A Review of Five Approaches of Quantum Potential Including Madelung Hydrodynamics Formulation**

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ABSTRACT

It has been long known that a year after Schrodinger published his equation, Madelung also published a hydrodynamics version of Schrodinger equation. But it is often misinterpreted by many contemporary physicists, especially after the famous Bohmian quantum potential. In this paper we will review quantum potential by five different approaches, including Madelung hydrodynamics, complex Madelung, and also Navier-Stokes hydrodynamics approach. In the last section we will also discuss a new expression of quantum potential based on complex Riccati equation. It is our hope that these methods can be verified and compared with experimental data. But we admit that more researches are needed to fill all the missing details.

Keywords: quantum hydrodynamics, quantum potential, quantum wave function, quantum-classical correspondence, Madelung equation, Navier-Stokes equations, complex Riccati equation.

1. Introduction

The Copenhagen interpretation of quantum mechanics is guilty for the quantum mystery and many strange phenomena such as the Schrödinger cat, parallel quantum and classical worlds, wave-particle duality, decoherence, collapsing wave function, etc.

The Copenhagen interpretation of QM was challenged not only by Schrödinger but also by a large group of physicists led by Albert Einstein who claimed that the quantum mechanical description of the physical reality cannot be considered complete, as shown in their famous EPR paper Einstein, Podolsky and Rosen. They concluded their derivations by stating that “*While we have thus shown that the wave function does not provide a complete description of the physical reality, we left open the question of whether or not such a description exists. We believe, however that such a theory is possible.”* Einstein did not object to the probabilistic description of sub-atomic phenomena in quantum mechanics. However, he believed that this probabilistic representation was a technique used to overcome the practical difficulties of dealing with a more complicated underlying physical reality, much in the same way he suggested earlier to deal with Brownian motion.[8]

Many scientists have tried, however, to put the quantum mechanics back on *ontological* foundations. For instance, Bohm proposed an alternative interpretation of quantum mechanics, which is able to overcome some puzzles of the Copenhagen interpretation. He developed further the de Broglie pilot-wave theory and, for this reason, the Bohmian mechanics is also known as the de Broglie-Bohm theory.[2]

Long before the Bohmian mechanics proposal, a year after Erwin Schrödinger published his celebrated equation, Erwin Madelung showed (in 1927) that it can be written in a hydrodynamic form. Madelung’s representation has a seemingly major disadvantage by transforming the linear Schrödinger equation into two nonlinear ones. Nonetheless, despite of its additional complexity, the hydrodynamic analogy provides important insights with regard to the Schrödinger equation.[6]

The Madelung equations (ME) describe a compressible fluid, and compressibility yields a linkage between hydrodynamic and thermodynamic effects. The work done by the pressure gradient force to expand the flow transforms internal thermal microscopic kinetic energy to the macroscopic hydrodynamic kinetic energy of the flow.[6]

In this paper we will review quantum potential by five different approaches, including Madelung hydrodynamics, complex Madelung, and also Navier-Stokes hydrodynamics approach. In the last section we will also discuss a new expression of quantum potential based on complex Riccati equation. It is shown that the Bohmian mechanics and the Madelung quantum hydrodynamics are different theories and the latter is a better ontological interpretation of quantum mechanics, but we also argue that quantum potentials derived from Navier-Stokes hydrodynamics and complex Riccati equation are also worthy to investigate.

It is our hope that these approaches can be verified and compared with observation data. But we admit that more researches are needed to fill all the missing details, for example we do not yet discuss comparison between quantum trajectories and classical trajectories such as in Wilson chamber experiments.

2. Bohmian quantum potential [2]

The evolution of the wave function of a quantum mechanical system consisting of N particles is supposed to be described by the Schrödinger equation:

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\nabla + U\right)\psi. \qquad (1)$$

The complex wave function can be presented generally in the polar form:

$$\psi = \sqrt{\rho}\exp\left(\frac{iS}{h}\right), \qquad (2)$$

Where $\rho = |\psi|^2$ is the N-particles distribution density and $\frac{S}{\hbar}$ is the wave function phase.

Introducing equation (2) into (1) one gets a set of equations:

$$\partial_t\rho = -\nabla.(\rho\nabla S/m), \qquad (3)$$

$$\partial_t S + \frac{(\nabla S)^2}{2m} + U + Q = 0, \tag{4}$$

Where quantum potential, Q, is defined as follows:

$$Q = -\frac{\hbar^2}{2m^2}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}. \tag{5}$$

Equation (5) is called Bohmian quantum potential.[2]

3. Madelung quantum potential[2]

If one starts with a different assumption that in equation (3) S is the hydrodynamic-like velocity potential, not the mechanical action as suggested by Bohm, then he can arrives at different relations, such as the two equations proposed by Madelung as follows:

$$\partial_t \rho = -\nabla.(\rho V), \tag{6}$$

$$m\partial V + mV \cdot \nabla V = -\nabla(U + Q), \tag{7}$$

Where

$$V = \nabla S / m. \tag{8}$$

Equations (6) and (7) are known as the Madelung quantum hydrodynamics.[2]

4. Complex Madelung quantum hydrodynamics[2]

An interesting alternative of the Madelung quantum hydrodynamics is the complex hydrodynamics with an irrotational complex hydrodynamic velocity:

$$\omega = i\hbar\frac{\nabla \ln \overline{\psi}}{m} = V + i\hbar\frac{\nabla \ln \rho}{2m},$$

(9)

introduced via the quantum momentum operator. Substituting this expression, the continuity Eq. (6) changes to a complex convective diffusion equation

$$\partial_t \rho + \nabla.(\rho\omega) = D\nabla^2 \rho, \tag{10}$$

with imaginary diffusion coefficient D is defined as follows:

$$D = \frac{i\hbar}{2m}.$$

(11)

Using further Eq. (9), the Madelung hydrodynamic force balance (7) acquires the form of a complex Navier-Stokes equation:

$$\partial_t \omega + \omega.\nabla\omega = -\frac{\nabla U}{m} + \nu\nabla^2 \omega,$$

(12)

with constant pressure and kinematic viscosity $\nu = -\frac{i\hbar}{2m}$. Thus, the weird quantum potential disappears completely and, hence, the Schrödinger equation reduces to classical diffusion and hydrodynamics but with complex transport coefficients. As is seen, the vacuum possesses purely imaginary diffusion and viscosity constants.[2]

Some exact solutions of the above Navier-Stokes interpretation of Schrodinger equation have been discussed by Kulish and Lage [4][5].

Now things become much more interesting because we find a plausible picture of quantum-classical correspondence via Navier-Stokes hydrodynamics, therefore in the next section we will discuss another approach using Navier-Stokes-inspired potential.

5. Alternative Navier-Stokes-inspired quantum potential

According to Cordoba, Isidro and Molina, Quantum mechanics has been argued to be a coarse-graining of some underlying deterministic theory. They support this view by establishing a map between certain solutions of the Schrödinger equation, and the corresponding solutions of the irrotational Navier–Stokes equation for viscous fluid flow. Nonetheless, Vadasz has shown that the complete Navier–Stokes equations render into an *extended generalized* version of Schrödinger equation.[8]

After some careful logical process, he arrives at an equation which represents the original Schrödinger equation, plus additional terms which are the result of converting the complete Navier–Stokes equations leading to an extended form of Schrödinger equation.

The equation proposed by Vadasz can be written as follows [8]:

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\nabla + U\right)\psi + Q, \tag{13}$$

Where:

$$Q = -\left(i\hbar(\nabla\psi) - \psi\frac{m}{2}(\nabla\times\chi)\right)\cdot(\nabla\times\chi) + \psi m\nabla^{-1}a.n.c.t \tag{14}$$

The reader should keep in mind that the new terms of quantum potential here are different from complex Madelung hydrodynamics as described in preceding section.

6. Complex Riccati-inspired quantum potential

Schuch argues that it can be shown where complex Riccati equations appear in time-dependent quantum mechanics and how they can be treated and compared with similar space-dependent Riccati equations in supersymmetric quantum mechanics. Furthermore, the time-independent Schrödinger equation can also be rewritten as a complex Riccati equation.

The corresponding complex Riccati equation is given by [9][10]:

$$\nabla C + C^2 + \frac{2m}{\hbar^2}(E - V) = 0, \tag{15}$$

Where

$$C = \left(\frac{\nabla\psi}{\psi}\right) = \frac{\nabla a}{a} + i\frac{1}{h}\nabla S. \tag{16}$$

It can be shown that equation (15) can be linearized to become TISE.[9][10]

7. Discussion and Concluding Remarks

We have discussed five different approaches of describing quantum potential, including Madelung hydrodynamics, complex Madelung, and also Navier-Stokes hydrodynamics approach. In the last section we will also discuss a new expression of quantum potential based on complex Riccati equation. It is our hope that these methods can be verified and compared with experimental data. But we admit that more researches are needed to fill all the missing details.

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