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The Smarandache Curves on H_0^2

Murat SAVAS[1] , Atakan Tugkan YAKUT[2,♠], Tugba TAMIRCI[1]

[1] *Gazi University, Faculty of Sciences, Department of Mathematics, 06500 Teknikokullar, Ankara, Turkey*
[2] *Nigde University, Faculty of Arts and Sciences, Department of Mathematics, 51350, Nigde, Turkey*

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ABSTRACT

In this study, we give special Smarandache curves according to the Sabban frame in hyperbolic space and new Smarandache partners in de Sitter space. The existence of duality between Smarandache curves in hyperbolic and de Sitter space is obtained. We also describe how we can depict picture of Smarandache partners in de Sitter space of a curve in hyperbolic space. Finally, two examples are given to illustrate our main results.

Key words: *Smarandache curves, de Sitter space, Sabban frame, Minkowski space.*

1. INTRODUCTION

Regular curves have an important role in the theory of curves in differential geometry and relativity theory. In the geometry of regular curves in Euclidean or Minkowskian spaces, it is well-known that one of the most important problem is the characterization and classification of these curves. In the theory of regular curves, there are some special curves, such as Bertrand, Mannheim, involute, evolute, and pedal curves. In the light of the literature, in [11] authors introduced a special curve by Frenet-Serret frame vector fields in Minkowski space-time. The new special curve, which is named Smarandache curve according to the Sabban frame in the Euclidean unit sphere, is defined by Turgut and Yilmaz in Minkowski space-time [11]. Smarandache curves in Euclidean or non-Euclidean

♠Corresponding author, e-mail: *sevaty@nigde.edu.tr*

spaces have been recently of particular interest for researchers. In Euclidean differential geometry, Smarandache curves of a curve are defined to be combination of its position, tangent, and normal vectors. These curves have been also studied widely [1, 4, 6, 9, 11, 12]. Smarandache curves play an important role in Smarandache geometry. They are the objects of Smarandache geometry, i.e. a geometry which has at least one Smarandachely denied axiom [2]. An axiom is said to be Smarandachely denied if it behaves in at least two different ways within the same space. Smarandache geometry has a significant role in the theory of relativity and parallel universes. Ozturk U., et al. studied Smarandache curves in hyperbolic space but they don't give dual Smarandache partners of these curves in de Sitter space [6]. We answer it for curves in hyperbolic space and show the Smarandache partners curve of these curves in de Sitter space. We explain the Smarandache de Sitter duality of curves in hyperbolic space. In this paper, we give the Smarandache partner curves in de Sitter space according to the Sabban frame $\{\alpha, t, \xi\}$ of a curve in hyperbolic space. We obtain the geodesic curvatures and the expressions for the Sabban frame's vectors of special Smarandache curves on de Sitter surface. In particular, we see that the timelike $\alpha\xi$-Smarandache curve of a curve α does not exist in de Sitter space. We give some examples of the Smarandache curves in hyperbolic space and its dual Smarandache curves in de Sitter space. Furthermore, we give some examples of special hyperbolic and de Sitter Smarandache curves, which are found in the study of Yakut et al. [12]. In her Master thesis [9], Tamirci also studied the curves in de Sitter and hyperbolic spaces using a similar framework.

2. PRELIMINARIES

In this section, we use the basic notions and results in Lorentzian geometry. For more detailed concepts, see [7,8]. Let $\mathbb{R}^3$ be the 3-dimensional vector space equipped with the scalar product $\langle\ ,\ \rangle$ which is defined by

$$\langle x, y\rangle_L = -x_1y_1 + x_2y_2 + x_3y_3.$$

The space $E_1^3 = (\mathbb{R}^3, \langle\ ,\ \rangle_L)$ is a pseudo-Euclidean space, or Minkowski 3-space. The unit pseudo-sphere (de Sitter space) with index one S_1^2 in E_1^3 is given by

$$S_1^2 = \{x\epsilon E_1^3 | \langle x, x\rangle_L = 1\}.$$

The unit pseudo-hyperbolic space

$$H_0^2 = \{x\epsilon E_1^3 | \langle x, x\rangle_L = -1\}$$

has two connected components $H_{0,+}^2$ and $H_{0,-}^2$. Each of them can be taken as a model for the 2-dimensional hyperbolic space H_0^2. In this paper, we take $H_{0,+}^2 = H_0^2$. Recall that a nonzero vector $x\epsilon E_1^3$ is spacelike if $\langle x, x\rangle_L > 0$, timelike if $\langle x, x\rangle_L < 0$, and null (lightlike) if $\langle x, x\rangle_L = 0$. The norm (length) of a vector $x\epsilon E_1^3$ is given by $\|x\|_L = \sqrt{|\langle x, x\rangle_L|}$ and two vectors x and y are said to be orthogonal if $\langle x, y\rangle_L = 0$. Next, we say that an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 can locally be spacelike, timelike, or null(lightlike) if all of its velocity vectors $\alpha'(s)$ are, respectively, spacelike, timelike, or null for all $s\epsilon\mathrm{I}$. If $\|\alpha'(s)\|_L \neq 0$ for every $s\epsilon\mathrm{I}$, then α is a regular curve in E_1^3. A spacelike(timelike) regular curve α is parameterized by a pseudo-arc length parameter s, which is given by $\alpha: \mathrm{I} \subset \mathbb{R} \to E_1^3$, and then the tangent vector $\alpha'(s)$ along α has unit length, that is

$$\langle\alpha'(s), \alpha'(s)\rangle_L = 1(\langle\alpha'(s), \alpha'(s)\rangle_L = -1)$$

for all $s\epsilon\mathrm{I}$. Let

$x = (x_1, x_2, x_3),\ \ y = (y_1, y_2, y_3),\ \ z = (z_1, z_2, z_3)\epsilon E_1^3$. The Lorentzian pseudo-vector cross product is defined as follows:

$$x \wedge y = (-x_2y_3 + x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1) \quad (1)$$

We remark that the following relations hold:

(i) $\langle x \wedge y, z\rangle_L = \det(x\ y\ z)$

(ii) $x \wedge (y \wedge z) = \langle x, y\rangle_L z - \langle x, z\rangle_L y$

Let $\alpha: \mathrm{I} \subset \mathbb{R} \to H_0^2$ be a regular unit speed curve lying fully in H_0^2. Then its position vector α is a timelike vector, which implies that the tangent vector $t = \alpha'$ and normal vector ξ are unit spacelike vector for all $s\epsilon\mathrm{I}$. We have the orthonormal Sabban frame $\{\alpha(s), t(s), \xi(s)\}$ along the curve α, where $\xi(s) = \alpha(s) \wedge t(s)$ is the unit spacelike vector. The corresponding Frenet formula of α, according to the Sabban frame, is given by

$$\begin{cases} \alpha'(s) = t(s) \\ t'(s) = \alpha(s) + \kappa_g(s)\ \xi(s) \\ \xi'(s) = -\kappa_g(s)t(s) \end{cases} \quad (2)$$

where $\kappa_g(s) = \det(\alpha(s), t(s), t'(s))$ is the geodesic curvature of α on H_0^2 and s is the arc length parameter of α. In particular, the following relations hold:

$$\xi = \alpha \wedge t,\ -\alpha = t \wedge \xi,\ t = \xi \wedge \alpha \quad (3)$$

Now we define a new curve $\beta: \mathrm{I} \subset \mathbb{R} \to S_1^2$ to be a regular unit speed curve lying fully on S_1^2 for all $s\epsilon\mathrm{I}$ such that its position vector β is a unit spacelike vector according to the combination of the position, tangent, and normal vectors of α. In this case $\beta' = t_\beta$ may be a unit timelike or spacelike vector.

Definition 2.1. A unit speed regular curve $\beta(\bar{s}(s))$ lying fully in Minkowski 3-space, whose position vector is associated with Sabban frame vectors on another regular curve $\alpha(s)$, is called a Smarandache curve[11].

In the light of this definition, if a regular unit speed curve $\alpha: \mathrm{I} \subset \mathbb{R} \to H_0^2$ is lying fully on H_0^2 for all $s\epsilon\mathrm{I}$ and its position vector α is a unit timelike vector, then the

Smarandache curve $\beta=\beta(\bar{s}(s))$ of the curve α is a regular unit speed curve lying fully in S_1^2 or H_0^2. In our work we are interested in curves lying in S_1^2 and so we have the following:

a) The Smarandache curve $\beta(\bar{s}(s))$ may be a spacelike curve on S_1^2 or,

b) The Smarandache curve $\beta(\bar{s}(s))$ may be a timelike curve on S_1^2 for all $s\epsilon$I.

Let $\{\alpha,t,\xi\}$ and $\{\beta,t_\beta,\xi_\beta\}$ be the moving Sabban frames of α and β, respectively. Then we have the following definitions and theorems of Smarandache curves $\beta=\beta(\bar{s}(s))$.

3. CURVES ON H_0^2 AND ITS SPACELIKE SMARANDACHE PARTNERS ON S_1^2

Let α be a regular unit speed curve on H_0^2. Then the Smarandache partner curve of α is either in de Sitter or in hyperbolic space. β is called de Sitter dual of α in hyperbolic space. In this section we obtain the spacelike Smarandache partners in de Sitter space of a curve in hyperbolic space.

Definition 3.1. Let $\alpha=\alpha(s)$ be a unit speed regular curve lying fully on H_0^2 with the moving Sabban frame $\{\alpha,t,\xi\}$. The curve $\beta\colon \mathrm{I}\subset\mathbb{R}\to S_1^2$ of α defined by

$$\beta(\bar{s}(s))=\frac{1}{\sqrt{2}}(c_1\alpha(s)+c_2\xi(s)) \tag{4}$$

is called the spacelike αξ-Smarandache curve of α and fully lies on S_1^2, where $c_1,c_2\epsilon\mathbb{R}\backslash\{0\}$ and $-c_1^2+c_2^2=2$.

Theorem 3.1. Let $\alpha\colon \mathrm{I}\subset\mathbb{R}\to H_0^2$ be a unit speed regular curve lying fully on H_0^2 with the Sabban frame $\{\alpha,t,\xi\}$ and geodesic curvature κ_g. If $\beta\colon \mathrm{I}\subset\mathbb{R}\to S_1^2$ is the αξ-Smarandache curve of α with the Sabban frame $\{\beta,t_\beta,\xi_\beta\}$ then the relationships between the Sabban frame of α and its αξ-Smarandache curve are given by

$$\begin{bmatrix}\beta\\ t_\beta\\ \xi_\beta\end{bmatrix}=\begin{bmatrix}\frac{c_1}{\sqrt{2}} & 0 & \frac{c_2}{\sqrt{2}}\\ 0 & \varepsilon & 0\\ \frac{c_2\varepsilon}{\sqrt{2}} & 0 & \frac{c_1\varepsilon}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\alpha\\ t\\ \xi\end{bmatrix} \tag{5}$$

where $\varepsilon=\pm1$ and its geodesic curvature κ_g^β is given by

$$\kappa_g^\beta=\frac{c_1\kappa_g-c_2}{|c_1-c_2\kappa_g|}. \tag{6}$$

Proof. Differentiating the equation (4) with respect to s and considering (2), we obtain

$$\frac{d\beta}{d\bar{s}}\frac{d\bar{s}}{ds}=\frac{1}{\sqrt{2}}(c_1-c_2\kappa_g)t.$$

This can be rewritten as

$$t_\beta\frac{d\bar{s}}{ds}=\frac{1}{\sqrt{2}}(c_1-c_2\kappa_g)t \tag{7}$$

where

$$\frac{d\bar{s}}{ds}=\frac{|c_1-c_2\kappa_g|}{\sqrt{2}} \tag{8}$$

By substituting (8) into (7) we obtain a simple form of Eq. 7 as follows,

$$t_\beta=\varepsilon t \tag{9}$$

where $\varepsilon=1$ if $c_1-c_2\kappa_g>0$ for all s and $\varepsilon=-1$ if $c_1-c_2\kappa_g<0$ for all s. It can be easily seen that the tangent vector t_β is a unit spacelike vector. Taking the Lorentzian vector cross product of (4) with (9) we have

$$\begin{aligned}\xi_\beta&=\beta\wedge t_\beta\\ &=\frac{\varepsilon}{\sqrt{2}}(c_2\alpha+c_1\xi)\end{aligned} \tag{10}$$

It is easily seen that ξ_β is a unit timelike vector. On the other hand, by taking the derivative of the equation (9) with respect to s, we find

$$\frac{dt_\beta}{d\bar{s}}\frac{d\bar{s}}{ds}=\varepsilon(\alpha+\kappa_g\xi) \tag{11}$$

By substituting (8) into (11) we find

$$t'_\beta=\frac{\sqrt{2}\varepsilon}{|c_1-c_2\kappa_g|}(\alpha+\kappa_g\xi). \tag{12}$$

Consequently, from (4), (9), and (12), the geodesic curvature κ_g^β of the curve $\beta=\beta(\bar{s}(s))$ is explicitly obtained by

$$\kappa_g^\beta=\det(\beta,t_\beta,t'_\beta)=\frac{c_1\kappa_g-c_2}{|c_1-c_2\kappa_g|} \tag{13}$$

Thus, the theorem is proved. In three theorems that follow, in a similar way as in Theorem 3.1 we obtain the Sabban frame $\{\beta,t_\beta,\xi_\beta\}$ and the geodesic curvature κ_g^β of a spacelike Smarandache curve. We omit the proofs of Theorems 3.2, 3.3, and 3.4, since they are analogous to the proof of Theorem 3.1.

Definition 3.2. Let $\alpha=\alpha(s)$ be a regular unit speed curve lying fully on H_0^2. Then the spacelike αt-Smarandache curve $\beta\colon \mathrm{I}\subset\mathbb{R}\to S_1^2$ of α defined by

$$\beta(\bar{s}(s))=\frac{1}{\sqrt{2}}(c_1\alpha(s)+c_2t(s)) \tag{14}$$

where $c_1,c_2\epsilon\mathbb{R}\backslash\{0\}$ and $-c_1^2+c_2^2=2$.

Theorem 3.2. Let $\alpha\colon \mathrm{I}\subset\mathbb{R}\to H_0^2$ be a regular unit speed curve lying fully on H_0^2 with the Sabban frame $\{\alpha,t,\xi\}$ and geodesic curvature κ_g. If $\beta\colon \mathrm{I}\subset\mathbb{R}\to S_1^2$ is the spacelike αt-Smarandache curve of α, then its frame $\{\beta,t_\beta,\xi_\beta\}$ is given by

$$\begin{bmatrix}\beta\\ t_\beta\\ \xi_\beta\end{bmatrix}=\begin{bmatrix}\frac{c_1}{\sqrt{2}} & \frac{c_2}{\sqrt{2}} & 0\\ \frac{c_2}{\sqrt{c_2^2\kappa_g^2-2}} & \frac{c_1}{\sqrt{c_2^2\kappa_g^2-2}} & \frac{c_2\kappa_g}{\sqrt{c_2^2\kappa_g^2-2}}\\ \frac{-c_2^2\kappa_g}{\sqrt{2(c_2^2\kappa_g^2-2)}} & \frac{-c_1c_2\kappa_g}{\sqrt{2(c_2^2\kappa_g^2-2)}} & \frac{-2}{\sqrt{2(c_2^2\kappa_g^2-2)}}\end{bmatrix}\begin{bmatrix}\alpha\\ t\\ \xi\end{bmatrix} \qquad (15)$$

The geodesic curvature κ_g^β of the curve β is given by

$$\kappa_g^\beta=\frac{1}{\left(c_2^2\kappa_g^2-2\right)^{5/2}}\left(c_2^2\kappa_g\lambda_1-c_1c_2\kappa_g\lambda_2-2\lambda_3\right) \qquad (16)$$

where $c_2^2\kappa_g^2>2$ and

$$\begin{cases}\lambda_1=-c_2^3\kappa_g\kappa_g'+c_1(c_2^2\kappa_g^2-2)\\ \lambda_2=-c_1c_2^2\kappa_g\kappa_g'+(c_2-c_2\kappa_g^2)(c_2^2\kappa_g^2-2)\\ \lambda_3=-c_2^3\kappa_g^2\kappa_g'+(c_1\kappa_g+c_2\kappa_g')(c_2^2\kappa_g^2-2)\end{cases} \qquad (17)$$

Definition 3.3. Let $\alpha:\mathrm{I}\subset\mathbb{R}\to H_0^2$ be a regular unit speed curve lying fully on H_0^2. Then the spacelike $t\xi$-Smarandache curve $\beta:\mathrm{I}\subset\mathbb{R}\to S_1^2$ of α defined by

$$\beta(\bar{s}(s))=\frac{1}{\sqrt{2}}(c_1t(s)+c_2\xi(s)) \qquad (18)$$

where $c_1,c_2\epsilon\mathbb{R}\backslash\{0\}$ and $c_1^2+c_2^2=2$.

Theorem 3.3. Let $\alpha:\mathrm{I}\subset\mathbb{R}\to H_0^2$ be a regular unit speed curve lying fully on H_0^2 with the Sabban frame $\{\alpha,t,\xi\}$ and geodesic curvature κ_g. If $\beta:\mathrm{I}\subset\mathbb{R}\to S_1^2$ is the spacelike $t\xi$-Smarandache curve of α, then its frame $\{\beta,t_\beta,\xi_\beta\}$ is given by

$$\begin{bmatrix}\beta\\ t_\beta\\ \xi_\beta\end{bmatrix}=\begin{bmatrix}0 & \frac{c_1}{\sqrt{2}} & \frac{c_2}{\sqrt{2}}\\ \frac{c_1}{\sqrt{2\kappa_g^2-c_1^2}} & \frac{-c_2\kappa_g}{\sqrt{2\kappa_g^2-c_1^2}} & \frac{c_1\kappa_g}{\sqrt{2\kappa_g^2-c_1^2}}\\ \frac{-2\kappa_g}{\sqrt{2(2\kappa_g^2-c_1^2)}} & \frac{c_1c_2}{\sqrt{2(2\kappa_g^2-c_1^2)}} & \frac{-c_1^2}{\sqrt{2(2\kappa_g^2-c_1^2)}}\end{bmatrix}\begin{bmatrix}\alpha\\ t\\ \xi\end{bmatrix} \qquad (19)$$

he geodesic curvature κ_g^β of the curve β is given by

$$\kappa_g^\beta=\frac{1}{\left(2\kappa_g^2-c_1^2\right)^{5/2}}\left(2\kappa_g\lambda_1+c_1c_2\lambda_2-c_1^2\lambda_3\right) \qquad (20)$$

where $c_1^2<2\kappa_g^2$ and

$$\begin{cases}\lambda_1=-2c_1\kappa_g\kappa_g'-c_2\kappa_g(2\kappa_g^2-c_1^2)\\ \lambda_2=2c_2\kappa_g^2\kappa_g'+\left(c_1-c_2\kappa_g'-c_1\kappa_g^2\right)(2\kappa_g^2-c_1^2)\\ \lambda_3=-2c_1\kappa_g^2\kappa_g'+\left(-c_2\kappa_g^2+c_1\kappa_g'\right)\left(2\kappa_g^2-c_1^2\right)\end{cases} \qquad (21)$$

Definition 3.4. Let $\alpha:\mathrm{I}\subset\mathbb{R}\to H_0^2$ be a regular unit speed curve lying fully on H_0^2. Then the spacelike $\alpha t\xi$-Smarandache curve $\beta:\mathrm{I}\subset\mathbb{R}\to S_1^2$ of α defined by

$$\beta(\bar{s}(s))=\frac{1}{\sqrt{3}}(c_1\alpha(s)+c_2t(s)+c_3\xi(s)) \qquad (22)$$

where $c_1,c_2,c_3\epsilon\mathbb{R}\backslash\{0\}$ and $-c_1^2+c_2^2+c_3^2=3$.

Theorem 3.4. Let $\alpha:\mathrm{I}\subset\mathbb{R}\to H_0^2$ be a regular unit speed curve lying fully on H_0^2 with the Sabban frame $\{\alpha,t,\xi\}$ and geodesic curvature κ_g. If $\beta:\mathrm{I}\subset\mathbb{R}\to S_1^2$ is the $\alpha t\xi$-Smarandache curve of α, then its frame $\{\beta,t_\beta,\xi_\beta\}$ is given by

$$\begin{bmatrix}\beta\\ t_\beta\\ \xi_\beta\end{bmatrix}=\begin{bmatrix}\frac{c_1}{\sqrt{3}} & \frac{c_2}{\sqrt{3}} & \frac{c_3}{\sqrt{3}}\\ \frac{c_2}{\sqrt{A}} & \frac{c_1-c_3\kappa_g}{\sqrt{A}} & \frac{c_2\kappa_g}{\sqrt{A}}\\ \frac{-c_2^2\kappa_g-c_3(-c_1+c_3\kappa_g)}{\sqrt{3A}} & \frac{c_2c_3-c_1c_2\kappa_g}{\sqrt{3A}} & \frac{c_1(c_1-c_3\kappa_g)-c_2^2}{\sqrt{3A}}\end{bmatrix}\times\begin{bmatrix}\alpha\\ t\\ \xi\end{bmatrix} \qquad (23)$$

where

$A=\left(c_1-c_3\kappa_g\right)^2-c_2^2+c_2^2\kappa_g^2$, $\left(c_1-c_3\kappa_g\right)^2>c_2^2-c_2^2\kappa_g^2$ and the Smarandache curve β is a spacelike curve. Furthermore, the geodesic curvature κ_g^β of curve β is given by

$$\kappa_g^\beta=\left(\left(c_2^2\kappa_g+c_3^2\kappa_g-c_1c_3\right)\lambda_1+\left(-c_1c_2\kappa_g+c_2c_3\right)\lambda_2+\left(c_1^2-c_1c_3\kappa_g-c_2^2\right)\lambda_3\right)\times\left(\left(\left(c_1-c_3\kappa_g\right)^2-c_2^2+c_2^2\kappa_g^2\right)^{5/2}\right)^{-1} \qquad (24)$$

where

$$\begin{cases}\lambda_1=c_2\left(c_3\kappa_g'\left(c_1-c_3\kappa_g\right)-c_2^2\kappa_g\kappa_g'\right)\\ \qquad +\left(c_1-c_3\kappa_g\right)\left(\left(c_1-c_3\kappa_g\right)^2-c_2^2+c_2^2\kappa_g^2\right)\\ \lambda_2=\left(c_1-c_3\kappa_g\right)\left(c_3\kappa_g'\left(c_1-c_3\kappa_g\right)-c_2^2\kappa_g\kappa_g'\right)\\ \qquad +\left(c_2-c_3\kappa_g'-c_2\kappa_g^2\right)\left(\left(c_1-c_3\kappa_g\right)^2-c_2^2+c_2^2\kappa_g^2\right)\\ \lambda_3=c_2\kappa_g\left(c_3\kappa_g'\left(c_1-c_3\kappa_g\right)-c_2^2\kappa_g\kappa_g'\right)\\ \qquad +\left(\kappa_g\left(c_1-c_3\kappa_g\right)+c_2\kappa_g'\right)\left(\left(c_1-c_3\kappa_g\right)^2-c_2^2+c_2^2\kappa_g^2\right)\end{cases} \qquad (25)$$

Example 3.1. Let us consider a regular unit speed curve α on H_0^2 defined by

$$\alpha(s)=\left(\frac{(s-1)^2}{2}+1,\ \frac{(s-1)^2}{2},\ s-1\right).$$

Then the orthonormal Sabban frame $\{\alpha(s),t(s),\xi(s)\}$ of α can be calculated as follows:

$$\begin{cases}\alpha(s)=\left(\frac{(s-1)^2}{2}+1,\ \frac{(s-1)^2}{2},\ s-1\right)\\ t(s)=(s-1,\ s-1,\ 1)\\ \xi(s)=\left(\frac{(s-1)^2}{2},\ \frac{(s-1)^2}{2}-1,\ s-1\right)\end{cases}$$

The geodesic curvature of α is -1. In terms of the definitions, we obtain the spacelike Smarandache curves on S_1^2 according to the Sabban frame on H_0^2.

First, when we take $c_1=1$ and $c_2=\sqrt{3}$, then the $\alpha\xi$-Smarandache curve is spacelike and given by

$$\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}\left(\left(\frac{1+\sqrt{3}}{2}\right)(s-1)^2+1,\ \left(\frac{1+\sqrt{3}}{2}\right)(s-1)^2-\sqrt{3},\ (s-1)(1+\sqrt{3})\right)$$

and the Sabban frame of the spacelike $\alpha\xi$-Smarandache curve is given by

$$\begin{bmatrix}\beta\\ t_\beta\\ \xi_\beta\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{\sqrt{3}}{\sqrt{2}}\\ 0 & 1 & 0\\ \frac{\sqrt{3}}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\alpha\\ t\\ \xi\end{bmatrix}$$

and its geodesic curvature κ_g^β is -1.

Second, when we take $c_1 = 1$ and $c_2 = \sqrt{3}$, then the αt-Smarandache curve is a spacelike and given by

$$\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}\left(\frac{(s-1)^2}{2}+\sqrt{3}(s-1)+1,\ \frac{(s-1)^2}{2}+\sqrt{3}(s-1), (s-1+\sqrt{3})\right)$$

and the Sabban frame of the spacelike αt-Smarandache curve is given by

$$\begin{bmatrix}\beta\\ t_\beta\\ \xi_\beta\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} & 0\\ \sqrt{3} & 1 & -\sqrt{3}\\ \frac{3}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} & -\sqrt{2}\end{bmatrix}\begin{bmatrix}\alpha\\ t\\ \xi\end{bmatrix}$$

and its geodesic curvature κ_g^β is -1.

Third, when we take $c_1 = 1$ and $c_2 = 1$, then the $t\xi$-Smarandache curve is a spacelike curve and given by

$$\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}\left(\frac{(s-1)^2}{2}+s-1,\ \frac{(s-1)^2}{2}+s-2,\ s\right)$$

and the Sabban frame of the spacalike $t\xi$-Smarandache curve is given by

$$\begin{bmatrix}\beta\\ t_\beta\\ \xi_\beta\end{bmatrix} = \begin{bmatrix}0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ 1 & 1 & -1\\ \sqrt{2} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\alpha\\ t\\ \xi\end{bmatrix}$$

and its geodesic curvature κ_g^β is -1.

Finally, when we take $c_1 = \sqrt{3}$, $c_2 = \sqrt{3}$ and $c_3 = \sqrt{3}$,

then the $\alpha t\xi$-Smarandache curve is a spacelike curve and given by

$$\beta(\bar{s}(s)) = ((s-1)^2+s, (s-1)^2+s-2, 2s-1)$$

and the Sabban frame of the spacelike $\alpha t\xi$-Smarandache curve is given by

$$\begin{bmatrix}\beta\\ t_\beta\\ \xi_\beta\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ \frac{1}{2} & 1 & -\frac{1}{2}\\ \frac{3}{2} & 1 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}\alpha\\ t\\ \xi\end{bmatrix}$$

and its geodesic curvature κ_g^β is -1.

We give a curve α in hyperbolic space and its Smarandache partners in de Sitter space in Figure 1.

4. CURVES IN HYPERBOLIC SPACE AND DUAL TIMELIKE SMARANDACHE PARTNERS

In this section we obtain the timelike Smarandache partners in de Sitter space of a curve in hyperbolic space.

Theorem 4.1. Let $\alpha: \mathrm{I} \subset \mathbb{R} \to H_0^2$ be a regular unit speed curve lying fully on H_0^2. Then the timelike $\alpha\xi$-Smarandache curve $\beta: \mathrm{I} \subset \mathbb{R} \to S_1^2$ of α does not exist.

Proof. Assume that $\alpha: \mathrm{I} \subset \mathbb{R} \to H_0^2$ be a regular unit speed curve lying fully on H_0^2. Then the timelike

$\alpha\xi$-Smarandache curve $\beta: \mathrm{I} \subset \mathbb{R} \to S_1^2$ of α is defined by

$$\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}(c_1\alpha(s) + c_2\xi(s)) \tag{26}$$

$c_1, c_2 \epsilon \mathbb{R}\backslash\{0\}$, $-c_1^2 + c_2^2 = 2$. Differentiating (26) with respect to s and using (2), we obtain

$$\beta'(s) = \frac{d\beta}{d\bar{s}}\frac{d\bar{s}}{ds} = \frac{1}{\sqrt{2}}(c_1 - c_2\kappa_g)t$$

$$t_\beta\frac{d\bar{s}}{ds} = \frac{1}{\sqrt{2}}(c_1 - c_2\kappa_g)t$$

where

$$\frac{d\bar{s}}{ds} = \sqrt{-\frac{(c_1-c_2\kappa_g)^2}{2}}$$

which is a contradiction.

In the corollaries which follow, in a similar way as in the previous section, we obtain the Sabban frame $\{\beta, t_\beta, \xi_\beta\}$ and the geodesic curvature κ_g^β of a timelike Smarandache curve. We omit the proofs of Theorems 4.2, 4.3, and 4.4.

Definition 4.1. Let $\alpha: \mathrm{I} \subset \mathbb{R} \to H_0^2$ be a regular unit speed curve lying fully on H_0^2. Then, the timelike αt-Smarandache curve $\beta: \mathrm{I} \subset \mathbb{R} \to S_1^2$ of α is defined by

$$\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}(c_1\alpha(s) + c_2 t(s))$$

where $c_1, c_2 \epsilon \mathbb{R}\backslash\{0\}$, and $-c_1^2 + c_2^2 = 2$.

Corollary 4.1. Let $\alpha: \mathrm{I} \subset \mathbb{R} \rightarrow H_0^2$ be a regular unit speed curve lying fully on H_0^2 with the Sabban frame $\{\alpha, t, \xi\}$ and the geodesic curvature κ_g. If $\beta: \mathrm{I} \subset \mathbb{R} \rightarrow S_1^2$ is the timelike αt-Smarandache curve of α, then its frame $\{\beta, t_\beta, \xi_\beta\}$ is given by

$$\begin{bmatrix} \beta \\ t_\beta \\ \xi_\beta \end{bmatrix} = \begin{bmatrix} \frac{c_1}{\sqrt{2}} & \frac{c_2}{\sqrt{2}} & 0 \\ \frac{c_2}{\sqrt{2-c_2^2\kappa_g^2}} & \frac{c_1}{\sqrt{2-c_2^2\kappa_g^2}} & \frac{c_2\kappa_g}{\sqrt{2-c_2^2\kappa_g^2}} \\ \frac{-c_2^2\kappa_g}{\sqrt{2(2-c_2^2\kappa_g^2)}} & \frac{-c_1c_2\kappa_g}{\sqrt{2(2-c_2^2\kappa_g^2)}} & \frac{-2}{\sqrt{2(2-c_2^2\kappa_g^2)}} \end{bmatrix} \begin{bmatrix} \alpha \\ t \\ \xi \end{bmatrix}$$

and the corresponding geodesic curvature κ_g^β is given by

$$\kappa_g^\beta = \frac{1}{(2-c_2^2\kappa_g^2)^{5/2}}\left(c_2^2\kappa_g\lambda_1 - c_1c_2\kappa_g\lambda_2 - 2\lambda_3\right)$$

where $-c_1^2 + c_2^2 = 2$ with $c_1, c_2 \epsilon \mathbb{R}\backslash\{0\}$, $c_2^2\kappa_g^2 < 2$ and

$$\begin{cases} \lambda_1 = c_2^3\kappa_g\kappa_g' + c_1(2 - c_2^2\kappa_g^2) \\ \lambda_2 = c_1c_2^2\kappa_g\kappa_g' + (c_2 - c_2\kappa_g^2)(2 - c_2^2\kappa_g^2) \\ \lambda_3 = c_2^3\kappa_g^2\kappa_g' + (c_1\kappa_g + c_2\kappa_g')(2 - c_2^2\kappa_g^2). \end{cases}$$

Definition 4.2. Let $\alpha: \mathrm{I} \subset \mathbb{R} \rightarrow H_0^2$ be a regular unit speed curve lying fully on H_0^2. Then, the timelike $t\xi$-Smarandache curve $\beta: \mathrm{I} \subset \mathbb{R} \rightarrow S_1^2$ of α is defined by

$$\beta(\bar{s}(s)) = \frac{1}{\sqrt{2}}\left(c_1 t(s) + c_2 \xi(s)\right),$$

where $c_1, c_2 \epsilon \mathbb{R}\backslash\{0\}$, and $c_1^2 + c_2^2 = 2$.

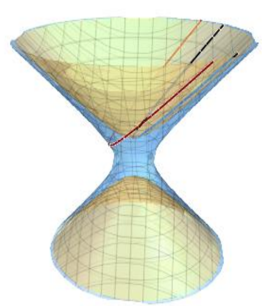
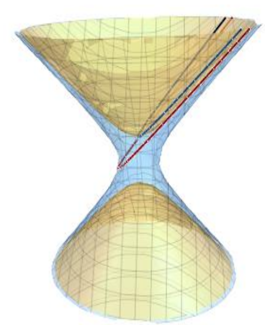
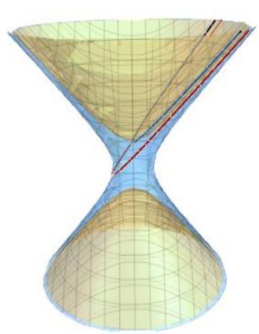
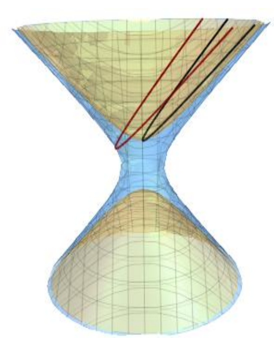
α is a unit speed curve on H_0^2 (black line)	
β is a unit speed spacelike curve on S_1^2 (red line)	
 $\alpha\xi$-Smarandache curve	 αt-Smarandache curve
 $t\xi$-Smarandache curve	 $\alpha t\xi$-Smarandache curve

Figure 1. Spacelike Smarandache partner curves of a curve α on H_0^2

Corollary 4.2. Let $\alpha: \mathrm{I} \subset \mathbb{R} \to H_0^2$ be a regular unit speed curve lying fully on H_0^2 with the Sabban frame $\{\alpha, t, \xi\}$ and geodesic curvature κ_g. If $\beta: \mathrm{I} \subset \mathbb{R} \to S_1^2$ is the timelike $t\xi$-Smarandache curve of α, then its frame $\{\beta, t_\beta, \xi_\beta\}$ is given by

$$\begin{bmatrix}\beta\\ t_\beta\\ \xi_\beta\end{bmatrix} = \begin{bmatrix} 0 & \frac{c_1}{\sqrt{2}} & \frac{c_2}{\sqrt{2}} \\ \frac{c_1}{\sqrt{c_1^2-2\kappa_g^2}} & \frac{-c_2\kappa_g}{\sqrt{c_1^2-2\kappa_g^2}} & \frac{c_1\kappa_g}{\sqrt{c_1^2-2\kappa_g^2}} \\ \frac{-2\kappa_g}{\sqrt{2\left(c_1^2-2\kappa_g^2\right)}} & \frac{c_1c_2}{\sqrt{2\left(c_1^2-2\kappa_g^2\right)}} & \frac{-c_1^2}{\sqrt{2\left(c_1^2-2\kappa_g^2\right)}} \end{bmatrix} \begin{bmatrix}\alpha\\ t\\ \xi\end{bmatrix}$$

and the corresponding geodesic curvature κ_g^β is given by

$$\kappa_g^\beta = \frac{1}{\left(c_1^2-2\kappa_g^2\right)^{5/2}}\left(2\kappa_g\lambda_1 + c_1c_2\lambda_2 + c_1^2\lambda_3\right)$$

where $c_1, c_2 \epsilon \mathbb{R}\backslash\{0\}$, $c_1^2 + c_2^2 = 2$, $c_1^2 > 2\kappa_g^2$ and

$$\begin{cases} \lambda_1 = 2c_1\kappa_g\kappa_g' - c_2\kappa_g\left(c_1^2 - 2\kappa_g^2\right) \\ \lambda_2 = -2c_2\kappa_g^2\kappa_g' + \left(c_1 - c_2\kappa_g' - c_1\kappa_g^2\right)\left(c_1^2 - 2\kappa_g^2\right) \\ \lambda_3 = 2c_1\kappa_g^2\kappa_g' + \left(c_1\kappa_g' - c_2\kappa_g^2\right)\left(c_1^2 - 2\kappa_g^2\right). \end{cases}$$

Definition 4.3. Let $\alpha: \mathrm{I} \subset \mathbb{R} \to H_0^2$ be a regular unit speed curve lying fully on H_0^2. Then $\alpha t\xi$-Smarandache curve $\beta: \mathrm{I} \subset \mathbb{R} \to S_1^2$ of α is defined by

$$\beta\left(\bar{s}(s)\right) = \frac{1}{\sqrt{3}}\left(c_1\alpha(s) + c_2t(s) + c_3\xi(s)\right)$$

$$c_1, c_2, c_3 \epsilon \mathbb{R}\backslash\{0\},\ -c_1^2 + c_2^2 + c_3^2 = 3.$$

Corollary 4.3. Let $\alpha: \mathrm{I} \subset \mathbb{R} \to H_0^2$ be a regular unit speed curve lying fully on H_0^2 with the Sabban frame $\{\alpha, t, \xi\}$ and geodesic curvature κ_g. If $\beta: \mathrm{I} \subset \mathbb{R} \to S_1^2$ is the timelike $\alpha t\xi$-Smarandache curve of α, then its frame $\{\beta, t_\beta, \xi_\beta\}$ is given by

$$\begin{bmatrix}\beta\\ t_\beta\\ \xi_\beta\end{bmatrix}$$
$$= \begin{bmatrix} \frac{c_1}{\sqrt{3}} & \frac{c_2}{\sqrt{3}} & \frac{c_3}{\sqrt{3}} \\ \frac{c_2}{\sqrt{A^*}} & \frac{c_1 - c_3\kappa_g}{\sqrt{A^*}} & \frac{c_2\kappa_g}{\sqrt{A^*}} \\ \frac{-c_2^2\kappa_g - c_3\left(-c_1 + c_3\kappa_g\right)}{\sqrt{3A^*}} & \frac{c_2c_3 - c_1c_2\kappa_g}{\sqrt{3A^*}} & \frac{c_1\left(c_1 - c_3\kappa_g\right) - c_2^2}{\sqrt{3A^*}} \end{bmatrix} \begin{bmatrix}\alpha\\ t\\ \xi\end{bmatrix}$$

where

$A^* = c_2^2 - \left(c_1 - c_3\kappa_g\right)^2 - c_2^2\kappa_g^2,\ \left(c_1 - c_3\kappa_g\right)^2 < c_2^2 - c_2^2\kappa_g^2$

and the corresponding geodesic curvature κ_g^β is given by

$$\kappa_g^\beta = \begin{pmatrix} \left(c_2^2\kappa_g + c_3^2\kappa_g - c_1c_3\right)\lambda_1 + \left(-c_1c_2\kappa_g + c_2c_3\right)\lambda_2 \\ +\left(c_1^2 - c_1c_3\kappa_g - c_2^2\right)\lambda_3 \end{pmatrix} \times \left(\left(c_2^2 - \left(c_1 - c_3\kappa_g\right)^2 - c_2^2\kappa_g^2\right)^{5/2}\right)^{-1}$$

where $c_1, c_2, c_3 \epsilon \mathbb{R}\backslash\{0\}$, $-c_1^2 + c_2^2 + c_3^2 = 3$ and

$$\begin{cases} \lambda_1 = c_2\left(-c_3\kappa_g'\left(c_1 - c_3\kappa_g\right) + c_2^2\kappa_g\kappa_g'\right) \\ \qquad +\left(c_1 - c_3\kappa_g\right)\left(c_2^2 - \left(c_1 - c_3\kappa_g\right)^2 - c_2^2\kappa_g^2\right) \\ \lambda_2 = \left(c_1 - c_3\kappa_g\right)\left(-c_3\kappa_g'\left(c_1 - c_3\kappa_g\right) + c_2^2\kappa_g\kappa_g'\right) \\ \qquad +\left(c_2 - c_3\kappa_g' - c_2\kappa_g^2\right)\left(c_2^2 - \left(c_1 - c_3\kappa_g\right)^2 - c_2^2\kappa_g^2\right) \\ \lambda_3 = c_2\kappa_g\left(-c_3\kappa_g'\left(c_1 - c_3\kappa_g\right) + c_2^2\kappa_g\kappa_g'\right) \\ \qquad +\left(c_2\kappa_g' + \kappa_g\left(c_1 - c_3\kappa_g\right)\right)\left(c_2^2 - \left(c_1 - c_3\kappa_g\right)^2 - c_2^2\kappa_g^2\right) \end{cases}$$

Example 4.1. Let us consider a regular unit speed curve α on H_0^2 defined by
$\alpha(s) = (\cosh s, \sinh s, 0)$.

Then the orthonormal Sabban frame $\{\alpha, t, \xi\}$ of the curve α and the orthonormal Sabban frame $\{\beta, t_\beta, \xi_\beta\}$ of the curve β and the geodesic curvature κ_g^β of a timelike Smarandache curve can be calculated as in the previous example. The curve α and its Smarandache partners are given in Figure 2.

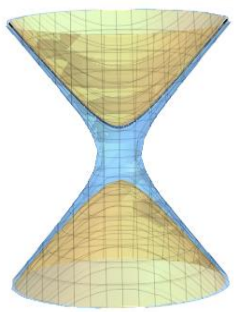
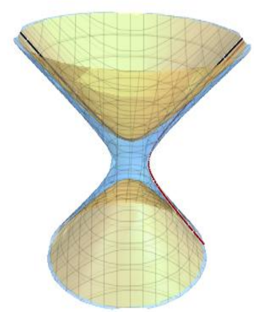
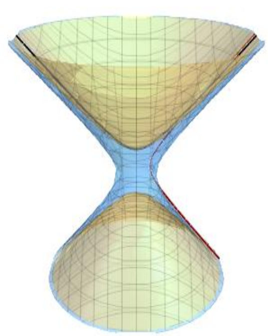
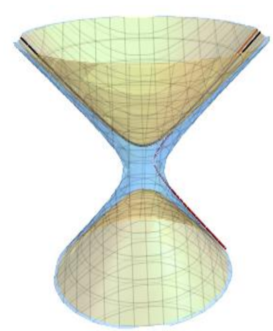
α is a unit speed curve on H_0^2 (black line)	
β is a unit speed timelike curve on S_1^2 (red line)	
 $\alpha\xi$-Smarandache curve is undefined	 αt-Smarandache curve
 $t\xi$-Smarandache curve	 $\alpha t\xi$-Smarandache curve

Figure 2. Timelike Smarandache partner curves of a curve α on H_0^2

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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