# Why are odd Bernoulli numbers equal to zero?*

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**Abstract**

In this note we will use Faulhaber's Formula to explain why the odd Bernoulli numbers are equal to zero.

For odd numbers greater than or equal to seven, why are the Bernoulli numbers equal to zero? Because Faulhaber's Formula tells us that $\sum_{k=1}^{n} k^{2m+1}$ is a polynomial in $\left(\sum_{k=1}^{n} k\right)^2$, and $\left(\sum_{k=1}^{n} k\right)^2 = \frac{n^2+2n^3+n^4}{4}$.

For positive integers $m \geqq 3$, Faulhaber's Formula is

$$\begin{aligned}\sum_{k=1}^{n} k^{2m+1} \quad &= \quad \frac{1}{m+1} \cdot \left[ 2^m \cdot \left(\frac{n(n+1)}{2}\right)^{m-1} - a_2 \cdot \left(\frac{n(n+1)}{2}\right)^{m-2} + a_3 \cdot \left(\frac{n(n+1)}{2}\right)^{m-3} \right. \\ &\qquad \left. \mp \cdots \mp a_{m-2} \cdot \left(\frac{n(n+1)}{2}\right)^2 \mp a_{m-1} \cdot \frac{1}{3} \cdot \left(4 \cdot \frac{n(n+1)}{2} - 1\right)\right] \cdot \left(\sum_{k=1}^{n} k\right)^2,\end{aligned}$$

where the $a_i$ are rational numbers. For the Bernoulli numbers $B_o, B_1, \ldots, B_{2m}, B_{2m+1}$, where $B_3 = B_5 = 0$, we can write $\sum_{k=1}^{n} k^{2m+1}$ as

$$\begin{aligned}\sum_{k=1}^{n} k^{2m+1} \quad &= \quad \frac{1}{2m+2} \cdot \left[ \binom{2m+2}{0} \cdot B_0 \cdot n^{2m+2} + \binom{2m+2}{1} \cdot B_1 \cdot n^{2m+1} \right. \\ &\qquad \left. + \cdots + \binom{2m+2}{2m} \cdot B_{2m} \cdot n^2 + \binom{2m+2}{2m+1} \cdot B_{2m+1} \cdot n \right].\end{aligned}$$

Suppose we set the expressions equal to one another. If we multiply out the first one, we see it does not contain the term $n$. That means the last term of the second one,

$$\frac{1}{2m+2} \cdot \binom{2m+2}{2m+1} \cdot B_{2m+1} \cdot n,$$

must be equal to zero. In other words, for all $m \geqq 3$, $B_{2m+1} = 0$.