Labeled Graph — A Mathematical Element

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Abstract: The universality of contradiction and connection of things in nature implies that a thing is nothing else but a labeled topological graph G^L with a labeling map $L: V(G)\bigcup E(G) \rightarrow \mathscr{L}$ in space, which concludes also that labeled graph should be an element for understanding things in the world. This fact proposes 2 directions on labeled graphs: (1) verify a graph family $\mathscr{G}$ whether or not they can be labeled by a labeling L constraint on special conditions, and (2) establish mathematical systems such as those of groups, rings, linear spaces or Banach spaces over graph G, i.e., view labeled graphs G^L as elements of that system. However, all results on labeled graphs are nearly concentrated on the first in past decades, which is in fact searching structure G of the labeling set $\mathscr{L}$. The main purpose of this report is to show the role of labeled graphs in extending mathematical systems over graphs G, particularly graphical tensors and $\overrightarrow{G}$-flows with conservation laws and applications to physics and other sciences such as those of labeled graphs with sets or Euclidean spaces $\mathbb{R}^n$ labeling, labeled graph solutions of non-solvable systems of differential equations with global stability and extended Banach or Hilbert $\overrightarrow{G}$-flow spaces. All of these makes it clear that holding on the reality of things by classical mathematics is partial or local, only on the coherent behaviors of things for itself homogenous without contradictions, but the mathematics over graphs G is applicable for contradictory systems over G because contradiction is universal in the nature, which can turn a contradictory system to a compatible one, i.e., mathematical combinatorics.

Key Words: Topological graph, labeling, group, ring, linear space, Banach space, Smarandache multispace, non-solvable equation, graphical tensor, $\overrightarrow{G}$-flow, mathematical combinatorics.

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§1. Introduction

Is our world continuous or discrete? Different peoples with different world views will answer this question differently, particularly for researchers on continuous or discrete sciences, for instance, the fluid mechanics or elementary particles with interactions. Actually, a natural thing T is complex, ever hybrid with other things on the eyes of human beings sometimes. Thus, holding on the true face of thing T is difficult, maybe result in disputation for persons standing on different views or positions for T, which also implies that all contradictions are man made, not the nature of things. For this fact, a typical example was shown once by the famous fable "the blind men with an elephant". In this fable, there are six blind men were asked to determine what an elephant looked like by feeling different parts of the elephant's body. The man touched the elephant's leg, tail, trunk, ear, belly or tusk respectively claims it's like a pillar, a rope, a tree branch, a hand fan, a wall or a solid pipe, such as those shown in Fig.1 following. Each of them insisted on his own and not accepted others. They then entered into an endless argument.

Fig.1

All of you are right! A wise man explains to them: *why are you telling it differently is because each one of you touched the different part of the elephant. So, actually the elephant has all those features what you all said.*

Thus, the best result on an elephant for these blind men is

$$\begin{aligned}\text{An elephant} &= \{4 \text{ pillars}\}\bigcup\{1 \text{ rope}\}\bigcup\{1 \text{ tree branch}\}\\ &\bigcup \{2 \text{ hand fans}\}\bigcup\{1 \text{ wall}\}\bigcup\{1 \text{ solid pipe}\}\end{aligned}$$

A thing T is usually identified with known characters on it at one time, and this process is advanced gradually by ours. For example, let $\mu_1, \mu_2, \cdots, \mu_n$ be the known and $\nu_i, i \geq 1$ the unknown characters at time t. Then, the thing T is understood by

$$T = \left(\bigcup_{i=1}^{n}\{\mu_i\}\right) \bigcup \left(\bigcup_{k\geq 1}\{\nu_k\}\right) \tag{1.1}$$

in logic and with an approximation $T^\circ = \bigcup_{i=1}^{n}\{\mu_i\}$ at time t. Particularly, *how can the wise man tell these blind men the visual image of an elephant in fable of the blind men with an elephant*? If the wise man is a discrete mathematician, he would tell the blind men that an elephant looks like nothing else but a labeled tree shown in Fig.2.

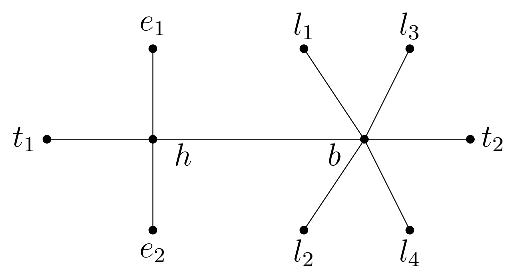

Fig.2

where, $\{t_1\}$ =tusk, $\{e_1, e_2\}$ =ears, $\{h\}$ =head, $\{b\}$ =belly, $\{l_1, l_2, l_3, l_4\}$ =legs and $\{t_2\}$ =tail. Hence, labeled graphs are elements for understanding things of the world in our daily life. *What is the philosophical meaning of this fable for understanding things in the world*? It lies in that the situation of human beings knowing things in the world is analogous to these blind men. We can only hold on things by canonical model (1.1), or the labeled tree in Fig.2.

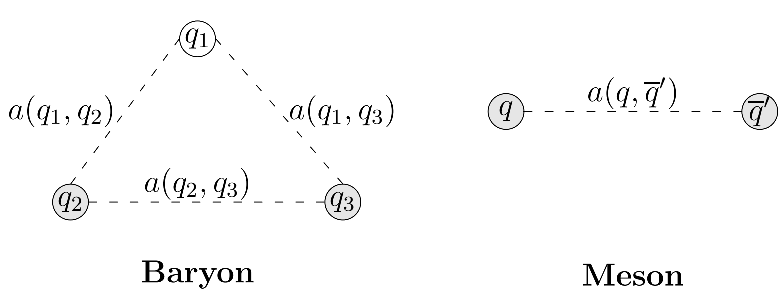

Fig.3

Notice that the elementary particle theory is indeed a discrete notion on matters in the nature. For example, a baryon is predominantly formed from three quarks, and a meson is mainly composed of a quark and an antiquark in the quark models of Sakata, or Gell-Mann and Ne'eman ([27], [32]) such as those shown in Fig.3, which said in fact that the baryon and meson are nothing else but a multiverse ([3]), graphs labeled by quark $q_i \in \{\mathbf{u}, \mathbf{d}, \mathbf{c}, \mathbf{s}, \mathbf{t}, \mathbf{b}\}$ for $i = 1, 2, 3$ and antiquark $\overline{q}' \in \left\{\overline{\mathbf{u}}, \overline{\mathbf{d}}, \overline{\mathbf{c}}, \overline{\mathbf{s}}, \overline{\mathbf{t}}, \overline{\mathbf{b}}\right\}$, where $a(q, q')$ denotes the strength between quarks q and q'.

Certainly, a natural thing can not exist out of the live space, the universe. Thus, the labeled graphs in Fig.2 and 3 are actually embedded in the Euclidean space $\mathbb{R}^3$, i.e. a labeled topological graph. Generally, a topological graph $\varphi(G)$ in a space $\mathscr{S}$ is an embedding of $\varphi: \ G \to \varphi(G) \subset \mathscr{S}$ with $\varphi(p) \neq \varphi(q)$ if $p \neq q$ for $\forall p, q \in G$, i.e., edges of G only intersect at vertices in $\mathscr{S}$. There is a well-known result on embedding of graphs without loops and multiple edges in $\mathbb{R}^n$ for $n \geq 3$ ([10]), i.e., *there always exists an embedding of G that all edges are straight segments in $\mathbb{R}^n$.*

Mathematically, a labeling on a graph G is a mapping $L: \ V(G) \bigcup E(G) \to \mathscr{L}$ with a labeling set $\mathscr{L}$ such as two labeled graphs on K_4 with integers in $\{1, 2, 3, 4\}$ shown in Fig.4,

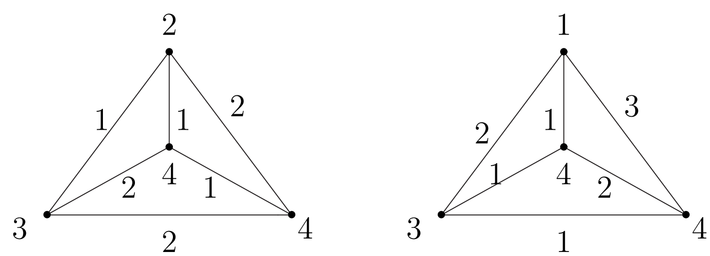

Fig.4

and they have been concentrated more attentions of researchers, particularly, the dynamical survey paper [4] first published. Usually, $\mathscr{L}$ is chosen to be a segment of integers $\mathbb{Z}$ and a labeling $L: \ V(G) \to \mathscr{L}$ with constraints on edges in $E(G)$. Only on the journal: *International Journal of Mathematical Combinatorics* in the past 9 years, we searched many papers on labeled graphs. For examples, the graceful, harmonic, Smarandache edge m-mean labeling ([29]) and quotient cordial labeling ([28]) are respectively with edge labeling $|L(u) - L(v)|$, $|L(u) + L(v)|$, $\left\lceil \dfrac{f(u) + f(v)}{m} \right\rceil$ for $m \geq 2$, $\left[\dfrac{f(u)}{f(v)}\right]$ or $\left[\dfrac{f(v)}{f(u)}\right]$ according $f(u) \geq f(v)$ or $f(v) > f(u)$ for $\forall uv \in E(G)$, and a Smarandache-Fibonacci or Lucas graceful labeling is such a labeling $L: V(G) \to \{S(0), S(1), S(2), \cdots, S(q)\}$ that the induced edge labeling is

$\{S(1), S(2), \cdots, S(q)\}$ by $L(uv) = |L(u) - L(v)|$ for $\forall uv \in E(G)$ for a Smarandache-Fibonacci or Lucas sequence $\{S(i),\ i \geq 1\}$ ([23]).

Similarly, an n-signed labeling is a n-tuple of $\{-1, +1\}^n$ or $\{0, 1\}$-vector labeling on edges of graph G with $|e_f(0) - e_f(1)| \leq 1$, where $e_f(0)$ and $e_f(1)$ respectively denote the number of edges labeled with even integer or odd integer([26]), and a graceful set labeling is a labeling $L: V(G) \rightarrow 2^X$ on vertices of G by subsets of a finite set X with induced edge labeling $L(uv) = L(u) \oplus L(v)$ for $\forall uv \in E(G)$, where " $\oplus$ " denotes the binary operation of taking the symmetric difference of the sets in 2^X ([30]). As a result, the combinatorial structures on $\mathscr{L}$ were partially characterized.

However, for understanding things in the world we should ask ourself: *what are labels on a labeled graph, is it just different symbols*? And *are such labeled graphs a mechanism for understanding the reality of things, or only a labeling game*? Clearly, labeled graphs G considered by researchers are graphs mainly with number labeling, vector symbolic labeling without operation, or finite set labeling, and with an additional assumption that *each vertex of G is mapped exactly into one point of space $\mathscr{S}$* in topology. However, labels all are space objects in Fig.2 and 3. If we put off this assumption, i.e., labeling a topological graph by geometrical spaces, or elements with operations in a linear space, *what will happens*? *Are these resultants important for understanding things in the world*? The answer is certainly YES because this step will enable one to pullback more characters of things, characterize more precisely and then hold on the reality of things in the world, i.e., combines continuous mathematics with the discrete, which is nothing else but the *mathematical combinatorics*.

The main purpose of this report is to survey the role of labeled graphs in extending mathematical systems over graphs G, particularly graphical tensors and $\overrightarrow{G}$-flows with conservation laws and applications to mathematics, physics and other sciences such as those of labeled graphs with sets or Euclidean spaces $\mathbb{R}^n$ labeling, labeled graph solutions of non-solvable systems of differential equations with global stability, labeled graph with elements in a linear space, and extended Banach or Hilbert $\overrightarrow{G}$-flow spaces, $\cdots$, etc. All of these makes it clear that holding on the reality of things by classical mathematics is partial, only on the coherent behaviors of things for itself homogenous without contradictions but the extended mathematics over graphs G can characterize contradictory systems, and accordingly can be applied to hold on the reality of things because contradiction is universal in the nature.

For terminologies and notations not mentioned here, we follow references [5] for functional analysis, [9]-[11] for graphs and combinatorial geometry, [2] for differential equations, [27] for elementary particles, and [1],[10] for Smarandache multispaces or multisystems.

§2. Graphs Labeled by Sets

Notice that the understanding form (1.1) of things is in fact a Smarandache multisystem following, which shows the importance of labeled graphs for things.

Definition 2.1([1],[10]) *Let $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ be m mathematical systems, different two by two. A Smarandache multisystem $\widetilde{\Sigma}$ is a union $\bigcup\limits_{i=1}^{m}\Sigma_i$ with rules $\widetilde{\mathcal{R}}=\bigcup\limits_{i=1}^{m}\mathcal{R}_i$ on $\widetilde{\Sigma}$, denoted by $(\widetilde{\Sigma};\widetilde{\mathcal{R}})$.*

Definition 2.2([9]-[11]) *For an integer $m\geq 1$, let $(\widetilde{\Sigma};\widetilde{\mathcal{R}})$ be a Smarandache multisystem consisting of m mathematical systems $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$. An inherited combinatorial structure $G^L[\widetilde{\Sigma};\widetilde{\mathcal{R}}]$ of $(\widetilde{\Sigma};\widetilde{\mathcal{R}})$ is a labeled topological graph defined following:*

$$V\left(G^L[\widetilde{\Sigma};\widetilde{\mathcal{R}}]\right)=\{\Sigma_1,\Sigma_2,\cdots,\Sigma_m\},$$
$$E\left(G^L[\widetilde{\Sigma};\widetilde{\mathcal{R}}]\right)=\{(\Sigma_i,\Sigma_j)|\Sigma_i\bigcap\Sigma_j\neq\emptyset,\ 1\leq i\neq j\leq m\}\ \textit{with labeling}$$
$$L:\ \Sigma_i\to L(\Sigma_i)=\Sigma_i\quad \textit{and}\quad L:\ (\Sigma_i,\Sigma_j)\to L(\Sigma_i,\Sigma_j)=\Sigma_i\bigcap\Sigma_j$$

for integers $1\leq i\neq j\leq m$.

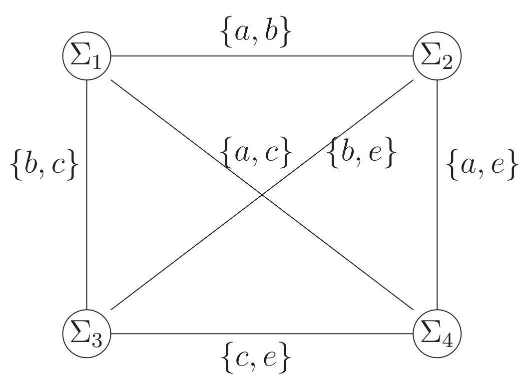

Fig.5

For example, let $\Sigma_1 = \{a,b,c\},\ \Sigma_2 = \{a,b,e\},\ \Sigma_3 = \{b,c,e\},\ \Sigma_4 = \{a,c,e\}$ and $\mathcal{R}_i = \emptyset$ for integers $1 \leq i \leq 4$. The multisystem $(\widetilde{\Sigma};\widetilde{\mathcal{R}})$ with $\widetilde{\Sigma} = \bigcup\limits_{i=1}^{4}\Sigma_i = \{a,b,c,d,e\}$ and $\widetilde{\mathscr{R}} = \emptyset$ is characterized by the labeled topological graph $G^L[\widetilde{\Sigma};\widetilde{\mathcal{R}}]$ shown in Fig.5.

2.1 Exact Labeling

A multiset $\widetilde{S} = \bigcup\limits_{i=1}^{m} S_i$ is *exact* if $S_i = \bigcup\limits_{j=1,j\neq i}^{m}(S_j \bigcap S_i)$ for any integer $1 \leq i \leq m$, i.e., for any vertex $v \in V\left(G^L[\widetilde{\Sigma};\widetilde{\mathcal{R}}]\right)$, $S_v = \bigcup\limits_{u\in N_{G^L}(v)}(S_v \bigcap S_u)$ such as those shown in Fig.5. Clearly, a multiset $\widetilde{S}$ uniquely determines a labeled graph G^L by Definition 2.2, and conversely, if G^L is a labeled graph by sets, we are easily get an exact multiset such that

$$\widetilde{S} = \bigcup_{v\in V(G^L)} S_v \quad \text{with} \quad S_v = \bigcup_{u\in N_{G^L}(v)}\left(S_v \bigcap S_u\right).$$

This concludes the following result.

Theorem 2.3([10]) *A multiset $\widetilde{S}$ uniquely determine a labeled graph $G^L[\widetilde{S}]$, and conversely, any labeled graph G^L uniquely determines an exact multiset $\widetilde{S}$.*

All labeling sets on edges of graph in Fig.4 are 2-sets. Generally, we know the result following.

Theorem 2.4 *For any graph G, if $\varepsilon(G) \leq \begin{pmatrix} |S| \\ k \end{pmatrix}$, there is a labeling L on G with k-set labels on all of its edges, and if $k \geq \Delta(G)$, there is a labeling L on G with l-sets labels on all of its vertices, where $\varepsilon(G), \Delta(G)$ are respectively the size and maximum valency of G.*

Furthermore, if G is an s-regular graph, there exist integers k, l such that there is a labeling L on G with k-set, l-set labels on its vertices and edges, respectively.

Proof Clearly, if $\varepsilon(G) \leq \begin{pmatrix} |S| \\ k \end{pmatrix}$, we can always choose $\varepsilon(G)$ k-sets and then label edges of G, and if $k \geq \Delta(G)$, we are easily find k-sets $S_v,\ v \in V(G)$ such that $S_v \bigcap S_u \neq \emptyset$ if and only if $uv \in E(G)$ because we can allocate different colors on edges of G.

Furthermore, if G is an s-regular graph, we can always allocate $\chi'(G)$ l-sets $\{C_1, C_2, \cdots, C_{\chi'(G)}\}$ with $C_i \bigcap C_j = \emptyset$ for integers $1 \leq i \neq j \leq \chi'(G)$ on edges in $E(G)$ such that colors on adjacent edges are different, and then label vertices v in G by $\bigcup\limits_{u \in N_G(v)} C(vu)$, which is a sl-set. The proof is complete for integer $k = sl$. $\Box$

2.2 Linear Space Labeling

Let $(\widetilde{V}; F)$ be a multilinear space consisting of subspaces V_i, $1 \leq i \leq |G|$ of linear space V over a field F. Such a multilinear space $(\widetilde{V}; F)$ is said to be *exact* if $V_i = \bigoplus\limits_{j \neq i} (V_i \bigcap V_j)$ holds for integers $1 \leq i \leq n$. According to linear algebra, two linear spaces V and V' over a field F are isomorphic if and only if $\dim V = \dim V'$, which enables one to characterize a vector V space by its basis $\mathscr{B}(V)$ and label edges of $G[\widetilde{V}; F]$ by $L: \ V_uV_v \to \mathscr{B}\left(V_u \bigcap V_v\right)$ for $\forall V_uV_v \in E\left(G[\widetilde{V}; F]\right)$ in Definition 2.2 such as those shown in Fig.6.

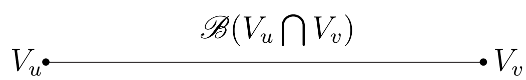

Fig.6

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Clearly, if $(\widetilde{V}; F)$ is exact, i.e., $V_i = \bigoplus\limits_{j \neq i} (V_i \bigcap V_j)$, then it is clear that

$$\mathscr{B}(V) = \bigcup_{VV' \in E\left(G[\widetilde{V};F]\right)} \mathscr{B}(V \bigcap V') \quad \text{and} \quad (\mathscr{B}(V \bigcap V')) \bigcap (\mathscr{B}(V \bigcap V'') = \emptyset$$

by definition. Conversely, if

$$\mathscr{B}(V) = \bigcup_{VV' \in E\left(G[\widetilde{V};F]\right)} \mathscr{B}(V \bigcap V') \quad \text{and} \quad \mathscr{B}\left(V \bigcap V'\right) \bigcap \mathscr{B}\left(V \bigcap V''\right) = \emptyset$$

for $V', V'' \in N_{G[\widetilde{V};F]}(V)$. Notice also that $VV' \in E\left(G[\widetilde{V}; F]\right)$ if and only if $V \bigcap V' \neq \emptyset$, we know that

$$V_i = \bigoplus_{j \neq i} (V_i \bigcap V_j)$$

for integers $1 \leq i \leq n$. This concludes the following result.

Theorem 2.5([10]) *Let $(\widetilde{V};F)$ be a multilinear space with $\widetilde{V} = \bigcup\limits_{i=1}^{n} V_i$. Then it is exact if and only if*

$$\mathscr{B}(V) = \bigcup_{VV' \in E\left(G[\widetilde{V};F]\right)} \mathscr{B}\left(V \bigcap V'\right) \quad \text{and} \quad \mathscr{B}\left(V \bigcap V'\right) \bigcap \mathscr{B}\left(V \bigcap V''\right) = \emptyset$$

for $V', V'' \in N_{G[\widetilde{V};F]}(V)$.

2.3 Euclidean Space Labeling

Let $\mathbf{R}^n$ be a Euclidean space with normal basis $\mathscr{B}(\mathbf{R}^n) = \{\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n\}$, where $\overline{\epsilon}_1 = (1, 0, \cdots, 0)$, $\overline{\epsilon}_2 = (0, 1, 0 \cdots, 0)$, $\cdots$, $\overline{\epsilon}_n = (0, \cdots, 0, 1)$ and let $(\widetilde{V};F)$ be a multilinear space with $\widetilde{V} = \bigcup\limits_{i=1}^{m} \mathbb{R}^{n_i}$ in Theorem 2.5, where $\mathbb{R}^{n_i} \bigcap \mathbb{R}^{n_j} \neq \mathbb{R}^{\min\{i,j\}}$ for integers $1 \leq i \neq j \leq n_m$. If the labeled graph $G[\widetilde{V};F]$ is known, we are easily determine the dimension of $\dim\widetilde{V}$. For example, let G^L be a labeled graph shown in Fig.7. We are easily finding that $\mathscr{B}(\widetilde{\mathbf{R}}) = \{\overline{\epsilon}_1, \overline{\epsilon}_2, \overline{\epsilon}_3, \overline{\epsilon}_4, \overline{\epsilon}_5, \overline{\epsilon}_6\}$, i.e., $\dim\widetilde{V} = 6$.

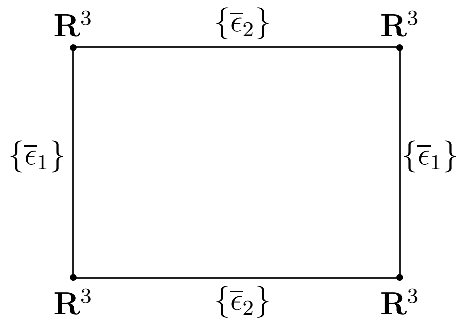

Fig.7

Notice that $\widetilde{V}$ is not exact in Fig.7 because basis $\overline{\epsilon}_3, \overline{\epsilon}_4, \overline{\epsilon}_5, \overline{\epsilon}_6$ are additional. Generally, we are easily know the result by the inclusion-exclusion principle.

Theorem 2.6([8]) *Let G^L be a graph labeled by $\mathbf{R}^{n_{v_1}}$, $\mathbf{R}^{n_{v_2}}$, $\cdots$, $\mathbf{R}^{n_{v_{|G|}}}$. Then*

$$\dim G^L = \sum_{\langle v_i \in V(G) | 1 \leq i \leq s \rangle \in CL_s(G)} (-1)^{s+1} \dim(\mathbf{R}^{n_{v_1}} \bigcap \mathbf{R}^{n_{v_2}} \bigcap \cdots \bigcap \mathbf{R}^{n_{v_s}}),$$

where $CL_s(G)$ consists of all complete graphs of order s in G^L.

However, if edge labelings $\mathscr{B}(\mathbb{R}^{n_u} \bigcap \mathbb{R}^{n_v})$ are not known for $uv \in E\left(G^L\right)$, *can we still determine the dimension* $\dim G^L$? In fact, we only get the maximum or minimum dimension $\dim_{max} G^L$, $\dim_{min} G^L$ in this case.

Theorem 2.7([8]) *Let G^L be a graph labeled by $\mathbf{R}^{n_{v_1}}$, $\mathbf{R}^{n_{v_2}}$, $\cdots$, $\mathbf{R}^{n_{v_{|G|}}}$. Then its maximum dimension* $\dim_{max}G^L$ *is*

$$\dim_{max}G^L = 1 - m + \sum_{v\in V(G^L)} n_v$$

with conditions $\dim(\mathbf{R}^{n_u}\cap\mathbf{R}^{n_v}) = 1$ *for* $\forall uv \in E\left(G^L\right)$.

However, for determining the minimum value $\dim_{min}G^L$ of graph G^L labeled by Euclidean spaces is a difficult problem in general. We only know the following result on labeled complete graphs $K_m, m \geq 3$.

Theorem 2.8([8]) *For any integer $r \geq 2$, let $K_m^L(r)$ be a complete graph K_m labeled by Euclidean space $\mathbb{R}^r$ on its vertices, and there exists an integer s, $0 \leq s \leq r-1$ such that*

$$\binom{r+s-1}{r} < m \leq \binom{r+s}{r}.$$

Then

$$\dim_{min}K_m^L(r) = r + s.$$

Particularly,

$$\dim_{min}K_m^L(3) = \begin{cases} 3, & \text{if} \quad m = 1, \\ 4, & \text{if} \quad 2 \leq m \leq 4, \\ 5, & \text{if} \quad 5 \leq m \leq 10, \\ 2 + \lceil\sqrt{m}\rceil, & \text{if} \quad m \geq 11. \end{cases}$$

All of these results presents a combinatorial model for characterizing things in the space $R^n, n \geq 4$, particularly, the G^L solution of equations in the next subsection.

2.4 G^L-Solution of Equations

Let $\mathbb{R}^m$, $\mathbb{R}^n$ be Euclidean spaces of dimensional $m,\ n \geq 1$ and let $T:\ \mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^m$ be a $\mathbb{C}^k,\ 1 \leq k \leq \infty$ mapping such that $T(\overline{x}_0, \overline{y}_0) = \overline{0}$ for $\overline{x}_0 \in \mathbb{R}^n$, $\overline{y}_0 \in \mathbb{R}^m$ and the $m\times m$ matrix $\partial T^j/\partial y^i(\overline{x}_0, \overline{y}_0)$ is non-singular, i.e.,

$$\det(\frac{\partial T^j}{\partial y^i})\Big|_{(\overline{x}_0,\overline{y}_0)} \neq 0,\ \text{where}\ 1 \leq i, j \leq m.$$

Then the implicit function theorem concludes that there exist opened neighborhoods $V \subset \mathbb{R}^n$ of $\overline{x}_0$, $W \subset \mathbb{R}^m$ of $\overline{y}_0$ and a $\mathbb{C}^k$ function $\phi : V \to W$ such that $T(\overline{x}, \phi(\overline{x})) = \overline{0}$. Thus there always exists solutions $\overline{y}$ for the equation $T(\overline{x}, \overline{y}) = \overline{0}$ in case.

By the implicit function theorem, we can always choose mappings $T_1, T_2, \cdots, T_m$ and subsets $S_{T_i} \subset \mathbb{R}^n$ where $S_{T_i} \neq \emptyset$ such that $T_i : \ S_{T_i} \to 0$ for integers $1 \leq i \leq m$. Consider the system of equations

$$\begin{cases} T_1(x_1, x_2, \cdots, x_n) = 0 \\ T_2(x_1, x_2, \cdots, x_n) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ T_m(x_1, x_2, \cdots, x_n) = 0 \end{cases} \qquad (ES_m)$$

in Euclidean space $\mathbb{R}^n, n \geq 1$. Clearly, the system (ES_m) is non-solvable or not dependent on

$$\bigcap_{i=1}^{m} S_{T_i} = \emptyset \quad \text{or} \quad \neq \emptyset.$$

This fact implies the following interesting result.

Theorem 2.9 *A system (ES_m) of equations is solvable if and only if $\bigcap\limits_{i=1}^{m} S_{T_i} \neq \emptyset$.*

Furthermore, if (ES_m) is solvable, it is obvious that $G^L[ES_m] \simeq K_m^L$. We conclude that (ES_m) is non-solvable if $G^L[ES_m] \not\simeq K_m^L$. Thus the case of solvable is special respect to the general case, non-solvable. However, the understanding on non-solvable case was abandoned in classical for a wrongly thinking, i.e., meaningless for hold on the reality of things.

By Definition 2.2, all spaces $S_{T_i}, 1 \leq i \leq m$ exist for the system (ES_m) and we are easily get a labeled graph $G^L[ES_m]$, which is in fact a combinatorial space, a really geometrical figure in $\mathbb{R}^n$. For example, in cases of linear algebraic equations, we can further determine $G^L[ES_m]$ whatever the system (ES_m) is solvable or not as follows.

A *parallel family* $\mathscr{C}$ of system (ES_m) of linear equations consists of linear equations in (ES_m) such that they are parallel two by two but there are no other linear equations parallel to any one in $\mathscr{C}$. We know a conclusion following on $G^L[ES_m]$ for linear algebraic systems.

Theorem 2.10([12]) *Let (ES_m) be a linear equation system for integers $m, n \geq 1$.*

Then

$$G[ES_m] \simeq K_{n_1,n_2,\cdots,n_s}$$

with $n_1 + n + 2 + \cdots + n_s = m$, *where* $\mathscr{C}_i$ *is the parallel family with* $n_i = |\mathscr{C}_i|$ *for integers* $1 \leq i \leq s$ *in* (ES_m) *and it is non-solvable if* $s \geq 2$.

Similarly, let

$$\dot{X} = A_1 X, \cdots, \dot{X} = A_k X, \cdots, \dot{X} = A_m X \qquad (LDES_m^1)$$

be a linear ordinary differential equation system of first order with

$$A_k = \begin{bmatrix} a_{11}^{[k]} & a_{12}^{[k]} & \cdots & a_{1n}^{[k]} \\ a_{21}^{[k]} & a_{22}^{[k]} & \cdots & a_{2n}^{[k]} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}^{[k]} & a_{n2}^{[k]} & \cdots & a_{nn}^{[k]} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix}$$

where each $a_{ij}^{[k]}$ is a real number for integers $0 \leq k \leq m,\ 1 \leq i, j \leq n$.

Notice that the solution space of the ith in $(LDES_m^1)$ is a linear space S_{A_i}. We know the result following.

Theorem 2.11([13], [14]) *Every linear system* $(LDES_m^1)$ *of homogeneous differential equations uniquely determines a labeled graph* $G^L[LDES_m^1]$, *and conversely, every graph* G^L *labeled by basis of linear spaces uniquely determines a homogeneous differential equation system* $(LDES_m^1)$ *such that* $G^L[LDES_m^1] \simeq G^L$.

For example, let $(LDES_m^1)$ be the system of linear homogeneous differential equations

$$\begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 & (1) \\ \ddot{x} - 5\dot{x} + 6x = 0 & (2) \\ \ddot{x} - 7\dot{x} + 12x = 0 & (3) \\ \ddot{x} - 9\dot{x} + 20x = 0 & (4) \\ \ddot{x} - 11\dot{x} + 30x = 0 & (5) \\ \ddot{x} - 7\dot{x} + 6x = 0 & (6) \end{cases}$$

where $\ddot{x} = \dfrac{d^2x}{dt^2}$ and $\dot{x} = \dfrac{dx}{dt}$. Then the solution basis of equations $(1) - (6)$ are respectively $\{e^t, e^{2t}\}$, $\{e^{2t}, e^{3t}\}$, $\{e^{3t}, e^{4t}\}$, $\{e^{4t}, e^{5t}\}$, $\{e^{5t}, e^{6t}\}$, $\{e^{6t}, e^t\}$ with a labeled graph shown in Fig.8.

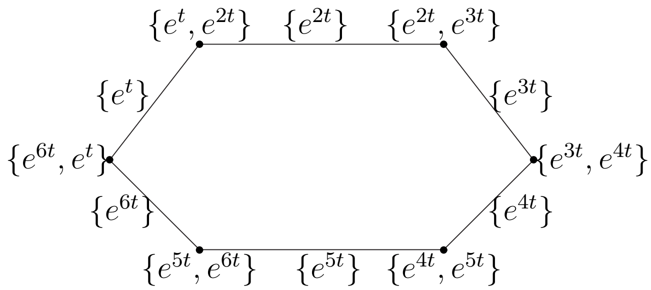

Fig.8

By definition, two linear systems $(LDES_m^1)$, $(LDES_m^1)'$ of homogeneous differential equations are called *combinatorially equivalent* if there is an isomorphism $\varphi : G^L[LDES_m^1] \to G^{L'}[LDES_m^1]$, i.e., there is an isomorphism $\varphi : G^L[LDES_m^1] \to G^{L'}[LDES_m^1]$ of graph and labelings L, L' such that $\varphi L(x) = L'\varphi(x)$ for $\forall x \in V(G^L[LDES_m^1]) \bigcup E(G^L[LDES_m^1])$.

Combinatorially classifying linear systems of differential equations enables one introduces integral labeled graphs. Generally, an integral labeled graph G^{L^I} is such a labeling $L^I : \ G \to \mathbb{Z}^+$ that $L^I(uv) \leq \min\{L^I(u), L^I(v)\}$ for $\forall uv \in E(G)$, and two integral labeled graphs $G_1^{L^I}$ and $G_2^{L'^I}$ are said to be identical, denoted by $G_1^{L^I} = G_2^{L'^I}$ if $G_1 \overset{\varphi}{\simeq} G_2$ and $L^I(x) = L'^I(\varphi(x))$ for any graph isomorphism φ and $\forall x \in V(G_1) \bigcup E(G_1)$. For example, these labeled graphs shown in Fig.9 are all integral on $K_4 - e$, and $G_1^{L_1^I} = G_2^{L_2^I}$ but $G_1^{L_1^I} \neq G_3^{L_3^I}$.

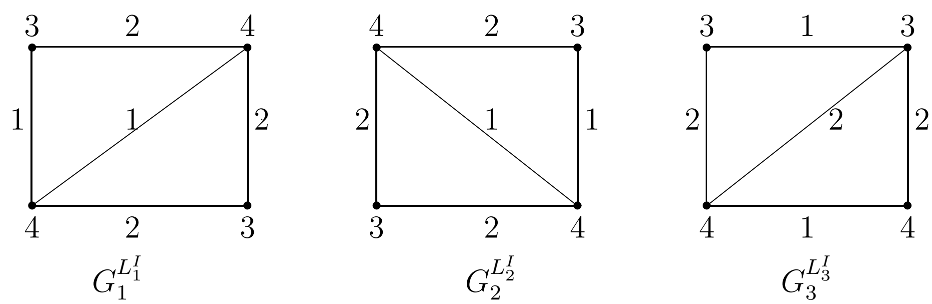

Fig.9

The following result was obtained in [13] first.

Theorem 2.12([13], [14]) *Let $(LDES_m^1)$, $(LDES_m^1)'$ be two linear system of homogeneous differential equations with integral labeled graphs $G^{L^I}[LDES_m^1]$, $G^{L'^I}$. Then $(LDES_m^1) \overset{\varphi}{\simeq} (LDES_m^1)'$ if and only if $G^{L^I}[LDES_m^1] = G^{L'^I}[LDES_m^1]'$.*

§3. Graphical Tensors

As shown in last section, labeled graphs by sets, particularly, geometrical sets such as those of Euclidean spaces $\mathbb{R}^n, n \geq 1$ are useful for holding on things characterized by non-solvable systems of equations. A further question on labeled graphs is

For labeled graphs G_1^L, G_2^L, G_3^L, is there a binary operation $o: \ (G_1^L, G_2^L) \to G_3^L$? And can we established algebra on labeled graphs?

Answer these questions enables us to extend linear space on graphs G hold with conservation laws on its each vertex and establish tensors underlying graph.

3.1 Action Flows

Let $(\mathscr{V}; +, \cdot)$ be a linear space over a field $\mathscr{F}$. An *action* flow $\left(\overrightarrow{G}; L, A\right)$ is an oriented embedded graph $\overrightarrow{G}$ in a topological space $\mathscr{S}$ associated with a mapping $L: (v,u) \to L(v,u)$, 2 end-operators $A_{vu}^+ : L(v,u) \to L^{A_{vu}^+}(v,u)$ and $A_{uv}^+ : L(u,v) \to L^{A_{uv}^+}(u,v)$ on $\mathscr{V}$ with $L(v,u) = -L(u,v)$ and $A_{vu}^+(-L(v,u)) = -L^{A_{vu}^+}(v,u)$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$

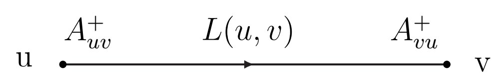

Fig.10

holding with conservation laws

$$\sum_{u \in N_G(v)} L^{A_{vu}^+}(v,u) = \mathbf{0} \quad \text{for} \quad \forall v \in V\left(\overrightarrow{G}\right)$$

such as those shown for vertex v in Fig.11 following

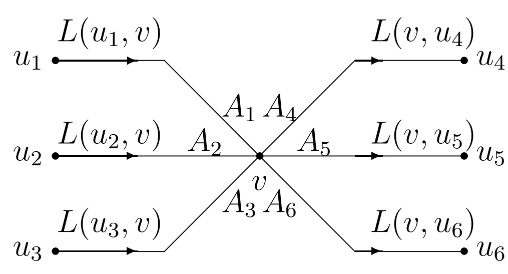

Fig.11

with a conservation law

$$-L^{A_1}(v,u_1)-L^{A_2}(v,u_2)-L^{A_4}(v,u_3)+L^{A_4}(v,u_4)+L^{A_5}(v,u_5)+L^{A_6}(v,u_6)=\mathbf{0},$$

and such a set $\{-L^{A_i}(v,u_i), 1\le i\le 3\}\bigcup\{L^{A_j}, 4\le j\le 6\}$ is called a conservation family at vertex v.

Action flow is a useful model for holding on natural things. It combines the discrete with that of analytical mathematics and therefore, it can help human beings understanding the nature.

For example, let $L:(v,u)\to L(v,u)\in\mathbb{R}^n\times\mathbb{R}^+$ with action operators $A_{vu}^+=a_{vu}\frac{\partial}{\partial t}$ and $a_{vu}:\ \mathbb{R}^n\to\mathbb{R}$ for any edge $(v,u)\in E\left(\overrightarrow{G}\right)$ in Fig.12.

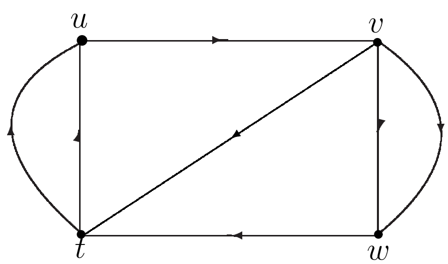

Fig.12

Then the conservation laws are partial differential equations

$$\left\{\begin{array}{l}
a_{tu^1}\frac{\partial L(t,u)^1}{\partial t}+a_{tu^2}\frac{\partial L(t,u)^2}{\partial t}=a_{uv}\frac{\partial L(u,v)}{\partial t}\\
a_{uv}\frac{\partial L(u,v)}{\partial t}=a_{vw^1}\frac{\partial L(v,w)^1}{\partial t}+a_{vw^2}\frac{\partial L(v,w)^2}{\partial t}+a_{vt}\frac{\partial L(v,t)}{\partial t}\\
a_{vw^1}\frac{\partial L(v,w)^1}{\partial t}+a_{vw^2}\frac{\partial L(v,w)^2}{\partial t}=a_{wt}\frac{\partial L(w,t)}{\partial t}\\
a_{wt}\frac{\partial L(w,t)}{\partial t}+a_{vt}\frac{\partial L(v,t)}{\partial t}=a_{tu^1}\frac{\partial L(t,u)^1}{\partial t}+a_{tu^2}\frac{\partial L(t,u)^2}{\partial t}
\end{array}\right.,$$

which maybe solvable or not but characterizes behavior of natural things.

If $A=\mathbf{1}_{\mathscr{V}}$, action flows $\left(\overrightarrow{G};L,\mathbf{1}_{\mathscr{V}}\right)$ are called $\overrightarrow{G}$-flow and denoted by $\overrightarrow{G}^L$ for simplicity. We naturally define

$$\overrightarrow{G}^{L_1}+\overrightarrow{G}^{L_2}=\overrightarrow{G}^{L_1+L_2}\ \text{ and }\ \lambda\cdot\overrightarrow{G}^L=\overrightarrow{G}^{\lambda\cdot L}$$

for $\forall\lambda\in\mathscr{F}$. All $\overrightarrow{G}$-flows $\overrightarrow{G}^{\mathscr{V}}$ on $\overrightarrow{G}$ naturally form a linear space $\left(\overrightarrow{G}^{\mathscr{V}};+,\cdot\right)$ because it hold with:

(1) A field $\mathscr{F}$ of scalars;

(2) A set $\overrightarrow{G}^{\mathscr{V}}$ of objects, called extended vectors;

(3) An operation "+", called extended vector addition, which associates with each pair of vectors $\overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2}$ in $\overrightarrow{G}^{\mathscr{V}}$ a extended vector $\overrightarrow{G}^{L_1+L_2}$ in $\overrightarrow{G}^{\mathscr{V}}$, called the sum of $\overrightarrow{G}^{L_1}$ and $\overrightarrow{G}^{L_2}$, in such a way that

(a) Addition is commutative, $\overrightarrow{G}^{L_1} + \overrightarrow{G}^{L_2} = \overrightarrow{G}^{L_2} + \overrightarrow{G}^{L_1}$;

(b) Addition is associative, $\left(\overrightarrow{G}^{L_1} + \overrightarrow{G}^{L_2}\right) + \overrightarrow{G}^{L_3} = \overrightarrow{G}^{L_1} + \left(\overrightarrow{G}^{L_2} + \overrightarrow{G}^{L_3}\right)$;

(c) There is a unique extended vector $\overrightarrow{G}^{\mathbf{0}}$, i.e., $\mathbf{0}(v,u) = \mathbf{0}$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ in $\overrightarrow{G}^{\mathscr{V}}$, called zero vector such that $\overrightarrow{G}^{L} + \overrightarrow{G}^{\mathbf{0}} = \overrightarrow{G}^{L}$ for all $\overrightarrow{G}^{L}$ in $\overrightarrow{G}^{\mathscr{V}}$;

(d) For each extended vector $\overrightarrow{G}^{L}$ there is a unique extended vector $\overrightarrow{G}^{-L}$ such that $\overrightarrow{G}^{L} + \overrightarrow{G}^{-L} = \overrightarrow{G}^{\mathbf{0}}$ in $\overrightarrow{G}^{\mathscr{V}}$;

(4) An operation "$\cdot$" , called scalar multiplication, which associates with each scalar k in F and an extended vector $\overrightarrow{G}^{L}$ in $\overrightarrow{G}^{\mathscr{V}}$ an extended vector $k \cdot \overrightarrow{G}^{L}$ in $\mathscr{V}$, called the product of k with $\overrightarrow{G}^{L}$, in such a way that

(a) $1 \cdot \overrightarrow{G}^{L} = \overrightarrow{G}^{L}$ for every $\overrightarrow{G}^{L}$ in $\overrightarrow{G}^{\mathscr{V}}$;

(b) $(k_1k_2) \cdot \overrightarrow{G}^{L} = k_1(k_2 \cdot \overrightarrow{G}^{L})$;

(c) $k \cdot (\overrightarrow{G}^{L_1} + \overrightarrow{G}^{L_2}) = k \cdot \overrightarrow{G}^{L_1} + k \cdot \overrightarrow{G}^{L_2}$;

(d) $(k_1 + k_2) \cdot \overrightarrow{G}^{L} = k_1 \cdot \overrightarrow{G}^{L} + k_2 \cdot \overrightarrow{G}^{L}$.

3.2 Dimension of Action Flow Space

Theorem 3.1 *Let $\mathscr{G}$ be all action flows $\left(\overrightarrow{G}; L, A\right)$ with $A \in \mathbf{O}(\mathscr{V})$. Then*

$$\dim\mathscr{G} = (\dim\mathbf{O}(\mathscr{V}) \times \dim\mathscr{V})^{\beta(\overrightarrow{G})}$$

if both $\mathscr{V}$ and $\mathbf{O}(\mathscr{V})$ are finite. Otherwise, $\dim\mathscr{G}$ *is infinite.*

Particularly, if operators $A \in \mathscr{V}^*$, the dual space of $\mathscr{V}$ on graph $\overrightarrow{G}$, then

$$\dim\mathscr{G} = (\dim\mathscr{V})^{2\beta(\overrightarrow{G})},$$

where $\beta\left(\overrightarrow{G}\right) = \varepsilon\left(\overrightarrow{G}\right) - \left|\overrightarrow{G}\right| + 1$ is the Betti number of $\overrightarrow{G}$.

Proof The infinite case is obvious. Without loss of generality, we assume $\overrightarrow{G}$ is connected with dimensions of $\mathscr{V}$ and $\mathbf{O}(\mathscr{V})$ both finite. Let $L(v) = \{L^{A^+_{vu}}(v,u) \in \mathscr{V}$ for some $u \in V\left(\overrightarrow{G}\right)\}$, $v \in V\left(\overrightarrow{G}\right)$ be the conservation families in $\mathscr{V}$ associated with $\left(\overrightarrow{G}; L, A\right)$ such that $L^{A^+_{vu}}(v,u) = -A^+_{uv}(L(u,v))$ and $L(v) \bigcap (-L(u)) =$

$L^{A^+_{vu}}(v,u)$ or $\emptyset$. An edge $(v,u) \in E\left(\overrightarrow{G}\right)$ is *flow freely* or not in $\overrightarrow{G}^{\mathscr{V}}$ if $L^{A^+_{vu}}(v,u)$ can be any vector in $\mathscr{V}$ or not. Notice that $L(v) = \{L^{A^+_{vu}}(v,u) \in \mathscr{V}$ for some $u \in V\left(\overrightarrow{G}\right)\}$, $v \in V\left(\overrightarrow{G}\right)$ are the conservation families associated with action flow $\left(\overrightarrow{G}; L, A\right)$. There is one flow non-freely edges for any vertex in $\overrightarrow{G}$ at least and $\dim\mathscr{G}$ is nothing else but the number of independent vectors $L(v,u)$ and independent end-operators A^+_{vu} on edges in $\overrightarrow{G}$ which can be chosen freely in $\mathscr{V}$.

We claim that all flow non-freely edges form a connected subgraph T in $\overrightarrow{G}$. If not, there are two components $C_1(T)$ and $C_2(T)$ in T such as those shown in Fig.13.

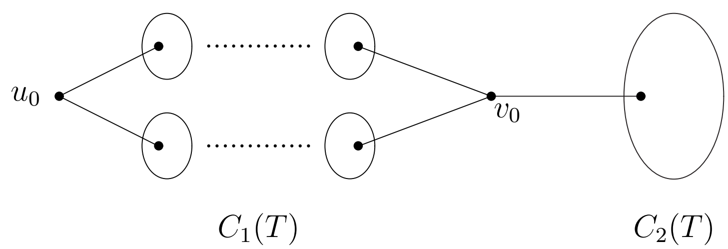

Fig.13

In this case, all edges between $C_1(T)$ and $C_2(T)$ are flow freely in $\overrightarrow{G}$. Let v_0 be such a vertex in $C_1(T)$ adjacent to a vertex in $C_2(T)$. Beginning from the vertex v_0 in $C_1(T)$, we choose vectors on edges in

$$E_G\left(v_0, N_G(v_0)\right) \bigcap \left\langle C_1(T)\right\rangle_G,$$
$$E_G\left(N_G(v_0) \setminus \{v_0\}, N_G(N_G(v_0)) \setminus N_G(v_0)\right) \bigcap \left\langle C_1(T)\right\rangle_G,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

in $\left\langle C_1(T)\right\rangle_G$ by conservation laws, and then finally arrive at a vertex $u_0 \in V(C_2(T))$ such that all flows from $V(C_1(T)) \setminus \{u_0\}$ to u_0 are fixed by conservation laws of vertices $N_G(u_0)$, which result in that there are no conservation law of flows on the vertex u_0, a contradiction. Hence, all flow freely edges form a connected subgraph in $\overrightarrow{G}$. Hence, we get that

$$\begin{aligned}\dim\mathscr{G} &\leq \dim\mathbf{O}(\mathscr{V}))^{|E(\overrightarrow{G})-E(T)|} \times (\dim\mathscr{V})^{|E(\overrightarrow{G})-E(T)|} \\ &= (\dim\mathbf{O}(\mathscr{V}) \times \dim\mathscr{V})^{\beta(\overrightarrow{G})}.\end{aligned}$$

We can indeed determine a flow non-freely tree T in $\overrightarrow{G}$ by programming following:

STEP 1. Define $X_1 = \{v_1\}$ for $\forall v_1 \in V\left(\overrightarrow{G}\right)$;

STEP 2. If $V\left(\overrightarrow{G}\right) \setminus X_1 \neq \emptyset$, choose $v_2 \in N_G(v_1) \setminus X_1$ and let (v_1, v_2) be a flow non-freely edge by conservation law on v_1 and define $X_2 = \{v_1, v_2\}$. Otherwise, $T = v_0$.

STEP 3. If $V\left(\overrightarrow{G}\right) \setminus X_2 \neq \emptyset$, choose $v_3 \in N_G(X_1) \setminus X_2$. Without loss of generality, assume v_3 adjacent with v_2 and let (v_2, v_3) be a flow non-freely edge by conservation law on v_2 with $X_3 = \{v_1, v_2, v_3\}$. Otherwise, $T = v_1v_2$.

STEP 4. For any integer $k \geq 2$, if X_k has been defined and $V\left(\overrightarrow{G}\right) \setminus X_k \neq \emptyset$, choose $v_{k+1} \in N_G(X_k) \setminus X_k$. Assume v_{k_1} adjacent with $v^k \in X_k$ and let (v^k, v_{k+1}) be a flow non-freely edge by conservation law on v^k with $X_{k+1} = X_k \bigcup \{v_{k+1}\}$. Otherwise, T is the flow non-freely tree spanned by $\langle X_k \rangle$ in $\overrightarrow{G}$.

STEP 5. The procedure is ended if $X_{|\overrightarrow{G}|}$ has been defined which enable one get a spanning flow non-freely tree T of $\overrightarrow{G}$.

Clearly, all edges in $E\left(\overrightarrow{G}\right) \setminus E(T)$ are flow freely in $\mathscr{V}$. We therefore know

$$\begin{aligned}
\dim\mathscr{G} &\geq (\dim\mathbf{O}(\mathscr{V}))^{\varepsilon\left(\overrightarrow{G}\right)-\varepsilon(T)} \times (\dim\mathscr{V})^{\varepsilon\left(\overrightarrow{G}\right)-\varepsilon(T)} \\
&= (\dim O(\mathscr{V}) \times \dim\mathscr{V})^{\varepsilon\left(\overrightarrow{G}\right)-\left|\overrightarrow{G}\right|+1} = (\dim\mathbf{O}(\mathscr{V}) \times \dim\mathscr{V})^{2\beta(\overrightarrow{G})}.
\end{aligned}$$

Thus,

$$\dim\mathscr{G} = (\dim\mathbf{O}(\mathscr{V}) \times \dim\mathscr{V})^{2\beta(\overrightarrow{G})}. \tag{3.1}$$

If operators $A \in \mathscr{V}^*$, $\dim\mathscr{V}^* = \dim\mathscr{V}$. We are easily get

$$\dim\mathscr{G} = (\dim\mathscr{V})^{2\beta(\overrightarrow{G})}$$

by the equation (3.1). This completes the proof. $\square$

Particularly, for action flows $\left(\overrightarrow{G}; L, \mathbf{1}_{\mathscr{V}}\right)$, i.e., $\overrightarrow{G}$-flow space we have a conclusion on its dimension following

Corollary 3.2 $\dim\overrightarrow{G}^{\mathscr{V}} = (\dim\mathscr{V})^{\beta(\overrightarrow{G})}$ *if $\mathscr{V}$ is finite. Otherwise,* $\dim\mathscr{H}$ *is infinite.*

3.3 Graphical Tensors

Definition 3.3 *Let $\left(\overrightarrow{G}_1; L_1, A_1\right)$ and $\left(\overrightarrow{G}_2; L_2, A_2\right)$ be action flows on linear space $\mathscr{V}$. Their tensor product $\left(\overrightarrow{G}_1; L_1, A_1\right) \otimes \left(\overrightarrow{G}_2; L_2, A_2\right)$ is defined on graph $\overrightarrow{G}_1 \otimes \overrightarrow{G}_2$*

with mapping

$$L: \ ((v_1,u_1),(v_2,u_2)) \to (L_1(v_1,u_1), L_2(v_2,u_2))$$

on edge $((v_1,u_1),(v_2,u_2)) \in E\left(\overrightarrow{G}_1 \otimes \overrightarrow{G}_2\right)$ *and end-operators* $A^+_{(v_1,u_1)(v_2,u_2)} = A^+_{v_1u_1} \otimes A^+_{v_2u_2}$, $A^+_{(v_2,u_2)(v_1,u_1)} = A^+_{u_1v_1} \otimes A^+_{u_2v_2}$, *such as those shown in Fig.14.*

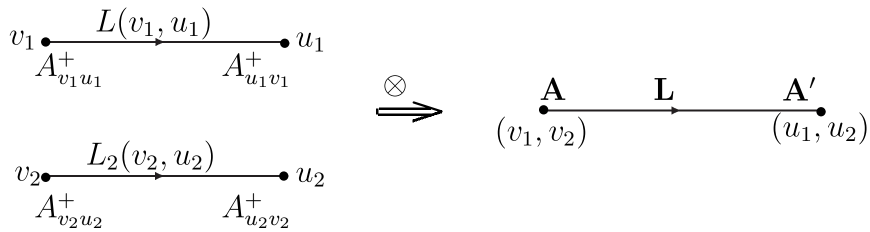

Fig.14

with $\mathbf{L} = (L_1(v_1,u_1), L_2(v_2,u_2))$ *and* $\mathbf{A} = A^+_{v_1u_1} \otimes A^+_{v_2u_2}$, $\mathbf{A}' = A^+_{u_1v_1} \otimes A^+_{u_2v_2}$, *where* $\overrightarrow{G}_1 \otimes \overrightarrow{G}_2$ *is the tensor product of* $\overrightarrow{G}_1$ *and* $\overrightarrow{G}_2$ *with*

$$V\left(\overrightarrow{G}_1 \otimes \overrightarrow{G}_2\right) = V\left(\overrightarrow{G}_1\right) \times V\left(\overrightarrow{G}_2\right)$$

and

$$\begin{aligned} E\left(\overrightarrow{G}_1 \otimes \overrightarrow{G}_2\right) &= \{((v_1,v_2),(u_1,u_2)) \mid if\ and\ only\ if \\ &\qquad (v_1,u_1) \in E\left(\overrightarrow{G}_1\right)\ and\ (v_2,u_2) \in E\left(\overrightarrow{G}_2\right)\} \end{aligned}$$

with an orientation $O^+: \ (v_1,v_2) \to (u_1,u_2)$ on $((v_1,v_2),(u_1,u_2)) \in E\left(\overrightarrow{G}_1 \otimes \overrightarrow{G}_2\right)$.

Indeed, $\left(\overrightarrow{G}_1; L_1, A_1\right) \otimes \left(\overrightarrow{G}_2; L_2, A_2\right)$ is an action flow with conservation laws on each vertex in $\overrightarrow{G}_1 \otimes \overrightarrow{G}_2$ because

$$\begin{aligned} &\sum_{(u_1,u_2)\in N_{G_1\otimes G_2}(v_1,v_2)} A^+_{v_1u_1} \otimes A^+_{v_2u_2}\left(L_1(v_1,u_1), L_2(v_2,u_2)\right) \\ =& \sum_{(u_1,u_2)\in N_{G_1\otimes G_2}(v_1,v_2)} A^+_{v_1u_1}\left(L_1(v_1,u_1)\right) A^+_{v_2u_2}\left(L_2(v_2,u_2)\right) \\ =& \left(\sum_{u_1\in N_{G_1}(v_1)} \left(L_1(v_1,u_1)\right)^{A^+_{v_1u_1}}\right) \times \left(\sum_{u_2\in N_{G_2}(v_2)} \left(L_2(v_2,u_2)\right)^{A^+_{v_2u_2}}\right) = \mathbf{0} \end{aligned}$$

for $\forall (v_1,v_2) \in V\left(\overrightarrow{G}_1 \otimes \overrightarrow{G}_2\right)$ by definition.

Theorem 3.4 *The tensor operation is associative, i.e.,*

$$\begin{aligned} &\left(\left(\overrightarrow{G}_1; L_1, A_1\right) \bigotimes \left(\overrightarrow{G}_2; L_2, A_2\right)\right) \bigotimes \left(\overrightarrow{G}_3; L_3, A_3\right) \\ &\qquad = \left(\overrightarrow{G}_1; L_1, A_1\right) \bigotimes \left(\left(\overrightarrow{G}_2; L_2, A_2\right) \bigotimes \left(\overrightarrow{G}_3; L_3, A_3\right)\right). \end{aligned}$$

Proof By definition, $\left(\overrightarrow{G}_1 \otimes \overrightarrow{G}_2\right) \otimes \overrightarrow{G}_3 = \overrightarrow{G}_1 \otimes \left(\overrightarrow{G}_2 \otimes \overrightarrow{G}_3\right)$. Let $(v_1, u_1) \in E\left(\overrightarrow{G}_1\right)$, $(v_2, u_2) \in E\left(\overrightarrow{G}_2\right)$ and $(v_3, u_3) \in E\left(\overrightarrow{G}_3\right)$. Then, $((v_1, v_2, v_3), (u_1, u_2, u_3)) \in E\left(\overrightarrow{G}_1 \otimes \overrightarrow{G}_2 \otimes \overrightarrow{G}_3\right)$ with flows $(L_1(v_1, u_1), L_2(v_2, u_2), L_3(v_3, u_3))$, and end-operators $\left(A^+_{v_1u_1} \otimes A^+_{v_2,u_2}\right) \otimes A^+_{v_3u_3}$ in $\left(\left(\overrightarrow{G}_1; L_1, A_1\right) \otimes \left(\overrightarrow{G}_2; L_2, A_2\right)\right) \otimes \left(\overrightarrow{G}_3; L_3, A_3\right)$ but $A^+_{v_1u_1} \otimes \left(A^+_{v_2,u_2} \otimes A^+_{v_3u_3}\right)$ in $\left(\overrightarrow{G}_1; L_1, A_1\right) \otimes \left(\left(\overrightarrow{G}_2; L_2, A_2\right) \otimes \left(\overrightarrow{G}_3; L_3, A_3\right)\right)$ on the vertex (v_1, v_2, v_3). However,

$$\left(A^+_{v_1u_1} \otimes A^+_{v_2,u_2}\right) \otimes A^+_{v_3u_3} = A^+_{v_1u_1} \otimes \left(A^+_{v_2,u_2} \otimes A^+_{v_3u_3}\right)$$

for tensors. This completes the proof. □

Theorem 3.4 enables one to define the product $\bigotimes\limits_{i=1}^{n} \left(\overrightarrow{G}_i; L_i, A_i\right)$. Clearly, if $\left\{\overrightarrow{G}_i^{L_{i1}}, \overrightarrow{G}_i^{L_{i2}}, \cdots, \overrightarrow{G}_i^{L_{in_i}}\right\}$ is a base of $\overrightarrow{G}_i^{\mathscr{V}}$, then $\overrightarrow{G}_1^{L_{1i_1}} \otimes \overrightarrow{G}_2^{L_{2i_2}} \otimes \cdots \otimes \overrightarrow{G}_n^{L_{ni_n}}$, $1 \leq i_j \leq n_i, 1 \leq i \leq n$ form a base of $\overrightarrow{G}_1^{\mathscr{V}_1} \otimes \overrightarrow{G}_2^{\mathscr{V}_2} \otimes \cdots \otimes \overrightarrow{G}_n^{\mathscr{V}_n}$. This implies the following result by Theorem 3.1 and Corollary 3.2.

Theorem 3.5 $$\dim \left(\bigotimes_{i=1}^{m} \left(\overrightarrow{G}_i; L_i, A_i\right)\right) = \prod_{i=1}^{m} \dim \mathscr{V}_i^{2\beta(\overrightarrow{G}_i)}.$$

Particularly, $\dim \left(\bigotimes\limits_{i=1}^{m} \overrightarrow{G}_i^{\mathscr{V}_i}\right) = \prod\limits_{i=1}^{m} \dim \mathscr{V}_i^{\beta(\overrightarrow{G}_i)}$ *and furthermore, if* $\mathscr{V}_i = \mathscr{V}$ *for integers* $1 \leq i \leq m$, *then*

$$\dim \left(\bigotimes_{i=1}^{m} \overrightarrow{G}_i^{\mathscr{V}}\right) = \dim \mathscr{V}^{\sum\limits_{i=1}^{m} \beta(\overrightarrow{G}_i)},$$

and if each $\overrightarrow{G}_i$ *is a circuit* $\overrightarrow{C}_{n_i}$, *or each* $\overrightarrow{G}_i$ *is a bouquet* $\overrightarrow{B}_{n_i}$ *for integers* $1 \leq i \leq m$, *then*

$$\dim \left(\bigotimes_{i=1}^{n} \overrightarrow{G}_i^{\mathscr{V}}\right) = \dim \mathscr{V}^{n} \quad \textit{or} \quad \dim \left(\bigotimes_{i=1}^{n} \overrightarrow{G}_i^{\mathscr{V}}\right) = \dim \mathscr{V}^{n_1+n_2+\cdot+n_m}.$$

§4. Banach $\overrightarrow{G}$-Flow Spaces

The Banach and Hilbert spaces are linear space $\mathscr{V}$ over a field $\mathbb{R}$ or $\mathbb{C}$ respectively equipped with a complete norm $\|\cdot\|$ or inner product $\langle\cdot,\cdot\rangle$, i.e., for every Cauchy sequence $\{x_n\}$ in $\mathscr{V}$, there exists an element x in $\mathscr{V}$ such that

$$\lim_{n\to\infty} \|x_n - x\|_{\mathscr{V}} = 0 \quad \text{or} \quad \lim_{n\to\infty} \langle x_n - x, x_n - x\rangle_{\mathscr{V}} = 0.$$

We extend Banach or Hilbert spaces on graph $\overrightarrow{G}$ by a kind of edge labeled graphs, i.e., $\overrightarrow{G}$-flows in this section.

4.1 Banach $\overrightarrow{G}$-Flow Spaces

Let $\mathscr{V}$ be a Banach space over a field $\mathscr{F}$ with $\mathscr{F} = \mathbb{R}$ or $\mathbb{C}$. For any $\overrightarrow{G}$-flow $\overrightarrow{G}^L \in \overrightarrow{G}^{\mathscr{V}}$, define

$$\left\|\overrightarrow{G}^L\right\| = \sum_{(v,u)\in E\left(\overrightarrow{G}\right)} \|L(v,u)\|,$$

where $\|L(v,u)\|$ is the norm of $L(v,u)$ in $\mathscr{V}$. Then it is easily to check that

(1) $\left\|\overrightarrow{G}^L\right\| \geq 0$ and $\left\|\overrightarrow{G}^L\right\| = 0$ if and only if $\overrightarrow{G}^L = \overrightarrow{G}^{\mathbf{0}}$.

(2) $\left\|\overrightarrow{G}^{\xi L}\right\| = \xi \left\|\overrightarrow{G}^L\right\|$ for any scalar ξ.

(3) $\left\|\overrightarrow{G}^{L_1} + \overrightarrow{G}^{L_2}\right\| \leq \left\|\overrightarrow{G}^{L_1}\right\| + \left\|\overrightarrow{G}^{L_2}\right\|$.

Whence, $\|\cdot\|$ is a norm on linear space $\overrightarrow{G}^{\mathscr{V}}$. Furthermore, if $\mathscr{V}$ is an inner space, define

$$\left\langle\overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2}\right\rangle = \sum_{(u,v)\in E\left(\overrightarrow{G}\right)} \langle L_1(v,u), L_2(v,u)\rangle.$$

Then

(4) $\left\langle\overrightarrow{G}^L, \overrightarrow{G}^L\right\rangle \geq 0$ and $\left\langle\overrightarrow{G}^L, \overrightarrow{G}^L\right\rangle = 0$ if and only if $L(v,u) = \mathbf{0}$ for $\forall (v,u) \in E(\overrightarrow{G})$, i.e., $\overrightarrow{G}^L = \overrightarrow{G}^{\mathbf{0}}$.

(5) $\left\langle\overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2}\right\rangle = \overline{\left\langle\overrightarrow{G}^{L_2}, \overrightarrow{G}^{L_1}\right\rangle}$ for $\forall \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \in \overrightarrow{G}^{\mathscr{V}}$.

(6) For $\overrightarrow{G}^L, \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \in \overrightarrow{G}^{\mathscr{V}}$, there is

$$\begin{aligned}\left\langle\lambda\overrightarrow{G}^{L_1}\right. &+ \left.\mu\overrightarrow{G}^{L_2}, \overrightarrow{G}^L\right\rangle \\ &= \lambda\left\langle\overrightarrow{G}^{L_1}, \overrightarrow{G}^L\right\rangle + \mu\left\langle\overrightarrow{G}^{L_2}, \overrightarrow{G}^L\right\rangle.\end{aligned}$$

Thus, $\overrightarrow{G}^{\mathscr{V}}$ is an inner space. As the usual, let

$$\left\|\overrightarrow{G}^L\right\| = \sqrt{\left\langle\overrightarrow{G}^L, \overrightarrow{G}^L\right\rangle}$$

for $\overrightarrow{G}^L \in \overrightarrow{G}^{\mathscr{V}}$. Then it is also a normed space.

If the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ are complete, then $\left\|\overrightarrow{G}^L\right\|$ and $\left\langle \overrightarrow{G}^L, \overrightarrow{G}^L \right\rangle$ are too also, i.e., any Cauchy sequence in $\overrightarrow{G}^{\mathscr{V}}$ is converges. In fact, let $\left\{\overrightarrow{G}^{L_n}\right\}$ be a Cauchy sequence in $\overrightarrow{G}^{\mathscr{V}}$. Then for any number $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that

$$\left\|\overrightarrow{G}^{L_n} - \overrightarrow{G}^{L_m}\right\| < \varepsilon$$

if $n, m \geq N(\varepsilon)$. By definition,

$$\|L_n(v,u) - L_m(v,u)\| \leq \left\|\overrightarrow{G}^{L_n} - \overrightarrow{G}^{L_m}\right\| < \varepsilon$$

i.e., $\{L_n(v,u)\}$ is also a Cauchy sequence for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$, which is converges in $\mathscr{V}$ by definition.

Now let $L(v,u) = \lim\limits_{n\to\infty} L_n(v,u)$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$. Clearly,

$$\lim_{n\to\infty} \overrightarrow{G}^{L_n} = \overrightarrow{G}^L.$$

Even so, we are needed to show that $\overrightarrow{G}^L \in \overrightarrow{G}^{\mathscr{V}}$. By definition,

$$\sum_{u \in N_G(v)} L_n(v,u) = \mathbf{0}, \quad v \in V\left(\overrightarrow{G}\right)$$

for any integer $n \geq 1$. If $n \to \infty$ on both sides, we are easily knowing that

$$\begin{aligned}
\lim_{n\to\infty}\left(\sum_{u \in N_G(v)} L_n(v,u)\right) &= \sum_{u \in N_G(v)} \lim_{n\to\infty} L_n(v,v) \\
&= \sum_{u \in N_G(v)} L(v,u) = \mathbf{0}.
\end{aligned}$$

Thus, $\overrightarrow{G}^L \in \overrightarrow{G}^{\mathscr{V}}$, which implies that the norm is complete, which can be also applied to the case of Hilbert space. Thus we get the following result.

Theorem 4.1([18], [22]) *For any graph $\overrightarrow{G}$, $\overrightarrow{G}^{\mathscr{V}}$ is a Banach space, and furthermore, if $\mathscr{V}$ is a Hilbert space, $\overrightarrow{G}^{\mathscr{V}}$ is a Hilbert space also.*

An operator $\mathbf{T} : \overrightarrow{G}^{\mathscr{V}} \to \overrightarrow{G}^{\mathscr{V}}$ is a *contractor* if

$$\left\|\mathbf{T}\left(\overrightarrow{G}^{L_1}\right) - \mathbf{T}\left(\overrightarrow{G}^{L_2}\right)\right\| \leq \xi \left\|\overrightarrow{G}^{L_1} - \overrightarrow{G}^{L_2})\right\|$$

for $\forall \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_1} \in \overrightarrow{G}^{\mathscr{V}}$ with $\xi \in [0, 1)$. The following result generalizes the fixed point theorem of Banach to Banach $\overrightarrow{G}$-flow space.

Theorem 4.2([18]) *Let $\mathbf{T} : \overrightarrow{G}^{\mathscr{V}} \to \overrightarrow{G}^{\mathscr{V}}$ be a contractor. Then there is a uniquely G-flow $\overrightarrow{G}^{L} \in \overrightarrow{G}^{\mathscr{V}}$ such that $\mathbf{T}\left(\overrightarrow{G}^{L}\right) = \overrightarrow{G}^{L}$.*

An operator $\mathbf{T} : \overrightarrow{G}^{\mathscr{V}} \to \overrightarrow{G}^{\mathscr{V}}$ is *linear* if

$$\mathbf{T}\left(\lambda \overrightarrow{G}^{L_1} + \mu \overrightarrow{G}^{L_2}\right) = \lambda \mathbf{T}\left(\overrightarrow{G}^{L_1}\right) + \mu \mathbf{T}\left(\overrightarrow{G}^{L_2}\right)$$

for $\forall \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \in \overrightarrow{G}^{\mathscr{V}}$ and $\lambda, \mu \in \mathscr{F}$. The following result generalizes the representation theorem of Fréchet and Riesz on linear continuous functionals of Hilbert space to Hilbert $\overrightarrow{G}$-flow space $\overrightarrow{G}^{\mathscr{V}}$.

Theorem 4.3([18], [22]) *Let $\mathbf{T} : \overrightarrow{G}^{\mathscr{V}} \to \mathbb{C}$ be a linear continuous functional. Then there is a unique $\overrightarrow{G}^{\widehat{L}} \in \overrightarrow{G}^{\mathscr{V}}$ such that $\mathbf{T}\left(\overrightarrow{G}^{L}\right) = \left\langle \overrightarrow{G}^{L}, \overrightarrow{G}^{\widehat{L}} \right\rangle$ for $\forall \overrightarrow{G}^{L} \in \overrightarrow{G}^{\mathscr{V}}$.*

4.3 Examples of Linear Operator on Banach $\overrightarrow{G}$-Flow Spaces

Let $\mathscr{H}$ be a Hilbert space consisting of measurable functions $f(x_1, x_2, \cdots, x_n)$ on a set

$$\Delta = \left\{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n | a_i \leq x_i \leq b_i, 1 \leq i \leq n\right\},$$

which is a functional space $L^2[\Delta]$, with inner product

$$\langle f(\mathbf{x}), g(\mathbf{x})\rangle = \int_{\Delta} \overline{f(\mathbf{x})} g(\mathbf{x}) d\mathbf{x} \quad \text{for} \quad f(\mathbf{x}), g(\mathbf{x}) \in L^2[\Delta],$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\overrightarrow{G}$ an oriented graph embedded in a topological space. As we shown in last section, we can extended $\mathscr{H}$ on graph $\overrightarrow{G}$ to get Hilbert $\overrightarrow{G}$-flow space $\overrightarrow{G}^{\mathscr{H}}$.

The *differential* and *integral operators*

$$D = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad \int_{\Delta}$$

on $\mathscr{H}$ are extended respectively by

$$D\overrightarrow{G}^{L} = \overrightarrow{G}^{DL(u^v)}$$

and

$$\int_{\Delta} \overrightarrow{G}^{L} \;=\; \int_{\Delta} K(\mathbf{x},\mathbf{y})\overrightarrow{G}^{L[\mathbf{y}]}d\mathbf{y} = \overrightarrow{G}^{\int_{\Delta} K(\mathbf{x},\mathbf{y})L(u^{v})[\mathbf{y}]d\mathbf{y}},$$

for $\forall(u,v)\in E\left(\overrightarrow{G}\right)$, where a_i, $\dfrac{\partial a_i}{\partial x_j}\in\mathbb{C}^0(\Delta)$ for integers $1\le i,j\le n$ and $K(\mathbf{x},\mathbf{y}):\Delta\times\Delta\to\mathbb{C}\in L^2(\Delta\times\Delta,\mathbb{C})$ with

$$\int_{\Delta\times\Delta} K(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y}<\infty.$$

Clearly,

$$\begin{aligned}
&D\left(\lambda\overrightarrow{G}^{L_1(v,u)}+\mu\overrightarrow{G}^{L_2(v,u)}\right)=D\left(\overrightarrow{G}^{\lambda L_1(v,u)+\mu L_2(v,u)}\right)\\
&=\overrightarrow{G}^{D(\lambda L_1(v,u)+\mu L_2(v,u))}=\overrightarrow{G}^{D(\lambda L_1(v,u))+D(\mu L_2(v,u))}\\
&=\overrightarrow{G}^{D(\lambda L_1(v,u))}+\overrightarrow{G}^{D(\mu L_2(v,u))}=D\left(\overrightarrow{G}^{(\lambda L_1(v,u))}+\overrightarrow{G}^{(\mu L_2(v,u))}\right)\\
&=\lambda D\left(\overrightarrow{G}^{L_1(v,u)}\right)+D\left(\mu\overrightarrow{G}^{L_2(v,u)}\right)
\end{aligned}$$

for $\overrightarrow{G}^{L_1},\overrightarrow{G}^{L_2}\in\overrightarrow{G}^{\mathscr{H}}$ and $\lambda,\mu\in\mathbb{R}$, i.e.,

$$D\left(\lambda\overrightarrow{G}^{L_1}+\mu\overrightarrow{G}^{L_2}\right)=\lambda D\overrightarrow{G}^{L_1}+\mu D\overrightarrow{G}^{L_2}.$$

Similarly, we can show also that

$$\int_{\Delta}\left(\lambda\overrightarrow{G}^{L_1}+\mu\overrightarrow{G}^{L_2}\right)=\lambda\int_{\Delta}\overrightarrow{G}^{L_1}+\mu\int_{\Delta}\overrightarrow{G}^{L_2},$$

i.e., the operators D and $\displaystyle\int_{\Delta}$ are linear on $\overrightarrow{G}^{\mathscr{H}}$.

Notice that $\overrightarrow{G}^{L(v,u)}\in\overrightarrow{G}^{\mathscr{H}}$, there must be

$$\sum_{u\in N_G(v)} L(v,u)=\mathbf{0}\ \text{ for }\ \forall v\in V\left(\overrightarrow{G}\right),$$

We therefore know that

$$\mathbf{0}=D\left(\sum_{u\in N_G(v)}L(v,u)\right)=\sum_{u\in N_G(v)}DL(v,u)$$

and

$$\mathbf{0}=\int_{\Delta}\left(\sum_{u\in N_G(v)}L(v,u)\right)=\sum_{u\in N_G(v)}\int_{\Delta}L(v,u)$$

for $\forall v \in V\left(\overrightarrow{G}\right)$. Consequently,

$$D: \ \overrightarrow{G}^{\mathscr{H}} \to \overrightarrow{G}^{\mathscr{H}}, \ \text{and} \ \int_{\Delta}: \ \overrightarrow{G}^{\mathscr{H}} \to \overrightarrow{G}^{\mathscr{H}}$$

are linear operators on $\overrightarrow{G}^{H}$.

For example, let $f(t) = t$, $g(t) = e^t$, $K(t,\tau) = 1$ for $\Delta = [0,1]$ and let $\overrightarrow{G}^{L}$ be the $\overrightarrow{G}$-flow shown on the left side in Fig.15. Calculation shows that $Df = 1, Dg = e^t$,

$$\int_0^1 K(t,\tau)f(\tau)d\tau = \int_0^1 \tau d\tau = \frac{1}{4}, \quad \int_0^1 K(t,\tau)g(\tau)d\tau = \int_0^1 e^{\tau}d\tau = e-1$$

and the actions $D\overrightarrow{G}^{L}$, $\int_{[0,1]} \overrightarrow{G}^{L}$ are shown on the right in Fig.15.

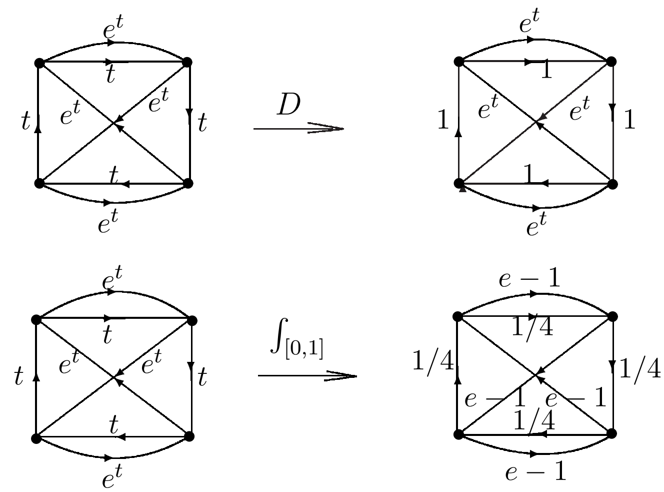

Fig.15

Particularly, the Cauchy problem on heat equation

$$\frac{\partial u}{\partial t} = c^2 \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

is solvable in $\mathbb{R}^n \times \mathbb{R}$ if $u(\mathbf{x}, t_0) = \varphi(\mathbf{x})$ is continuous and bounded in $\mathbb{R}^n$, and c is a non-zero constant in $\mathbb{R}$. Certainly, we can also consider the Cauchy problem in $\overrightarrow{G}^{\mathscr{H}}$, i.e.,

$$\frac{\partial X}{\partial t} = c^2 \sum_{i=1}^{n} \frac{\partial^2 X}{\partial x_i^2}$$

with initial values $X|_{t=t_0}$, and get the following result.

Theorem 4.4([18]) *For* $\forall \overrightarrow{G}^{L'} \in \overrightarrow{G}^{\mathbb{R}^n\times\mathbb{R}}$ *and a non-zero constant* c *in* $\mathbb{R}$*, the Cauchy problems on differential equations*

$$\frac{\partial X}{\partial t} = c^2 \sum_{i=1}^{n} \frac{\partial^2 X}{\partial x_i^2}$$

with initial value $X|_{t=t_0} = \overrightarrow{G}^{L'} \in \overrightarrow{G}^{\mathbb{R}^n\times\mathbb{R}}$ *is solvable in* $\overrightarrow{G}^{\mathbb{R}^n\times\mathbb{R}}$ *if* $L'(v,u)$ *is continuous and bounded in* $\mathbb{R}^n$ *for* $\forall (v,u) \in E\left(\overrightarrow{G}\right)$*.*

Fortunately, if the graph $\overrightarrow{G}$ is prescribed with special structures for instance the circuit decomposition, we can always solve the Cauchy problem on an equation in Hilbert $\overrightarrow{G}$-flow space $\overrightarrow{G}^{\mathscr{H}}$ if it is solvable is $\mathscr{H}$.

Theorem 4.5([18], [22]) If the graph $\overrightarrow{G}$ is strong-connected with circuit decomposition

$$\overrightarrow{G} = \bigcup_{i=1}^{l} \overrightarrow{C}_i$$

such that $L(v,u) = L_i(\mathbf{x})$ for $\forall (v,u) \in E\left(\overrightarrow{C}_i\right)$, $1 \le i \le l$ and the Cauchy problem

$$\left\{\begin{array}{l} \mathscr{F}_i\left(\mathbf{x}, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots\right) = 0 \\ u|_{\mathbf{x}_0} = L_i(\mathbf{x}) \end{array}\right.$$

is solvable in a Hilbert space $\mathscr{H}$ on domain $\Delta \subset \mathbb{R}^n$ for integers $1 \le i \le l$, then the Cauchy problem

$$\left\{\begin{array}{l} \mathscr{F}_i\left(\mathbf{x}, X, X_{x_1}, \cdots, X_{x_n}, X_{x_1x_2}, \cdots\right) = 0 \\ X|_{\mathbf{x}_0} = \overrightarrow{G}^{L} \end{array}\right.$$

such that $L(v,u) = L_i(\mathbf{x})$ for $\forall (v,u) \in X\left(\overrightarrow{C}_i\right)$ is solvable for $X \in \overrightarrow{G}^{\mathscr{H}}$.

§5. **Applications**

Notice that labeled graph combines the discrete with that of analytic mathematics. This character implies that it can be used as a model for living things in the nature and contributes to system control, gravitational field, interaction fields, economics, traffic flows, ecology, epidemiology and other sciences. But we only introduce 2 applications of labeled graphs for limitation of the space, i.e., global stability and spacetime in this section. More its applications can be found in references [6]-[7], [13]-[23].

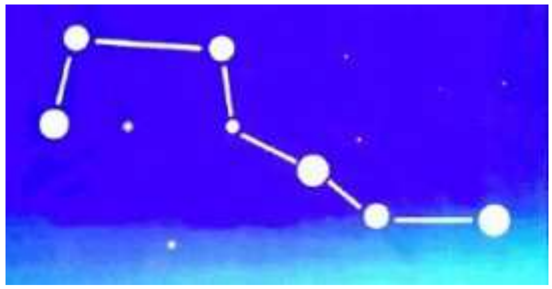

Fig.16

5.1 Global Stability

The stability of systems characterized by differential equations (ES_m) addresses the stability of solutions of (ES_m) and the trajectories of systems with small perturbations on initial values, such as those shown for Big Dipper in Fig.16.

In mathematics, a solution of system of differential equations (ES_m) is called stable or asymptotically stable ([25]) if for all solutions $Y(t)$ of the differential equations (ES_m) with

$$|Y(0) - X(0)| < \delta$$

exists for all $t \geq 0$,

$$|Y(t) - X(t)| < \varepsilon,$$

or furthermore,

$$\lim_{t \to 0} |Y(t) - X(t)| = 0.$$

However, by Theorem 2.9 if $\bigcap\limits_{i=1}^{m} S_{T_i} = \emptyset$ there are no solutions of (ES_m). Thus, the classical theory of stability is failed to apply. Then *how can one characterizes the stability of system (ES_m)*? As we have shown in Subsection 2.4, we always get a labeled graph solution $G^L[ES_m]$ of system (ES_m) whenever it is solvable or not, which can be applied to characterize the stability of system (ES_m).

Without loss of generality, assume $G^L(t)$ be a solution of (ES_m) with initial values $G^L(t_0)$ and let $\omega : V\left(G^L[ES_m]\right) \to \mathbb{R}$ be an index function. It is said to be *ω-stable* if there exists a number $\delta(\varepsilon)$ for any number $\varepsilon > 0$ such that

$$\left\|\omega\left(G^{L_1(t) - L_2(t)}\right)\right\| < \varepsilon,$$

or furthermore, *asymptotically ω-stable* if

$$\lim_{t\to\infty}\left\|\omega\left(G^{L_1(t)-L_2(t)}\right)\right\| = 0$$

if initial values holding with

$$\|L_1(t_0)(v) - L_2(t_0)(v)\| < \delta(\varepsilon)$$

for $\forall v) \in V\left(\overrightarrow{G}\right)$. If there is a Liapunov ω-function $L(\omega(t)) : \mathscr{O} \to \mathbb{R}, n \geq 1$ on $\overrightarrow{G}$ with $\mathscr{O} \subset \mathbb{R}^n$ open such that $L(\omega(t)) \geq 0$ with equality hold only if $(x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0)$ and if $t \geq t_0$, $\dfrac{dL(\omega)}{dt} \leq 0$, likewise Theorem 4.2 in [22] for the ω-stability of $\overrightarrow{G}$-flow, we know a result on ω-stability of (ES_M) following.

Theorem 5.1 *If there is a Liapunov ω-function $L(\omega(t)) : \mathscr{O} \to \mathbf{R}$ on $G^L[ES_m]$ of system (ES_m), then $G^L[ES_m]$ is ω-stable, and furthermore, if $\dot{L}(\omega(t)) < 0$ for $G^L[ES_m] \neq G^{\mathbf{0}}[ES_m]$, then $G^L[ES_m]$ is asymptotically ω-stable.*

For linear differential equations $(LDES_m^1)$, we can further introduce the sum-table subgraph following.

Definition 5.2 *Let H be a spanning subgraph of $G[LDES_m^1]$ of systems $(LDES_m^1)$ with initial value $X_v(0), v \in V\left(G[LDES_m^1]\right)$. Then $G[LDES_m^1]$ is called sum-stable or asymptotically sum-stable on H if for all solutions $Y_v(t)$, $v \in V(H)$ of the linear differential equations of $(LDES_m^1)$ with $|Y_v(0) - X_v(0)| < \delta_v$ exists for all $t \geq 0$,*

$$\left|\sum_{v\in V(H)} Y_v(t) - \sum_{v\in V(H)} X_v(t)\right| < \varepsilon,$$

or furthermore,

$$\lim_{t\to 0}\left|\sum_{v\in V(H)} Y_v(t) - \sum_{v\in V(H)} X_v(t)\right| = 0.$$

We get a result on the global stability for G-solutions of $(LDES_m^1)$ following.

Theorem 5.3([13]) *A labeled graph solution $G^{\mathbf{0}}[LDES_m^1]$ of linear homogenous differential equation systems $(LDES_m^1)$ is asymptotically sum-stable on a spanning subgraph H of $G[LDES_m^1]$ if and only if $\mathrm{Re}\alpha_v < 0$ for each $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$, $\forall v \in V(H)$ in $G^L\left[LDES_m^1\right]$.*

Example 5.4 Let a labeled graph solution $G^L[LDES_m^1]$ of $(LDES_m^1)$ be shown in Fig.17, where $v_1 = \{e^{-2t}, e^{-3t}, e^{3t}\}$, $v_2 = \{e^{-3t}, e^{-4t}\}$, $v_3 = \{e^{-4t}, e^{-5t}, e^{3t}\}$, $v_4 = \{e^{-5t}, e^{-6t}, e^{-8t}\}$, $v_5 = \{e^{-t}, e^{-6t}\}$, $v_6 = \{e^{-t}, e^{-2t}, e^{-8t}\}$. Then the labeled graph solution $G^{\mathbf{0}}[LDES_m^1]$ is sum-stable on the triangle $v_4v_5v_6$, but it is not on the triangle $v_1v_2v_3$.

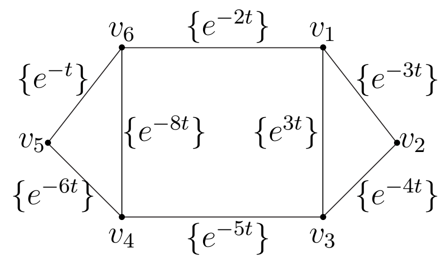

Fig.17

Similarly, let the system $(PDES_m^C)$ of linear partial differential equations be

$$\left.\begin{array}{l}\dfrac{\partial u}{\partial t} = H_i(t, x_1, \cdots, x_{n-1}, p_1, \cdots, p_{n-1}) \\ u|_{t=t_0} = u_0^{[i]}(x_1, x_2, \cdots, x_{n-1})\end{array}\right\} \quad 1 \leq i \leq m \qquad (APDES_m^C)$$

A point $X_0^{[i]} = (t_0, x_{10}^{[i]}, \cdots, x_{(n-1)0}^{[i]})$ with $H_i(t_0, x_{10}^{[i]}, \cdots, x_{(n-1)0}^{[i]}) = 0$ for $1 \leq i \leq m$ is called an *equilibrium point* of the ith equation in $(APDES_m)$. Then we know the following result, which can be applied to the ecological mathematics for the number of species more than 2 ([31]).

Theorem 5.5([17]) *Let $X_0^{[i]}$ be an equilibrium point of the ith equation in $(APDES_m)$ for integers $1 \leq i \leq m$. If $\sum\limits_{i=1}^{m} H_i(X) > 0$ and $\sum\limits_{i=1}^{m} \dfrac{\partial H_i}{\partial t} \leq 0$ for $X \neq \sum\limits_{i=1}^{m} X_0^{[i]}$, then the labeled graph solution $G^L[APDES_m]$ of system $(APDES_m)$ is sum-stable. Furthermore, if $\sum\limits_{i=1}^{m} \dfrac{\partial H_i}{\partial t} < 0$ for $X \neq \sum\limits_{i=1}^{m} X_0^{[i]}$, then the labeled graph solution $G^L[APDES_m]$ of system $(APDES_m)$ is asymptotically sum-stable.*

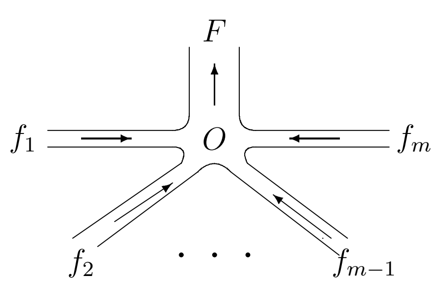

Fig.18

An immediately application of Theorem 5.5 is the control of traffic flows. For example, let O be a node in N incident with m in-flows and 1 out-flow such as those shown in Fig.18. Then, *what conditions will make sure the flow F being stable*? Denote the density of flow F by $\rho^{[F]}$ and f_i by $\rho^{[i]}$ for integers $1 \leq i \leq m$, respectively. Then, by traffic theory,

$$\frac{\partial \rho^{[i]}}{\partial t} + \phi_i(\rho^{[i]})\frac{\partial \rho^{[i]}}{\partial x} = 0,\ 1 \leq i \leq m.$$

We prescribe the initial value of $\rho^{[i]}$ by $\rho^{[i]}(x, t_0)$ at time t_0. Replacing each $\rho^{[i]}$ by ρ in these flow equations of $f_i,\ 1 \leq i \leq m$, we get a non-solvable system $(PDES_m^C)$ of partial differential equations

$$\left.\begin{array}{l} \dfrac{\partial \rho}{\partial t} + \phi_i(\rho)\dfrac{\partial \rho}{\partial x} = 0 \\ \rho\,|_{t=t_0} = \rho^{[i]}(x, t_0) \end{array}\right\}\ 1 \leq i \leq m.$$

Denote an equilibrium point of the ith equation by $\rho_0^{[i]}$, i.e., $\phi_i(\rho_0^{[i]})\dfrac{\partial \rho_0^{[i]}}{\partial x} = 0$. By Theorem 5.5, if

$$\sum_{i=1}^{m} \phi_i(\rho) < 0 \ \text{ and } \ \sum_{i=1}^{m} \phi_{(}\rho) \left[\frac{\partial^2 \rho}{\partial t \partial x} - \phi'(\rho)\left(\frac{\partial \rho}{\partial x}\right)^2\right] \geq 0$$

for $X \neq \sum\limits_{k=1}^{m} \rho_0^{[i]}$, then the flow F is stable, and furthermore, if

$$\sum_{i=1}^{m} \phi_{(}\rho) \left[\frac{\partial^2 \rho}{\partial t \partial x} - \phi'(\rho)\left(\frac{\partial \rho}{\partial x}\right)^2\right] < 0$$

for $X \neq \sum\limits_{k=1}^{m} \rho_0^{[i]}$, it is asymptotically stable.

5.2 Spacetime

Usually, different spacetime determine different structure of the universe, particularly for the solutions of Einstein's gravitational equations

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \lambda g^{\mu\nu} = -8\pi G T^{\mu\nu},$$

where $R^{\mu\nu} = R_{\alpha}^{\mu\alpha\nu} = g_{\alpha\beta}R^{\alpha\mu\beta\nu}$, $R = g_{\mu\nu}R^{\mu\nu}$ are the respective Ricci tensor, Ricci scalar curvature, $G = 6.673 \times 10^{-8} cm^3/gs^2$, $\kappa = 8\pi G/c^4 = 2.08 \times 10^{-48} cm^{-1} \cdot g^{-1} \cdot s^2$ ([24]).

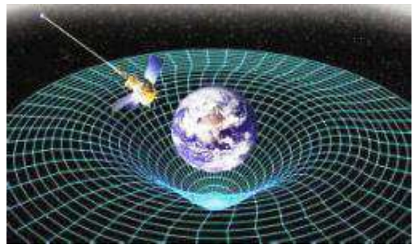

Fig.19

Certainly, Einstein's general relativity is suitable for use only in one spacetime $\mathbb{R}^4$, which implies a curved spacetime shown in Fig.19. But, if the dimension of the universe>4,

How can we characterize the structure of spacetime for the universe?

Generally, we understanding a thing by observation, i.e., the received information via hearing, sight, smell, taste or touch of our sensory organs and verify results on it in $\mathbb{R}^3 \times \mathbb{R}$. If the dimension of the universe>4, all these observations are nothing else but a projection of the true faces on our six organs, a partially truth. As a discrete mathematicians, the combinatorial notion should be his world view. A combinatorial spacetime $(\mathscr{C}_G|\overline{t})$ ([7]) is in fact a graph G^L labeled by Euclidean spaces $\mathbb{R}^n, n \geq 3$ evolving on a time vector $\overline{t}$ under smooth conditions in geometry. We can characterize the spacetime of the universe by a complete graph K_m^L labeled by $\mathbb{R}^4$ (See [9]-[11] for details).

For example, if $m = 4$, there are 4 Einstein's gravitational equations for $\forall v \in V\left(K_4^L\right)$. We solve it locally by spherically symmetric solutions in $\mathbb{R}^4$ and construct a graph K_4^L-solution labeled by $S_{f_1}, S_{f_2}, S_{f_3}$ and S_{f_4} of Einstein's gravitational equations, such as those shown in Fig.20,

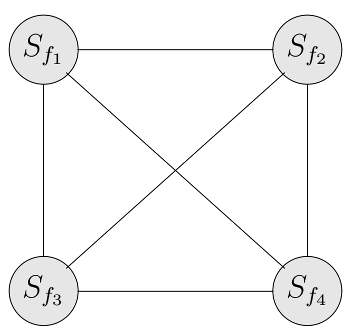

Fig.20

where, each S_{f_i} is a geometrical space determined by Schwarzschild spacetime

$$ds^2 = f(t)\left(1 - \frac{r_s}{r}\right) dt^2 - \frac{1}{1-\frac{r_s}{r}} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

for integers $1 \leq i \leq 4$. Certainly, its global behavior also depends on the intersections $S_{f_i} \bigcap S_{f_j}, 1 \leq i \neq j \leq 4$.

Notice that $m = 4$ is only an assumption. We do not know its exact value at present. Similarly, by Theorem 4.5, we also get a conclusion on spacetime of the Einstein's gravitational equations and we do not know also which labeled graph structure is the real spacetime of the universe.

Theorem 5.6([17]) *There are infinite many $\overrightarrow{G}$-flow solutions on Einstein's gravitational equations*

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = -8\pi G T^{\mu\nu}$$

in $\overrightarrow{G}^{\mathbb{C}}$, particularly on those graphs with circuit-decomposition $\overrightarrow{G} = \bigcup_{i=1}^{m} \overrightarrow{C}_i$ with Schwarzschild spacetime on their edges.

For example, let $\overrightarrow{G} = \overrightarrow{C}_4$. We are easily find $\overrightarrow{C}_4$-flow solution of Einstein's gravitational equations,such as those shown in Fig.7.

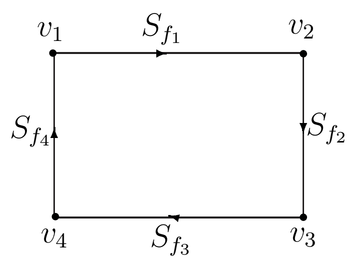

Fig.21

Then, the spacetime of the universe is nothing else but a curved ring such as those shown in Fig.22.

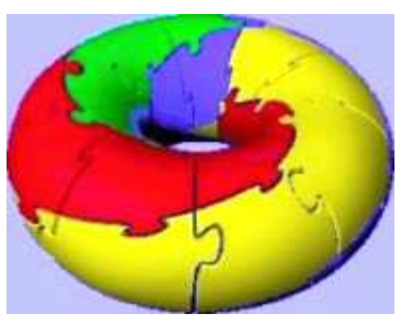

Fig.22

Generally, if G can be decomposed into m circuits, then Theorem 5.6 implies such a spacetime of Einstein's gravitational equations consisting of m curved rings underlying graph G in space.

§6. Conclusion

What are the elements of mathematics? Certainly, the mathematics consists of elements, include numbers $1, 2, 3, \cdots$, maps, functions $f(\mathbf{x})$, vectors, matrices, points, lines, opened sets $\cdots$, etc. with relations. However, these elements are not enough for understanding the reality of things because they must be a system without contradictions in its subfield of classical mathematics, i.e., a compatible system but contradictions exist everywhere, things are all in full of contradiction in the world. Thus, turn a systems with contradictions to mathematics is an important step for hold on the reality of things in the world. For such an objective, labeled graphs G^L are elements because a non-mathematics in classical is in fact a mathematics over a graph G, i.e., mathematical combinatorics. Thus, one should pay more attentions to labeled graphs, not only as a labeling technique on graphs but also a really mathematical element.

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