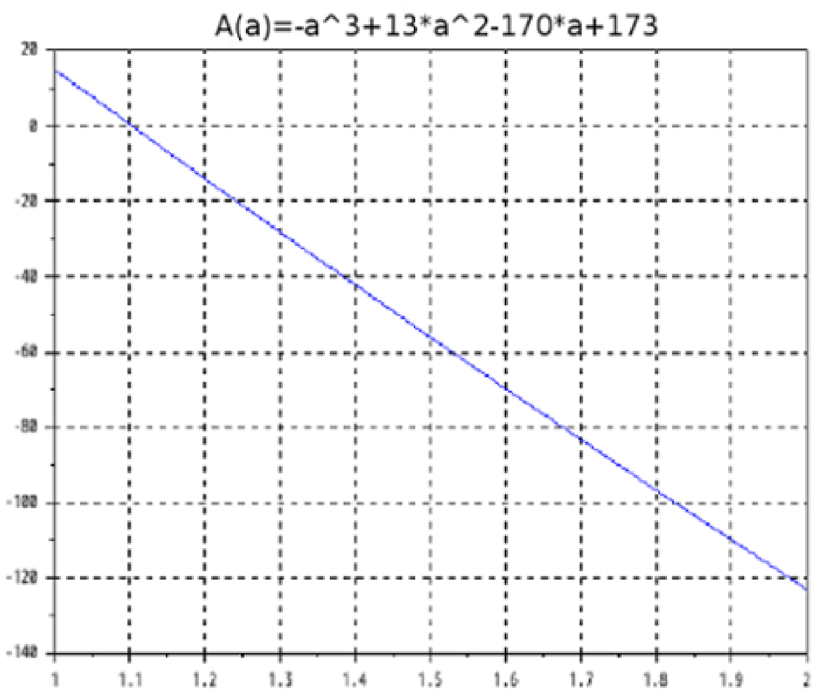

Fig. 1. *A*(*a*) visualized by Scilab

```
$ wxmaxima
(%i1)  plot2d(-a^3+13*a^2-170*a+173,[a,1,2]);
```

(%o1)

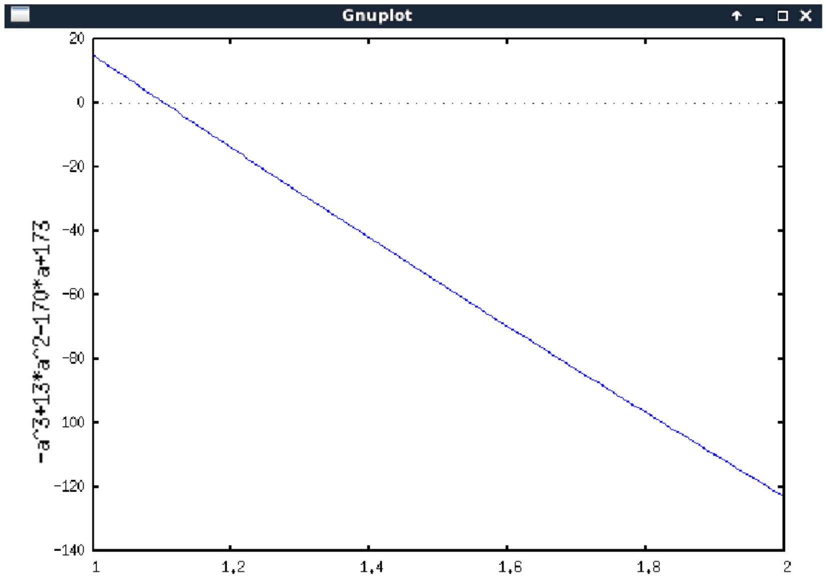

Fig. 2. $A(a)$ visualized by wxMaxima

These figures help us narrow down the range of α to $(1.0, 1.2)$. So a solution satisfying (1) is $(A_1e^{\alpha t}, A_2e^{\alpha t}, A_3e^{\alpha t})^{\mathrm{T}}$, with $1.0 < \alpha < 1.2$. $\|\cdot\|$ denoting the Euclidean norm , we obtain $\|x(t)\| = \sqrt{\{x_1(t)\}^2 + \{x_2(t)\}^2 + \{x_3(t)\}^2} = \sqrt{(A_1e^{\alpha t})^2 + (A_2e^{\alpha t})^2 + (A_3e^{\alpha t})^2} = \sqrt{A_1^2 + A_2^2 + A_3^2}\ |\ e^{\alpha t}\ | =$[11] $\sqrt{A_1^2 + A_2^2 + A_3^2}\, e^{\alpha t}$.

$\|x(t)\|$ is thus an exponentially increasing function of t, since $\sqrt{A_1^2 + A_2^2 + A_3^2}$ is greater than 0.[12] Indeed, $\|x(t)\|$ increases monotonously, since $\|x(t)\|' = \alpha \sqrt{A_1^2 + A_2^2 + A_3^2}\, e^{\alpha t} >$[13] 0 for all $t \in \mathbb{R}$.

[11] $|\ e^{\alpha t}\ | = e^{\alpha t}$, because $e^{\alpha t} > 0$ ($\alpha, t \in \mathbb{R}$). See **Appendix**.

[12] See footnote 8.

[13] Ditto.

2.3 A bit meticulous version

Although we have expressed the components of an eigenvector of A simply by A_i, we would like to be a bit meticulous in this subsection. Let $\mathbf{v_1} = (v_{11}, v_{21}, v_{31})^{\mathrm{T}}$ be an eigenvector of A with eigenvalue $\alpha \in \mathbb{R}$. Then, we have

$$\begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \end{pmatrix} = \alpha \begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \end{pmatrix},^{14}$$

which we rewrite as

$$\begin{cases} (1-\alpha)v_{11} + 6v_{21} + v_{31} = 0, & (7) \\ -4v_{11} + (4-\alpha)v_{21} + 11v_{31} = 0, & (8) \\ -3v_{11} - 9v_{21} + (8-\alpha)v_{31} = 0. & (9) \end{cases}$$

Since $(7) \times 11 - (8)$ gives $(15 - 11\alpha)v_{11} + (62 + \alpha)v_{21} = 0$, we get the ratio $v_{11} : v_{21} = \alpha + 62 : 11\alpha - 15$. Therefore, $(v_{11},\, v_{21}) = (B(\alpha+62),\, B(11\alpha - 15))$, where B is a nonzero constant. [15] Substituting v_{11} and v_{21} into (9), after some rearrangements we get $v_{31} = \frac{3B(\alpha+62)+9B(11\alpha-15)}{8-\alpha} = \frac{B(102\alpha+51)}{8-\alpha}$. [16] Replacing the numerator by $B(-\alpha^3 + 13\alpha^2 - 68\alpha + 224)$, we obtain $\frac{B(-\alpha^3+13\alpha^2-68\alpha+224)}{8-\alpha} = \frac{B(8-\alpha)(\alpha^2-5\alpha+28)}{8-\alpha} = B(\alpha^2 - 5\alpha + 28)$. [17] Hence, $\mathbf{v_1} = (B(\alpha+62),\, B(11\alpha-15),\, B(\alpha^2-5\alpha+28))^{\mathrm{T}}$.

We thus have

$$\begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix} \begin{pmatrix} B(\alpha+62) \\ B(11\alpha-15) \\ B(\alpha^2-5\alpha+28) \end{pmatrix} = \alpha \begin{pmatrix} B(\alpha+62) \\ B(11\alpha-15) \\ B(\alpha^2-5\alpha+28) \end{pmatrix}. \tag{10}$$

Multiplying both sides of the above by $e^{\alpha t}$, we get

$$\begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix} \begin{pmatrix} B(\alpha+62)e^{\alpha t} \\ B(11\alpha-15)e^{\alpha t} \\ B(\alpha^2-5\alpha+28)e^{\alpha t} \end{pmatrix} = \alpha \begin{pmatrix} B(\alpha+62)e^{\alpha t} \\ B(11\alpha-15)e^{\alpha t} \\ B(\alpha^2-5\alpha+28)e^{\alpha t} \end{pmatrix}.$$

[14] See also (3).

[15] If $B = 0$, $(v_{11},\, v_{21}) = (0,\, 0)$, which we substitute into (7) to get $v_{31} = 0$. Then, $\mathbf{v_1}$ becomes trivial.

[16] We can make the division by $8 - \alpha$, since $1.0 < \alpha < 1.2$. See Figs. 1 and 2.

[17] Since α is a root of (5), $-\alpha^3 + 13\alpha^2 - 170\alpha + 173 = 0$. Adding $102\alpha + 51$ to each side of this equation yields the relation $-\alpha^3 + 13\alpha^2 - 68\alpha + 224 = 102\alpha + 51$.

Rewriting the above as

$$\frac{d}{dt}\begin{pmatrix} B(\alpha+62)e^{\alpha t} \\ B(11\alpha-15)e^{\alpha t} \\ B(\alpha^2-5\alpha+28)e^{\alpha t} \end{pmatrix} = \begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix} \begin{pmatrix} B(\alpha+62)e^{\alpha t} \\ B(11\alpha-15)e^{\alpha t} \\ B(\alpha^2-5\alpha+28)e^{\alpha t} \end{pmatrix}$$

makes us notice that $(B(\alpha+62)e^{\alpha t}, B(11\alpha-15)e^{\alpha t}, B(\alpha^2-5\alpha+28)e^{\alpha t})^{\mathrm{T}}$ satisfies (1). In a sense, this solution is the same as the RHS of (2), since both $(A_1, A_2, A_3)^{\mathrm{T}}$ and $(B(\alpha+62), B(11\alpha-15), B(\alpha^2-5\alpha+28))^{\mathrm{T}}$ can be regarded as eigenvectors of A.[18]

So we can view this version as equivalent to subsection **2.2**, if we wish.

Acknowledgment. We would like to thank the developers of the free softwares used herein for their indirect help which enabled us to verify some of our computations.

References

[1] de Souza, P. N. and Silva, J.-N., "Berkeley problems in mathematics. 2nd ed.," Springer-Verlag New York Inc. 2001 p49.

[2] Suzuki, K., "Answering math problems," viXra:1605.0003 [v1] .

[3] Braun, M., "Differential equations and their applications. 4th ed.," Springer-Verlag New York Inc. 1993 p345.

[18] Compare (3) with (10).

3 Appendix

We 'forget zeros' such as $\lim_{x\to+\infty} \frac{1}{e^x} = 0$, $\lim_{x\to-\infty} e^{2x} = 0$, and so on, and explain why $e^{ax} > 0$ ($a, x \in \mathbb{R}$). We consider cases **3.1**, **3.2**, and **3.3**.

3.1 $a > 0$

In this case, we further consider the following two subcases.

3.1.1 $x \geq 0$

It is well-known that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$. Substituting ax for x, we get

$$e^{ax} = 1 + ax + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \dots.$$

By the way, $ax \geq 0$, since $a > 0$ and $x \geq 0$. So the RHS of the above is greater than or equal to 1, as is its LHS . Hence, $e^{ax} > 0$ for $x \geq 0$.

3.1.2 $x < 0$

In this case, we replace x by $-y$, where $y \in \mathbb{R}_{>0}$, to consider $e^{-ay} = \frac{1}{e^{ay}}$. Reading the variable x in **3.1.1** as y, we have $e^{ay} > 0$ for $y \geq 0$. So $\frac{1}{e^{ay}} = e^{-ay} > 0$ for $y > 0$. Substituting ax and $-x$ for $-ay$ and y, respectively, gives $e^{ax} > 0$ for $-x > 0$. Hence, $e^{ax} > 0$ for $x < 0$.

3.2 $a = 0$

In this case, $e^{0\cdot x} = e^0 = 1 > 0$.

3.3 $a < 0$

Like **3.1**, we consider two subcases.

3.3.1 $x > 0$

Substituting $-b$, where $b \in \mathbb{R}_{>0}$, for a, we consider $e^{-bx} = \frac{1}{e^{bx}}$. Reading the constant a in **3.1.1** as b, we have $e^{bx} > 0$ for $x > 0$. So $\frac{1}{e^{bx}} = e^{-bx} > 0$ for $x > 0$. Replacing $-bx$ by ax, we have $e^{ax} > 0$ for $x > 0$. Incidentally, since $ax < 0$, arguments we have made in this subsubsection are essentially the same as those in **3.1.2**.

3.3.2 $x \leq 0$

In this case, $ax \geq 0$, since $a < 0$ and $x \leq 0$. So it follows from **3.1.1** that $e^{ax} > 0$.

N.B. Cases **3.1 – 3.3** and their subcases exhaust classification which depends on the values of $a, x \in \mathbb{R}$.