Some formulas related with theta functions and pi constant

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Abstract
This note presents some formulas for pi constant.

Introduction

Some formulas related with theta functions:

1. $$(a\,;q)_\infty = \prod_{n=0}^{\infty} (1 - a\,q^n) \quad ,\ |q| < 1$$

2. $$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q\,;-q)_\infty}{(q\,;-q)_\infty}$$

3. $$\psi(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2\,;q^2)_\infty}{(q\,;q^2)_\infty}$$

The Ramanujan-Göllnitz-Gordon continued fraction:

4. $$H(q) = \cfrac{q^{1/2}}{1+q+\cfrac{q^2}{1+q^3+\cfrac{q^4}{1+q^5+\cfrac{q^6}{1+q^7+\ldots}}}} \quad ,\ |q| < 1$$

5. $$H(q) = q^{1/2}\,\frac{(q\,;q^8)_\infty (q^7\,;q^8)_\infty}{(q^3\,;q^8)_\infty (q^5\,;q^8)_\infty}$$

6. $$H\left(e^{-\pi}\right) = \sqrt{4+2\sqrt{2}} - \sqrt{3+2\sqrt{2}}$$

Ramanujan´s cubic continued fraction:

7. $$V(q)=\cfrac{q^{1/3}}{1+\cfrac{q+q^2}{1+\cfrac{q^2+q^4}{1+\ldots}}}\ ,\ |q|<1$$

8. $$V\left(-e^{-\pi}\right)=\frac{1-\sqrt{3}}{2}$$

Notation : $\lim_{n\to\infty} x_n = x \Leftrightarrow x_n \to x$

Formulas for pi

9. $$\pi=2\ln\left(\sqrt{4+2\sqrt{2}}+\sqrt{3+2\sqrt{2}}\right)-2\sum_{n=1}^{\infty}\frac{e^{-n\pi}-e^{-3n\pi}}{n\left(1+e^{-4n\pi}\right)}$$

10. $$\pi=4\ln\left(\sqrt{1+\sqrt{2}}+\sqrt{\sqrt{2}}\right)-4\sum_{n=1}^{\infty}\frac{e^{-\frac{n\pi}{2}}-e^{-\frac{3n\pi}{2}}}{n\left(1+e^{-2n\pi}\right)}$$

11. $$\pi=\sqrt{2}\ln\left(\left(\sqrt{2+\sqrt{2}}+\sqrt{1+\sqrt{2}}\right)\left(\sqrt{1+\sqrt{2}}+\sqrt{\sqrt{2}}\right)\right)-\sqrt{2}\sum_{n=1}^{\infty}\frac{e^{-n\pi\sqrt{2}}-e^{-3n\pi\sqrt{2}}}{n\left(1+e^{-4n\pi\sqrt{2}}\right)}$$

12. $$\pi=\left(\Gamma\left(\frac{3}{4}\right)\right)^4\left(\varphi\left(e^{-\pi}\right)\right)^4=\left(\Gamma\left(\frac{3}{4}\right)\right)^4\sum_{n=0}^{\infty}c_n\,e^{-n\pi}$$

13. $$c_0=1\ ,\ c_n=\frac{2}{n}\sum_{k=1}^{n}(5k-n)\,a_k\,c_{n-k}$$

14. $$a_k=\begin{cases}1 & \{\sqrt{k}\}=0\\ 0 & \{\sqrt{k}\}>0\end{cases}\ ,\ k\in\mathbb{N}\ ,\ \{x\}=\textit{fractional part of } x$$

15. $$c_0=1\ ,\ c_n=\frac{2}{n}\sum_{k=1}^{[\sqrt{n}]}\left(5k^2-n\right)c_{n-k^2}$$

16. $[x]=\textit{Integer part of } x$

17. $c_n=\{1, 8, 24, 32, 24, 48, 96, 64, 24, 104, \ldots\}$

18. $$\ln(\pi)=4\ln\left(\Gamma\left(\frac{3}{4}\right)\right)+8\sum_{n=1}^{\infty}\frac{e^{-(2n-1)\pi}}{(2n-1)\left(1+e^{-(2n-1)\pi}\right)}$$

19. $$\ln(\pi)=4\ln\left(\Gamma\left(\frac{3}{4}\right)\right)+8\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{n+k}}{2k+1}e^{-(2k+1)(n-k+1)\pi}$$

20. $$\pi=5\ln(2)-8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}e^{-n\pi}}{n\left(1+e^{-n\pi}\right)}$$

21. $$\pi = \ln\left(12 + 8\sqrt{2}\right) - \sum_{n=1}^{\infty}\left(\frac{4\,e^{-(4n-2)\pi}}{(2n-1)\left(1+e^{-(4n-2)\pi}\right)} - \frac{2\,e^{-4n\pi}}{n\left(1+e^{-4n\pi}\right)}\right)$$

Let $\varphi(q) = \dfrac{(-q;-q)_\infty}{(q;-q)_\infty}$, $|q| < 1$, *we have*

22. $$x_{n+1} = \frac{1}{2}\left(2x_n - \varphi(x_n) + \sqrt{5\sqrt{5}-10}\;\varphi\left(x_n^5\right)\right),\ x_1 = 0 \Rightarrow x_n \to e^{-\pi}$$

23. $$x_{n+1} = \frac{1}{2}\left(2x_n - \varphi(x_n) + \sqrt[4]{6\sqrt{3}-9}\;\varphi\left(x_n^3\right)\right),\ x_1 = 0 \Rightarrow x_n \to e^{-\pi}$$

24. $$x_{n+1} = \frac{1}{2}\left(2x_n - \varphi(x_n) + \sqrt[4]{5}\;\varphi\left(x_n^5\right)\right),\ x_1 = 0 \Rightarrow x_n \to e^{-\frac{\pi}{\sqrt{5}}}$$

25. $$x_{n+1} = \frac{1}{2}\left(2x_n - \varphi(x_n) + \sqrt[4]{3}\;\varphi\left(x_n^3\right)\right),\ x_1 = 0 \Rightarrow x_n \to e^{-\frac{\pi}{\sqrt{3}}}$$

26. $$x_{n+1} = \frac{1}{8}\left(8x_n - \left(\varphi(x_n)\right)^2 + 3\left(\varphi\left(x_n^9\right)\right)^2\right),\ x_1 = 0 \Rightarrow x_n \to e^{-\frac{\pi}{3}}$$

27. $$x_{n+1} = \frac{1}{84}\left(84x_n - 5\left(\varphi(x_n)\right)^4 + 15\left(\varphi\left(x_n^3\right)\right)^4\right),\ x_1 = 0 \Rightarrow x_n \to e^{-\frac{\pi}{\sqrt{3}}}$$

28. $$x_{n+1} = \frac{1}{55}\left(55x_n - 2\left(\varphi(x_n)\right)^4 + 10\left(\varphi\left(x_n^5\right)\right)^4\right),\ x_1 = 0 \Rightarrow x_n \to e^{-\frac{\pi}{\sqrt{5}}}$$

Let $h(q) = 1 + q + \cfrac{q^2}{1 + q^3 + \cfrac{q^4}{1 + q^5 + \cfrac{q^6}{1 + q^7 + \dots}}}$, $|q| < 1$, *we have*

29. $$x_{n+1} = \left(\sqrt{4+2\sqrt{2}} - \sqrt{3+2\sqrt{2}}\right)^2\left(h(x_n)\right)^2,\ x_1 = 0 \Rightarrow x_n \to e^{-\pi}$$

30. $$x_{n+1} = \left(\sqrt{1+\sqrt{2}} - \sqrt{\sqrt{2}}\right)^2\left(h(x_n)\right)^2,\ x_1 = 0 \Rightarrow x_n \to e^{-\frac{\pi}{2}}$$

Let $u(q) = 1 + \cfrac{q + q^2}{1 + \cfrac{q^2 + q^4}{1 + \cfrac{q^3 + q^6}{1 + \dots}}}$, $|q| < 1$, *we have*

31. $$x_{n+1} = -\left(\frac{3\sqrt{3}-5}{4}\right)\left(u(x_n)\right)^3,\ x_1 = 0 \Rightarrow x_n \to -e^{-\pi}$$

32. $$\pi = \frac{3}{2}\ln(3) + 3\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{e^{-\frac{n\pi}{3}}}{1 + (-1)^n e^{-\frac{n\pi}{3}}} - \frac{e^{-3n\pi}}{1 + (-1)^n e^{-3n\pi}}\right)$$

33. $$\pi = \ln\left(9 + 6\sqrt{3}\right) + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}e^{-n\pi}\left(1 - (-1)^n e^{-n\pi}\right)}{n\left(1 + (-1)^n e^{-3n\pi}\right)}$$

34. $$\pi = \sqrt{3}\,\ln\left(\frac{3}{\sqrt[3]{4}-1}\right) + \sqrt{3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,e^{-n\pi/\sqrt{3}}\left(1-e^{-8n\pi/\sqrt{3}}\right)}{n\left(1+(-1)^n e^{-n\pi/\sqrt{3}}\right)\left(1+(-1)^n e^{-3\sqrt{3}\,n\pi}\right)}$$

35. $$\pi = \frac{2\sqrt{5}}{3}\,\ln\left(\frac{\sqrt[4]{5}\cdot(1+\sqrt{3})(\sqrt{3}+\sqrt{5})}{2}\right) + \frac{2\sqrt{5}}{3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,e^{-3n\pi/\sqrt{5}}\left(1-e^{-12n\pi/\sqrt{5}}\right)}{n\left(1+(-1)^n e^{-3n\pi/\sqrt{5}}\right)\left(1+(-1)^n e^{-3\sqrt{5}\,n\pi}\right)}$$

Let $\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}$, $|q| < 1$, *we have*

36. $$x_{n+1} = \frac{1}{3}\left(3\,x_n - \psi(x_n) - \sqrt{3}\,x_n\,\psi\left(x_n^9\right)\right) \;,\; x_1 = 0 \Rightarrow x_n \to -e^{-\pi/3}$$

37. $$\pi = \frac{4}{3}\ln(2) + 2\ln\left(\sqrt{2}+\sqrt[4]{3}\right) - \frac{2}{3}\ln\left(\sqrt{3}-1\right) + 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,e^{-3n\pi}}{n\left(1-e^{-3n\pi}\right)}$$

Remark: $\Gamma(x) = \int_0^{\infty} e^{-t}\,t^{x-1}\,dt$, $x > 0$, $\Gamma(x+1) = x\,\Gamma(x)$

References

[1] C. Adiga , T. Kim , M.S. Mahadeva Naika , and H.S. Madhusudhan : On Ramanujan's cubic continued fraction and explicit evaluations of theta-functions. arXiv:math/0502323v1 [math.NT] 15 Feb 2005.

[2] H.H. Chan and S.-S. Huang : On the Ramanujan-Göllnitz-Gordon continued fraction, Ramanujan J. 1 (1997) , 75-90 .