Smarandache Isotopy Of Second Smarandache Bol Loops *†

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Abstract

The pair $(G_H,\cdot)$ is called a special loop if $(G,\cdot)$ is a loop with an arbitrary subloop $(H,\cdot)$ called its special subloop. A special loop $(G_H,\cdot)$ is called a second Smarandache Bol loop($S_{2^{\mathrm{nd}}}$BL) if and only if it obeys the second Smarandache Bol identity $(xs\cdot z)s = x(sz\cdot s)$ for all x, z in G and s in H. The popularly known and well studied class of loops called Bol loops fall into this class and so $S_{2^{\mathrm{nd}}}$BLs generalize Bol loops. The Smarandache isotopy of $S_{2^{\mathrm{nd}}}$BLs is introduced and studied for the first time. It is shown that every Smarandache isotope(S-isotope) of a special loop is Smarandache isomorphic(S-isomorphic) to a S-principal isotope of the special loop. It is established that every special loop that is S-isotopic to a $S_{2^{\mathrm{nd}}}$BL is itself a $S_{2^{\mathrm{nd}}}$BL. A special loop is called a Smarandache G-special loop(SGS-loop) if and only if every special loop that is S-isotopic to it is S-isomorphic to it. A $S_{2^{\mathrm{nd}}}$BL is shown to be a SGS-loop if and only if each element of its special subloop is a $S_{1^{\mathrm{st}}}$ companion for a $S_{1^{\mathrm{st}}}$ pseudo-automorphism of the $S_{2^{\mathrm{nd}}}$BL. The results in this work generalize the results on the isotopy of Bol loops as can be found in the Ph.D. thesis of D. A. Robinson.

1 Introduction

The study of the Smarandache concept in groupoids was initiated by W. B. Vasantha Kandasamy in [24]. In her book [22] and first paper [23] on Smarandache concept in loops, she defined a Smarandache loop(S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. The present author has contributed to the study of S-quasigroups and S-loops in [5, 6, 7, 8, 9, 10, 11, 12] by introducing some new concepts immediately after the works of Muktibodh [15, 16]. His recent monograph [14] gives

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inter-relationships and connections between and among the various Smarandache concepts and notions that have been developed in the aforementioned papers.

But in the quest of developing the concept of Smarandache quasigroups and loops into a theory of its own just as in quasigroups and loop theory(see [1, 2, 3, 4, 17, 22]), there is the need to introduce identities for types and varieties of Smarandache quasigroups and loops. This led Jaíyéọlá [13] to the introduction of second Smarandache Bol loop($S_{2^{nd}}$BL) described by the second Smarandache Bol identity $(xs \cdot z)s = x(sz \cdot s)$ for all x, z in G and s in H where the pair $(G_H, \cdot)$ is called a special loop if $(G, \cdot)$ is a loop with an arbitrary subloop $(H, \cdot)$. For now, a Smarandache loop or Smarandache quasigroup will be called a first Smarandache loop($S_{1^{st}}$-loop) or first Smarandache quasigroup($S_{1^{st}}$-quasigroup).

Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : if $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If the system of equations ; $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. For each $x \in L$, the elements $x^\rho = xJ_\rho, x^\lambda = xJ_\lambda \in L$ such that $xx^\rho = e^\rho$ and $x^\lambda x = e^\lambda$ are called the right, left inverses of x respectively. Furthermore, if there exists a unique element $e = e_\rho = e_\lambda$ in L called the identity element such that for all x in L, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop. We write xy instead of $x \cdot y$, and stipulate that $\cdot$ has lower priority than juxtaposition among factors to be multiplied. For instance, $x \cdot yz$ stands for $x(yz)$. A loop is called a right Bol loop(Bol loop in short) if and only if it obeys the identity

$$(xy \cdot z)y = x(yz \cdot y).$$

This class of loops was the first to catch the attention of loop theorists and the first comprehensive study of this class of loops was carried out by Robinson [19].

The popularly known and well studied class of loops called Bol loops fall into the class of $S_{2^{nd}}$BLs and so $S_{2^{nd}}$BLs generalize Bol loops. The aim of this work is to introduce and study for the first time, the Smarandache isotopy of $S_{2^{nd}}$BLs. It is shown that every Smarandache isotope(S-isotope) of a special loop is Smarandache isomorphic(S-isomorphic) to a S-principal isotope of the special loop. It is established that every special loop that is S-isotopic to a $S_{2^{nd}}$BL is itself a $S_{2^{nd}}$BL. A $S_{2^{nd}}$BL is shown to be a Smarandache G-special loop if and only if each element of its special subloop is a $S_{1^{st}}$ companion for a $S_{1^{st}}$ pseudo-automorphism of the $S_{2^{nd}}$BL. The results in this work generalize the results on the isotopy of Bol loops as can be found in the Ph.D. thesis of D. A. Robinson.

2 Preliminaries

Definition 2.1 *Let $(G, \cdot)$ be a quasigroup with an arbitrary non-trivial subquasigroup $(H, \cdot)$. Then, $(G_H, \cdot)$ is called a special quasigroup with special subquasigroup $(H, \cdot)$. If $(G, \cdot)$ is a loop with an arbitrary non-trivial subloop $(H, \cdot)$. Then, $(G_H, \cdot)$ is called a special loop with special subloop $(H, \cdot)$. If $(H, \cdot)$ is of exponent 2, then $(G_H, \cdot)$ is called a special loop of Smarandache exponent 2.*

A special quasigroup $(G_H, \cdot)$ is called a second Smarandache right Bol quasigroup($S_{2^{nd}}$-right Bol quasigroup) or simply a second Smarandache Bol quasigroup($S_{2^{nd}}$-Bol quasigroup) *and abbreviated $S_{2^{nd}}RBQ$ or $S_{2^{nd}}BQ$ if and only if it obeys the second Smarandache Bol identity($S_{2^{nd}}$-Bol identity) i.e $S_{2^{nd}}BI$*

$$(xs \cdot z)s = x(sz \cdot s) \text{ for all } x, z \in G \text{ and } s \in H. \tag{1}$$

Hence, if $(G_H, \cdot)$ is a special loop, and it obeys the $S_{2^{nd}}BI$, it is called a second Smarandache Bol loop($S_{2^{nd}}$-Bol loop) and abbreviated $S_{2^{nd}}BL$.

Remark 2.1 *A Smarandache Bol loop(i.e a loop with at least a non-trivial subloop that is a Bol loop) will now be called a first Smarandache Bol loop($S_{1^{st}}$-Bol loop). It is easy to see that a $S_{2^{nd}}BL$ is a $S_{1^{st}}BL$. But the converse is not generally true. So $S_{2^{nd}}BL$s are particular types of $S_{1^{st}}BL$. Their study can be used to generalise existing results in the theory of Bol loops by simply forcing H to be equal to G.*

Definition 2.2 *Let $(G, \cdot)$ be a quasigroup(loop). It is called a right inverse property quasigroup(loop)[RIPQ(RIPL)] if and only if it obeys the right inverse property(RIP) $yx \cdot x^\rho = y$ for all $x, y \in G$. Similarly, it is called a left inverse property quasigroup(loop)[LIPQ(LIPL)] if and only if it obeys the left inverse property(LIP) $x^\lambda \cdot xy = y$ for all $x, y \in G$. Hence, it is called an inverse property quasigroup(loop)[IPQ(IPL)] if and only if it obeys both the RIP and LIP.*

$(G, \cdot)$ is called a right alternative property quasigroup(loop)[RAPQ(RAPL)] if and only if it obeys the right alternative property(RAP) $y \cdot xx = yx \cdot x$ for all $x, y \in G$. Similarly, it is called a left alternative property quasigroup(loop)[LAPQ(LAPL)] if and only if it obeys the left alternative property(LAP) $xx \cdot y = x \cdot xy$ for all $x, y \in G$. Hence, it is called an alternative property quasigroup(loop)[APQ(APL)] if and only if it obeys both the RAP and LAP.

The bijection $L_x : G \to G$ defined as $yL_x = x \cdot y$ for all $x, y \in G$ is called a left translation(multiplication) of G while the bijection $R_x : G \to G$ defined as $yR_x = y \cdot x$ for all $x, y \in G$ is called a right translation(multiplication) of G. Let

$$x\backslash y = yL_x^{-1} = y\mathbb{L}_x \qquad \text{and} \qquad x/y = xR_y^{-1} = x\mathbb{R}_y$$

and note that

$$x\backslash y = z \Longleftrightarrow x \cdot z = y \qquad \text{and} \qquad x/y = z \Longleftrightarrow z \cdot y = x.$$

The operations $\backslash$ and $/$ are called the left and right divisions respectively. We stipulate that $/$ and $\backslash$ have higher priority than $\cdot$ among factors to be multiplied. For instance, $x \cdot y/z$ and $x \cdot y\backslash z$ stand for $x(y/z)$ and $x \cdot (y\backslash z)$ respectively.

$(G, \cdot)$ is said to be a right power alternative property loop(RPAPL) if and only if it obeys the right power alternative property(RPAP)

$$xy^n = \underbrace{(((xy)y)y)y \cdots y}_{n\text{-times}} \text{ i.e. } R_{y^n} = R_y^n \text{ for all } x, y \in G \text{ and } n \in \mathbb{Z}.$$

The right nucleus of G denoted by $N_\rho(G, \cdot) = N_\rho(G) = \{a \in G : y \cdot xa = yx \cdot a\ \forall\ x, y \in G\}$.

Let $(G_H,\cdot)$ be a special quasigroup(loop). It is called a second Smarandache right inverse property quasigroup(loop)[$S_{2^{nd}}RIPQ(S_{2^{nd}}RIPL)$] if and only if it obeys the second Smarandache right inverse property($S_{2^{nd}}RIP$) $ys\cdot s^\rho = y$ for all $y\in G$ and $s\in H$. Similarly, it is called a second Smarandache left inverse property quasigroup(loop)[$S_{2^{nd}}LIPQ(S_{2^{nd}}LIPL)$] if and only if it obeys the second Smarandache left inverse property($S_{2^{nd}}LIP$) $s^\lambda\cdot sy = y$ for all $y\in G$ and $s\in H$. Hence, it is called a second Smarandache inverse property quasigroup(loop)[$S_{2^{nd}}IPQ(S_{2^{nd}}IPL)$] if and only if it obeys both the $S_{2^{nd}}RIP$ and $S_{2^{nd}}LIP$.

$(G_H,\cdot)$ is called a third Smarandache right inverse property quasigroup(loop)[$S_{3^{rd}}RIPQ(S_{3^{rd}}RIPL)$] if and only if it obeys the third Smarandache right inverse property($S_{3^{rd}}RIP$) $sy\cdot y^\rho = s$ for all $y\in G$ and $s\in H$.

$(G_H,\cdot)$ is called a second Smarandache right alternative property quasigroup(loop)[$S_{2^{nd}}RAPQ(S_{2^{nd}}RAPL)$] if and only if it obeys the second Smarandache right alternative property($S_{2^{nd}}RAP$) $y\cdot ss = ys\cdot s$ for all $y\in G$ and $s\in H$. Similarly, it is called a second Smarandache left alternative property quasigroup(loop)[$S_{2^{nd}}LAPQ(S_{2^{nd}}LAPL)$] if and only if it obeys the second Smarandache left alternative property($S_{2^{nd}}LAP$) $ss\cdot y = s\cdot sy$ for all $y\in G$ and $s\in H$. Hence, it is called an second Smarandache alternative property quasigroup(loop)[$S_{2^{nd}}APQ(S_{2^{nd}}APL)$] if and only if it obeys both the $S_{2^{nd}}RAP$ and $S_{2^{nd}}LAP$.

$(G_H,\cdot)$ is said to be a Smarandache right power alternative property loop(SRPAPL) if and only if it obeys the Smarandache right power alternative property(SRPAP)

$$xs^n = \underbrace{(((xs)s)s)s\cdots s}_{n\text{-}times}\ \textit{i.e. } R_{s^n} = R_s^n \textit{ for all } x\in G,\ s\in H \textit{ and } n\in\mathbb{Z}.$$

The Smarandache right nucleus of G_H denoted by $SN_\rho(G_H,\cdot) = SN_\rho(G_H) = N_\rho(G)\cap H$. G_H is called a Smarandache right nuclear square special loop if and only if $s^2\in SN_\rho(G_H)$ for all $s\in H$.

Remark 2.2 *A Smarandache; RIPQ or LIPQ or IPQ(i.e a loop with at least a non-trivial subquasigroup that is a RIPQ or LIPQ or IPQ) will now be called a first Smarandache; RIPQ or LIPQ or IPQ($S_{1^{st}}RIPQ$ or $S_{1^{st}}LIPQ$ or $S_{1^{st}}IPQ$). It is easy to see that a $S_{2^{nd}}RIPQ$ or $S_{2^{nd}}LIPQ$ or $S_{2^{nd}}IPQ$ is a $S_{1^{st}}RIPQ$ or $S_{1^{st}}LIPQ$ or $S_{1^{st}}IPQ$ respectively. But the converse is not generally true.*

Definition 2.3 *Let $(G,\cdot)$ be a quasigroup(loop). The set $SYM(G,\cdot) = SYM(G)$ of all bijections in G forms a group called the permutation(symmetric) group of G. The triple (U,V,W) such that $U,V,W\in SYM(G,\cdot)$ is called an autotopism of G if and only if*

$$xU\cdot yV = (x\cdot y)W\ \forall\ x,y\in G.$$

The group of autotopisms of G is denoted by $AUT(G,\cdot) = AUT(G)$.

Let $(G_H,\cdot)$ be a special quasigroup(loop). The set $SSYM(G_H,\cdot) = SSYM(G_H)$ of all Smarandache bijections(S-bijections) in G_H i.e $A\in SYM(G_H)$ such that $A : H\to H$ forms a group called the Smarandache permutation(symmetric) group[S-permutation group] *of G_H. The triple (U,V,W) such that $U,V,W\in SSYM(G_H,\cdot)$ is called a first Smarandache autotopism($S_{1^{st}}$ autotopism) of G_H if and only if*

$$xU\cdot yV=(x\cdot y)W\ \forall\ x,y\in G_H.$$

If their set forms a group under componentwise multiplication, it is called the first Smarandache autotopism group($S_{1^{st}}$ autotopism group) of G_H and is denoted by $S_{1^{st}}AUT(G_H,\cdot)=S_{1^{st}}AUT(G_H)$.

The triple (U,V,W) such that $U,W\in SYM(G,\cdot)$ and $V\in SSYM(G_H,\cdot)$ is called a second right Smarandache autotopism($S_{2^{nd}}$ right autotopism) of G_H if and only if

$$xU\cdot sV=(x\cdot s)W\ \forall\ x\in G\ \textit{and}\ s\in H.$$

If their set forms a group under componentwise multiplication, it is called the second right Smarandache autotopism group($S_{2^{nd}}$ right autotopism group) of G_H and is denoted by $S_{2^{nd}}RAUT(G_H,\cdot)=S_{2^{nd}}RAUT(G_H)$.

The triple (U,V,W) such that $V,W\in SYM(G,\cdot)$ and $U\in SSYM(G_H,\cdot)$ is called a second left Smarandache autotopism($S_{2^{nd}}$ left autotopism) of G_H if and only if

$$sU\cdot yV=(s\cdot y)W\ \forall\ y\in G\ \textit{and}\ s\in H.$$

If their set forms a group under componentwise multiplication, it is called the second left Smarandache autotopism group($S_{2^{nd}}$ left autotopism group) of G_H and is denoted by $S_{2^{nd}}LAUT(G_H,\cdot)=S_{2^{nd}}LAUT(G_H)$.

Let $(G_H,\cdot)$ be a special quasigroup(loop) with identity element e. A mapping $T\in SSYM(G_H)$ is called a first Smarandache semi-automorphism($S_{1^{st}}$ semi-automorphism) if and only if $eT=e$ and

$$(xy\cdot x)T=(xT\cdot yT)xT\ \textit{for all}\ x,y\in G.$$

A mapping $T\in SSYM(G_H)$ is called a second Smarandache semi-automorphism($S_{2^{nd}}$ semi-automorphism) if and only if $eT=e$ and

$$(sy\cdot s)T=(sT\cdot yT)sT\ \textit{for all}\ y\in G\ \textit{and all}\ s\in H.$$

A special loop $(G_H,\cdot)$ is called a first Smarandache semi-automorphic inverse property loop($S_{1^{st}}SAIPL$) if and only if J_ρ is a $S_{1^{st}}$ semi-automorphism.

A special loop $(G_H,\cdot)$ is called a second Smarandache semi-automorphic inverse property loop($S_{2^{nd}}SAIPL$) if and only if J_ρ is a $S_{2^{nd}}$ semi-automorphism.

Let $(G_H,\cdot)$ be a special quasigroup(loop). A mapping $A\in SSYM(G_H)$ is a

1. *first Smarandache pseudo-automorphism($S_{1^{st}}$ pseudo-automorphism) of G_H if and only if there exists a $c\in H$ such that $(A,AR_c,AR_c)\in S_{1^{st}}AUT(G_H)$. c is reffered to as the first Smarandache companion($S_{1^{st}}$ companion) of A. The set of such A's is denoted by $S_{1^{st}}PAUT(G_H,\cdot)=S_{1^{st}}PAUT(G_H)$.*

2. *second right Smarandache pseudo-automorphism($S_{2^{nd}}$ right pseudo-automorphism) of G_H if and only if there exists a $c \in H$ such that $(A, AR_c, AR_c) \in S_{2^{nd}}RAUT(G_H)$. c is reffered to as the second right Smarandache companion($S_{2^{nd}}$ right companion) of A. The set of such A's is denoted by $S_{2^{nd}}RPAUT(G_H, \cdot) = S_{2^{nd}}RPAUT(G_H)$.*

3. *second left Smarandache pseudo-automorphism($S_{2^{nd}}$ left pseudo-automorphism) of G_H if and only if there exists a $c \in H$ such that $(A, AR_c, AR_c) \in S_{2^{nd}}LAUT(G_H)$. c is reffered to as the second left Smarandache companion($S_{2^{nd}}$ left companion) of A. The set of such A's is denoted by $S_{2^{nd}}LPAUT(G_H, \cdot) = S_{2^{nd}}LPAUT(G_H)$.*

Let $(G_H, \cdot)$ be a special loop. A mapping $A \in SSYM(G_H)$ is a

1. *first Smarandache automorphism($S_{1^{st}}$ automorphism) of G_H if and only if $A \in S_{1^{st}}PAUT(G_H)$ such that $c = e$. Their set is denoted by $S_{1^{st}}AUM(G_H, \cdot) = S_{1^{st}}AUM(G_H)$.*

2. *second right Smarandache automorphism($S_{2^{nd}}$ right automorphism) of G_H if and only if $A \in S_{2^{nd}}RPAUT(G_H)$ such that $c = e$. Their set is denoted by $S_{2^{nd}}RAUM(G_H, \cdot) = S_{2^{nd}}RAUM(G_H)$.*

3. *second left Smarandache automorphism($S_{2^{nd}}$ left automorphism) of G_H if and only if $A \in S_{2^{nd}}LPAUT(G_H)$ such that $c = e$. Their set is denoted by $S_{2^{nd}}LAUM(G_H, \cdot) = S_{2^{nd}}LAUM(G_H)$.*

A special loop $(G_H, \cdot)$ is called a first Smarandache automorphism inverse property loop($S_{1^{st}}$AIPL) if and only if $(J_\rho, J_\rho, J_\rho) \in AUT(H, \cdot)$.

A special loop $(G_H, \cdot)$ is called a second Smarandache right automorphic inverse property loop($S_{2^{nd}}$RAIPL) if and only if J_ρ is a $S_{2^{nd}}$ right automorphism.

A special loop $(G_H, \cdot)$ is called a second Smarandache left automorphic inverse property loop($S_{2^{nd}}$LAIPL) if and only if J_ρ is a $S_{2^{nd}}$ left automorphism.

Definition 2.4 *Let $(G, \cdot)$ and $(L, \circ)$ be quasigroups(loops). The triple (U, V, W) such that $U, V, W : G \to L$ are bijections is called an isotopism of G onto L if and only if*

$$xU \circ yV = (x \cdot y)W \ \forall \ x, y \in G. \tag{2}$$

Let $(G_H, \cdot)$ and $(L_M, \circ)$ be special groupoids. G_H and L_M are Smarandache isotopic(S-isotopic)[and we say $(L_M, \circ)$ is a Smarandache isotope of $(G_H, \cdot)$] if and only if there exist bijections $U, V, W : H \to M$ such that the triple $(U, V, W) : (G_H, \cdot) \to (L_M, \circ)$ is an isotopism. In addition, if $U = V = W$, then $(G_H, \cdot)$ and $(L_M, \circ)$ are said to be Smarandache isomorphic(S-isomorphic)[and we say $(L_M, \circ)$ is a Smarandache isomorph of $(G_H, \cdot)$ and thus write $(G_H, \cdot) \succsim (L_M, \circ)$.].

$(G_H, \cdot)$ is called a Smarandache G-special loop(SGS-loop) if and only if every special loop that is S-isotopic to $(G_H, \cdot)$ is S-isomorphic to $(G_H, \cdot)$.

Theorem 2.1 *(Jaíyéọlá [13])*

Let the special loop $(G_H, \cdot)$ be a $S_{2^{nd}}$BL. Then it is both a $S_{2^{nd}}$RIPL and a $S_{2^{nd}}$RAPL.

Theorem 2.2 *(Jaíyéọlá [13])*

Let $(G_H, \cdot)$ be a special loop. $(G_H, \cdot)$ is a $S_{2^{nd}}BL$ if and only if $(R_s^{-1}, L_sR_s, R_s) \in S_{1^{st}}AUT(G_H, \cdot)$.

3 Main Results

Lemma 3.1 *Let $(G_H, \cdot)$ be a special quasigroup and let $s, t \in H$. For all $x, y \in G$, let*

$$x \circ y = xR_t^{-1} \cdot yL_s^{-1}. \tag{3}$$

Then, $(G_H, \circ)$ is a special loop and so $(G_H, \cdot)$ and $(G_H, \circ)$ are S-isotopic.

Proof

It is easy to show that $(G_H, \circ)$ is a quasigroup with a subquasigroup $(H, \circ)$ since $(G_H, \cdot)$ is a special quasigroup. So, $(G_H, \circ)$ is a special quasigroup. It is also easy to see that $s \cdot t \in H$ is the identity element of $(G_H, \circ)$. Thus, $(G_H, \circ)$ is a special loop. With $U = R_t$, $V = L_s$ and $W = I$, the triple $(U, V, W) : (G_H, \cdot) \to (G_H, \circ)$ is an S-isotopism.

Remark 3.1 *$(G_H, \circ)$ will be called a Smarandache principal isotopism(S-principal isotopism) of $(G_H, \cdot)$.*

Theorem 3.1 *If the special quasigroup $(G_H, \cdot)$ and special loop $(L_M, \circ)$ are S-isotopic, then $(L_M, \circ)$ is S-isomorphic to a S-principal isotope of $(G_H, \cdot)$.*

Proof

Let e be the identity element of the special loop $(L_M, \circ)$. Let U, V and W be 1-1 S-mappings of G_H onto L_M such that

$$xU \circ yV = (x \cdot y)W \ \forall \ x, y \in G_H.$$

Let $t = eV^{-1}$ and $s = eU^{-1}$. Define $x * y$ for all $x, y \in G_H$ by

$$x * y = (xW \circ yW)W^{-1}. \tag{4}$$

From (2), with x and y replaced by xWU^{-1} and yWV^{-1} respectively, we get

$$(xW \circ yW)W^{-1} = xWU^{-1} \cdot yWV^{-1} \ \forall \ x, y \in G_H. \tag{5}$$

In (5), with $x = eW^{-1}$, we get $WV^{-1} = L_s^{-1}$ and with $y = eW^{-1}$, we get $WU^{-1} = R_t^{-1}$. Hence, from (4) and (5),

$$x * y = xR_t^{-1} \cdot yL_s^{-1} \text{ and } (x * y)W = xW \circ yW \ \forall \ x, y \in G_H.$$

That is, $(G_H, *)$ is a S-principal isotope of $(G_H, \cdot)$ and is S-isomorphic to $(L_M, \circ)$.

Theorem 3.2 *Let $(G_H,\cdot)$ be a $S_{2^{nd}}RIPL$. Let $f,g\in H$ and let $(G_H,\circ)$ be a S-principal isotope of $(G_H,\cdot)$. $(G_H,\circ)$ is a $S_{2^{nd}}RIPL$ if and only if $\alpha(f,g)=(R_g, L_fR_g^{-1}L_{f\cdot g}^{-1}, R_g^{-1})\in S_{2^{nd}}RAUT(G_H,\cdot)$ for all $f,g\in H$.*

Proof
Let $(G_H,\cdot)$ be a special loop that has the $S_{2^{nd}}$RIP and let $f,g\in H$. For all $x,y\in G$, define $x\circ y = xR_g^{-1}\cdot yL_f^{-1}$ as in (3). Recall that $f\cdot g$ is the identity in $(G_H,\circ)$, so $x\circ x^{\rho'}=f\cdot g$ where $xJ'_\rho = x^{\rho'}$ i.e the right identity element of x in $(G_H,\circ)$. Then, for all $x\in G$, $x\circ x^{\rho'} = xR_g^{-1}\cdot xJ'_\rho L_f^{-1}=f\cdot g$ and by the $S_{2^{nd}}$RIP of $(G_H,\cdot)$, since $sR_g^{-1}\cdot sJ'_\rho L_f^{-1}=f\cdot g$ for all $s\in H$, then $sR_g^{-1}=(f\cdot g)\cdot(sJ'_\rho L_f^{-1})J_\rho$ because $(H,\cdot)$ has the RIP. Thus,

$$sR_g^{-1}=sJ'_\rho L_f^{-1}J_\rho L_{f\cdot g}\Rightarrow sJ'_\rho = sR_g^{-1}L_{f\cdot g}^{-1}J_\lambda L_f. \tag{6}$$

$(G_H,\circ)$ has the $S_{2^{nd}}$RIP iff $(x\circ s)\circ sJ'_\rho = s$ for all $s\in H$, $x\in G_H$ iff $(xR_g^{-1}\cdot sL_f^{-1})R_g^{-1}\cdot sJ'_\rho L_f^{-1}=x$, for all $s\in H$, $x\in G_H$. Replace x by $x\cdot g$ and s by $f\cdot s$, then $(x\cdot s)R_g^{-1}\cdot(f\cdot s)J'_\rho L_f^{-1}=x\cdot g$ iff $(x\cdot s)R_g^{-1}=(x\cdot g)\cdot(f\cdot s)J'_\rho L_f^{-1}J_\rho$ for all $s\in H$, $x\in G_H$ since $(G_H,\cdot)$ has the $S_{2^{nd}}$RIP. Using (6),

$$(x\cdot s)R_g^{-1}=xR_g\cdot(f\cdot s)R_g^{-1}L_{f\cdot g}^{-1}\Leftrightarrow (x\cdot s)R_g^{-1}=xR_g\cdot sL_fR_g^{-1}L_{f\cdot g}^{-1}\Leftrightarrow$$

$$\alpha(f,g)=(R_g, L_fR_g^{-1}L_{f\cdot g}^{-1}, R_g^{-1})\in \mathrm{S}_{2^{nd}}RAUT(G_H,\cdot)\text{ for all }f,g\in H.$$

Theorem 3.3 *If a special loop $(G_H,\cdot)$ is a $S_{2^{nd}}BL$, then any of its S-isotopes is a $S_{2^{nd}}RIPL$.*

Proof
By virtue of Theorem 3.1, we need only to concern ourselves with the S-principal isotopes of $(G_H,\cdot)$. $(G_H,\cdot)$ is a $S_{2^{nd}}$BL iff it obeys the $S_{2^{nd}}$BI iff $(xs\cdot z)s = x(sz\cdot s)$ for all $x,z\in G$ and $s\in H$ iff $L_{xs}R_s = L_sR_sL_x$ for all $x\in G$ and $s\in H$ iff $R_s^{-1}L_{xs}^{-1}=L_x^{-1}R_s^{-1}L_s^{-1}$ for all $x\in G$ and $s\in H$ iff

$$R_s^{-1}L_s^{-1}=L_xR_s^{-1}L_{xs}^{-1}\text{ for all }x\in G\text{ and }s\in H. \tag{7}$$

Assume that $(G_H,\cdot)$ is a $S_{2^{nd}}$BL. Then, by Theorem 2.2,

$$(R_s^{-1}, L_sR_s, R_s)\in \mathrm{S}_{1^{st}}AUT(G_H,\cdot)\Rightarrow (R_s^{-1}, L_sR_s, R_s)\in \mathrm{S}_{2^{nd}}RAUT(G_H,\cdot)\Rightarrow$$

$$(R_s^{-1}, L_sR_s, R_s)^{-1}=(R_s, R_s^{-1}L_s^{-1}, R_s^{-1})\in \mathrm{S}_{2^{nd}}RAUT(G_H,\cdot).$$

By (7), $\alpha(x,s)=(R_s, L_xR_s^{-1}L_{xs}^{-1}, R_s^{-1})\in \mathrm{S}_{2^{nd}}RAUT(G_H,\cdot)$ for all $f,g\in H$. But $(G_H,\cdot)$ has the $S_{2^{nd}}$RIP by Theorem 2.1. So, following Theorem 3.2, all special loops that are S-isotopic to $(G_H,\cdot)$ are $S_{2^{nd}}$RIPLs.

Theorem 3.4 *Suppose that each special loop that is S-isotopic to $(G_H,\cdot)$ is a $S_{2^{nd}}RIPL$, then the identities:*

1. $(fg)\backslash f = (xg)\backslash x$;

2. $g\backslash(sg^{-1}) = (fg)\backslash[(fs)g^{-1}]$

are satisfied for all $f, g, s \in H$ *and* $x \in G$.

Proof

In particular, $(G_H, \cdot)$ has the $\mathrm{S}_{2^{\mathrm{nd}}}$RIP. Then by Theorem 3.1, $\alpha(f,g) = (R_g, L_f R_g^{-1} L_{f\cdot g}^{-1}, R_g^{-1}) \in \mathrm{S}_{2^{\mathrm{nd}}}RAUT(G_H, \cdot)$ for all $f, g \in H$. Let

$$Y = L_f R_g^{-1} L_{f\cdot g}^{-1}. \tag{8}$$

Then,

$$xg \cdot sY = (xs)R_g^{-1}. \tag{9}$$

Put $s = g$ in (9), then $xg \cdot gY = (xg)R_g^{-1} = x$. But, $gY = gL_f R_g^{-1} L_{f\cdot g}^{-1} = (fg)\backslash[(fg)g^{-1}] = (fg)\backslash f$. So, $xg \cdot (fg)\backslash f = x \Rightarrow (fg)\backslash f = (xg)\backslash x$.

Put $x = e$ in (9), then $sYL_g = sR_g^{-1} \Rightarrow sY = sR_g^{-1}L_g^{-1}$. So, combining this with (8), $sR_g^{-1}L_g^{-1} = sL_f R_g^{-1} L_{f\cdot g}^{-1} \Rightarrow g\backslash(sg^{-1}) = (fg)\backslash[(fs)g^{-1}]$.

Theorem 3.5 *Every special loop that is S-isotopic to a* $S_{2^{nd}}BL$ *is itself a* $S_{2^{nd}}BL$.

Proof

Let $(G_H, \circ)$ be a special loop that is S-isotopic to an $\mathrm{S}_{2^{\mathrm{nd}}}$BL $(G_H, \cdot)$. Assume that $x \cdot y = x\alpha \circ y\beta$ where $\alpha, \beta : H \to H$. Then the $\mathrm{S}_{2^{\mathrm{nd}}}$BI can be written in terms of $(\circ)$ as follows. $(xs \cdot z)s = x(sz \cdot s)$ for all $x, z \in G$ and $s \in H$.

$$[(x\alpha \circ s\beta)\alpha \circ z\beta]\alpha \circ s\beta = x\alpha \circ [(s\alpha \circ z\beta)\alpha \circ s\beta]\beta. \tag{10}$$

Replace $x\alpha$ by $\overline{x}$, $s\beta$ by $\overline{s}$ and $z\beta$ by $\overline{z}$, then

$$[(\overline{x} \circ \overline{s})\alpha \circ \overline{z}]\alpha \circ \overline{s} = \overline{x} \circ [(\overline{s}\beta^{-1}\alpha \circ \overline{z})\alpha \circ \overline{s}]\beta. \tag{11}$$

If $\overline{x} = e$, then

$$(\overline{s}\alpha \circ \overline{z})\alpha \circ \overline{s} = [(\overline{s}\beta^{-1}\alpha \circ \overline{z})\alpha \circ \overline{s}]\beta. \tag{12}$$

Substituting (12) into the RHS of (11) and replacing $\overline{x}$, $\overline{s}$ and $\overline{z}$ by x, s and z respectively, we have

$$[(x \circ s)\alpha \circ z]\alpha \circ s = x \circ [(s\alpha \circ z)\alpha \circ s]. \tag{13}$$

With $s = e$, $(x\alpha \circ z)\alpha = x \circ (e\alpha \circ z)\alpha$. Let $(e\alpha \circ z)\alpha = z\delta$, where $\delta \in SSYM(G_H)$. Then,

$$(x\alpha \circ z)\alpha = x \circ z\delta. \tag{14}$$

Applying (14), then (13) to the expression $[(x \circ s) \circ z\delta] \circ s$, that is

$$[(x \circ s) \circ z\delta] \circ s = [(x \circ s)\alpha \circ z]\alpha \circ s = x \circ [(s\alpha \circ z)\alpha \circ s] = x \circ [(s \circ z\delta) \circ s].$$

implies

$$[(x \circ s) \circ z\delta] \circ s = x \circ [(s \circ z\delta) \circ s].$$

Replace $z\delta$ by z, then

$$[(x \circ s) \circ z] \circ s = x \circ [(s \circ z) \circ s].$$

Theorem 3.6 *Let $(G_H,\cdot)$ be a $S_{2^{nd}}BL$. Each special loop that is S-isotopic to $(G_H,\cdot)$ is S-isomorphic to a S-principal isotope $(G_H,\circ)$ where $x\circ y = xR_f\cdot yL_f^{-1}$ for all $x,y\in G$ and some $f\in H$.*

Proof

Let e be the identity element of $(G_H,\cdot)$. Let $(G_H,*)$ be any S-principal isotope of $(G_H,\cdot)$ say $x*y = xR_v^{-1}\cdot yL_u^{-1}$ for all $x,y\in G$ and some $u,v\in H$. Let e' be the identity element of $(G_H,*)$. That is, $e'=u\cdot v$. Now, define $x*y$ by

$$x\circ y = [(xe')*(ye')]e'^{-1} \text{ for all } x,y\in G.$$

Then $R_{e'}$ is an S-isomorphism of $(G_H,\circ)$ onto $(G_H,*)$. Observe that e is also the identity element for $(G_H,\circ)$ and since $(G_H,\cdot)$ is a $\mathrm{S}_{2^{nd}}\mathrm{BL}$,

$$(pe')(e'^{-1}q\cdot e'^{-1}) = pq\cdot e'^{-1} \text{ for all } p,q\in G. \tag{15}$$

So, using (15),

$$x\circ y = [(xe')*(ye')]e'^{-1} = [xR_{e'}R_v^{-1}\cdot yR_{e'}L_u^{-1}]e'^{-1} = xR_{e'}R_v^{-1}R_{e'}\cdot yR_{e'}L_u^{-1}L_{e'^{-1}}R_{e'^{-1}}$$

implies that

$$x\circ y = xA\cdot yB,\ A = R_{e'}R_v^{-1}R_{e'} \text{ and } B = R_{e'}L_u^{-1}L_{e'^{-1}}R_{e'^{-1}}. \tag{16}$$

Let $f=eA$. then, $y = e\circ y = eA\cdot yB = f\cdot yB$ for all $y\in G$. So, $B = L_f^{-1}$. In fact, $eB = f^\rho = f^{-1}$. Then, $x = x\circ e = xA\cdot eB = xA\cdot f^{-1}$ for all $x\in G$ implies $xf = (xA\cdot f^{-1})f$ implies $xf = xA(\mathrm{S}_{2^{nd}}\mathrm{RIP})$ implies $A = R_f$. Now, (16) becomes $x\circ y = xR_f\cdot yL_f^{-1}$.

Theorem 3.7 *Let $(G_H,\cdot)$ be a $S_{2^{nd}}BL$ with the $S_{2^{nd}}RAIP$ or $S_{2^{nd}}LAIP$, let $f\in H$ and let $x\circ y = xR_f\cdot yL_f^{-1}$ for all $x,y\in G$. Then $(G_H,\circ)$ is a $S_{1^{st}}AIPL$ if and only if $f\in N_\lambda(H,\cdot)$.*

Proof

Since $(G_H,\cdot)$ is a $\mathrm{S}_{2^{nd}}\mathrm{BL}$, $J=J_\lambda=J_\rho$ in $(H,\cdot)$. Using (6) with $g=f^{-1}$,

$$sJ'_\rho = sR_fJL_f. \tag{17}$$

$(G_H,\circ)$ is a $\mathrm{S}_{1^{st}}\mathrm{AIPL}$ iff $(x\circ y)J'_\rho = xJ'_\rho\circ yJ'_\rho$ for all $x,y\in H$ iff

$$(xR_f\cdot yL_f^{-1})J'_\rho = xJ'_\rho R_f\cdot yJ'_\rho L_f^{-1}. \tag{18}$$

Let $x = uR_f^{-1}$ and $y = vL_f$ and use (16), then (18) becomes $(uv)R_fJL_f = uJL_fR_f\cdot vL_fR_fJ$ iff $\alpha = (JL_fR_f, L_fR_fJ, R_fJL_f)\in AUT(H,\cdot)$. Since $(G_H,\cdot)$ is a $\mathrm{S}_{1^{st}}\mathrm{AIPL}$, so $(J,J,J)\in AUT(H,\cdot)$. So, $\alpha\in AUT(H,\cdot)\Leftrightarrow \beta = \alpha(J,J,J)(R_{f^{-1}}^{-1}, L_{f^{-1}}R_{f^{-1}}, R_{f^{-1}})\in AUT(H,\cdot)$. Since $(G_H,\cdot)$ is a $\mathrm{S}_{2^{nd}}\mathrm{BL}$,

$xL_fR_fL_{f^{-1}}R_{f^{-1}} = [f^{-1}(fx\cdot f)]f^{-1} = [(f^{-1}f\cdot x)f]f^{-1} = x$ for all $x\in G$. That is, $L_fR_fL_{f^{-1}}R_{f^{-1}} = I$ in $(G_H,\cdot)$. Also, since $J\in AUM(H,\cdot)$, then $R_fJ = JR_{f^{-1}}$ and $L_fJ = JL_{f^{-1}}$ in $(H,\cdot)$. So,

$$\beta = (JL_fR_fJR_{f^{-1}}^{-1}, L_fR_fJ^2L_{f^{-1}}R_{f^{-1}}, R_fJL_fJR_{f^{-1}}) =$$
$$(JL_fJR_{f^{-1}}R_{f^{-1}}^{-1}, L_fR_fL_{f^{-1}}R_{f^{-1}}, R_fL_{f^{-1}}R_{f^{-1}}) = (L_{f^{-1}}, I, R_fL_{f^{-1}}R_{f^{-1}}).$$

Hence, $(G_H, \circ)$ is a $\mathrm{S_{1^{st}}}$AIPL iff $\beta \in AUT(H, \cdot)$.

Now, assume that $\beta \in AUT(H, \cdot)$. Then , $xL_{f^{-1}} \cdot y = (xy)R_fL_{f^{-1}}R_{f^{-1}}$ for all $x, y \in H$. For $y = e$, $L_{f^{-1}} = R_fL_{f^{-1}}R_{f^{-1}}$ in $(H, \cdot)$. so, $\beta = (L_{f^{-1}}, I, L_{f^{-1}}) \in AUT(H, \cdot) \Rightarrow f^{-1} \in N_\lambda(H, \cdot) \Rightarrow f \in N_\lambda(H, \cdot)$.

On the other hand, if $f \in N_\lambda(H, \cdot)$, then, $\gamma = (L_f, I, L_f) \in AUT(H, \cdot)$. But $f \in N_\lambda(H, \cdot) \Rightarrow L_f^{-1} = L_{f^{-1}} = R_fL_{f^{-1}}R_{f^{-1}}$ in $(H, \cdot)$. Hence, $\beta = \gamma^{-1}$ and $\beta \in AUT(H, \cdot)$.

Corollary 3.1 *Let $(G_H, \cdot)$ be a $S_{2^{nd}}BL$ and a $S_{1^{st}}AIPL$. Then, for any special loop $(G_H, \circ)$ that is S-isotopic to $(G_H, \cdot)$, $(G_H, \circ)$ is a $S_{1^{st}}AIPL$ iff $(G_H, \cdot)$ is a $S_{1^{st}}$-loop and a $S_{1^{st}}$ commutative loop.*

Proof

Suppose every special loop that is S-isotopic to $(G_H, \cdot)$ is a $\mathrm{S_{1^{st}}}$AIPL. Then, $f \in N_\lambda(H, \cdot)$ for all $f \in H$ by Theorem 3.7. So, $(G_H, \cdot)$ is a $\mathrm{S_{1^{st}}}$-loop. Then, $y^{-1}x^{-1} = (xy)^{-1} = x^{-1}y^{-1}$ for all $x, y \in H$. So, $(G_H, \cdot)$ is a $\mathrm{S_{1^{st}}}$ commutative loop.

The proof of the converse is as follows. If $(G_H, \cdot)$ is a $\mathrm{S_{1^{st}}}$-loop and a $\mathrm{S_{1^{st}}}$ commutative loop, then for all $x, y \in H$ such that $x \circ y = xR_f \cdot yL_f^{-1}$,

$$(x \circ y) \circ z = (xR_f \cdot yL_f^{-1})R_f \cdot zL_f^{-1} = (xf \cdot f^{-1}y)f \cdot f^{-1}z.$$
$$x \circ (y \circ z) = xR_f \cdot (yR_f \cdot zL_f^{-1})L_f^{-1} = xf \cdot f^{-1}(yf \cdot f^{-1}z).$$

So, $(x \circ y) \circ z = x \circ (y \circ z)$. Thus, $(H, \circ)$ is a group. Furthermore,

$$x \circ y = xR_f \cdot yL_f^{-1} = xf \cdot f^{-1}y = x \cdot y = y \cdot x = yf \cdot f^{-1}x = y \circ x.$$

So, $(H, \circ)$ is commutative and so has the AIP. Therefore, $(G_H, \circ)$ is a $\mathrm{S_{1^{st}}}$AIPL.

Lemma 3.2 *Let $(G_H, \cdot)$ be a $S_{2^{nd}}BL$. Then, every special loop that is S-isotopic to $(G_H, \cdot)$ is S-isomorphic to $(G_H, \cdot)$ if and only if $(G_H, \cdot)$ obeys the identity $(x \cdot fg)g^{-1} \cdot f\backslash(y \cdot fg) = (xy) \cdot (fg)$ for all $x, y \in G_H$ and $f, g \in H$.*

Proof

Let $(G_H, \circ)$ be an arbitrary S-principal isotope of $(G_H, \cdot)$. It is claimed that $(G_H, \cdot) \overset{R_{fg}}{\succsim} (G_H, \circ)$ iff $xR_{fg} \circ yR_{fg} = (x \cdot y)R_{fg}$ iff $(x \cdot fg)R_g^{-1} \cdot (y \cdot fg)L_f^{-1} = (x \cdot y)R_{fg}$ iff $(x \cdot fg)g^{-1} \cdot f\backslash(y \cdot fg) = (xy) \cdot (fg)$ for all $x, y \in G_H$ and $f, g \in H$.

Theorem 3.8 *Let $(G_H, \cdot)$ be a $S_{2^{nd}}BL$, let $f \in H$, and let $x \circ y = xR_f \cdot yL_f^{-1}$ for all $x, y \in G$. Then, $(G_H, \cdot) \succsim (G_H, \circ)$ if and only if there exists a $S_{1^{st}}$ pseudo-automorphism of $(G_H, \cdot)$ with $S_{1^{st}}$ companion f.*

Proof
$(G_H,\cdot) \succsim (G_H,\circ)$ if and only if there exists $T\in SSYM(G_H,\cdot)$ such that $xT\circ yT=(x\cdot y)T$ for all $x,y\in G$ iff $xTR_f\cdot yTL_f^{-1}=(x\cdot y)T$ for all $x,y\in G$ iff $\alpha=(TR_f,TL_f^{-1},T)\in$ $\mathrm{S}_{1^{\mathrm{st}}}AUT(G_H)$.

Recall that by Theorem 2.2, $(G_H,\cdot)$ is a $\mathrm{S}_{2^{\mathrm{nd}}}$BL iff $(R_f^{-1},L_fR_f,R_f)\in \mathrm{S}_{1^{\mathrm{st}}}AUT(G_H,\cdot)$ for each $f\in H$. So,

$$\alpha\in \mathrm{S}_{1^{\mathrm{st}}}AUT(G_H)\Leftrightarrow \beta=\alpha(R_f^{-1},L_fR_f,R_f)=$$
$$(T,TR_f,TR_f)\in \mathrm{S}_{1^{\mathrm{st}}}AUT(G_H,\cdot)\Leftrightarrow T\in \mathrm{S}_{1^{\mathrm{st}}}PAUT(G_H)$$

with $\mathrm{S}_{1^{\mathrm{st}}}$ companion f.

Corollary 3.2 *Let $(G_H,\cdot)$ be a $S_{2^{nd}}BL$, let $f\in H$ and let $x\circ y=xR_f\cdot yL_f^{-1}$ for all $x,y\in G_H$. If $f\in N_\rho(H,\cdot)$, then, $(G_H,\cdot)\succsim(G_H,\circ)$.*

Proof
Following Theorem 3.8, $f\in N_\rho(H,\cdot)\Rightarrow T\mathrm{S}_{1^{\mathrm{st}}}PAUT(G_H)$ with $\mathrm{S}_{1^{\mathrm{st}}}$ companion f.

Corollary 3.3 *Let $(G_H,\cdot)$ be a $S_{2^{nd}}BL$. Then, every special loop that is S-isotopic to $(G_H,\cdot)$ is S-isomorphic to $(G_H,\cdot)$ if and only if each element of H is a $S_{1^{st}}$ companion for a $S_{1^{st}}$ pseudo-automorphism of $(G_H,\cdot)$.*

Proof
This follows from Theorem 3.6 and Theorem 3.8.

Corollary 3.4 *Let $(G_H,\cdot)$ be a $S_{2^{nd}}BL$. Then, $(G_H,\cdot)$ is a SGS-loop if and only if each element of H is a $S_{1^{st}}$ companion for a $S_{1^{st}}$ pseudo-automorphism of $(G_H,\cdot)$.*

Proof
This is an immediate consequence of Corollary 3.4.

Remark 3.2 *Every Bol loop is a $S_{2^{nd}}BL$. Most of the results on isotopy of Bol loops in chapter 3 of [19] can easily be deduced from the results in this paper by simply forcing H to be equal to G.*

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