The Number $z = LambertW\left(\sqrt{-1}\right)$

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abstract

This note presents some formulas related with the number $z = LambertW(i)$, where $LambertW(x)$ is the Lambert function.

Introduction. Lambert function $W(x) \equiv LambertW(x)$.

The $W(x)$ function satisfies

$$W(x)e^{W(x)} = x \tag{1}$$

As the equation $ye^y = x$ has an infinite number of solutions $'y'$ for each value of $'x'$ $(x \neq 0)$, $W(x)$ has an infinite number of branches. Exactly one of branches is analytic at 0 . this branch is referred to as the principal branch of $W(x)$. the other branches are denoted by $W(k,x), k \neq 0, k$ integer. for details see ref.[1].

This note presents some formulas related with the number:

$$z = u + iv = W\left(\sqrt{-1}\right) = W(i) = 0.3746... + i \times 0.5764... \tag{2}$$

Formulas.

$$ze^z = i = \sqrt{-1} \tag{3}$$

$$z = u + iv = 0.3746990207371174... + i \times 0.5764127230314352... \tag{4}$$

$$z^2 e^{2z} + 1 = 0 \tag{5}$$

$$w_{n+1} = ie^{-w_n} \; , w_1 = 0 \Rightarrow \lim_{n\to\infty} w_n = z \tag{6}$$

$$z = ie^{-ie^{-ie^{-i...}}} = i\exp\left(-i\exp\left(-i\exp\left(-i...\right)\right)\right) \tag{7}$$

$$w_{n+1}=\frac{w_n^2+ie^{-w_n}}{1+w_n}\quad ,w_1=0\Rightarrow \lim_{n\to\infty} w_n=z \tag{8}$$

$$w_{n+1}=\frac{w_n+1}{1-ie^{w_n}}\quad ,w_1=0\Rightarrow \lim_{n\to\infty} w_n=z \tag{9}$$

$$w_{n+1}=\frac{w_n^2+2w_n^3-e^{-2w_n}}{2w_n\left(1+w_n\right)}\quad ,w_1=\frac{1+i}{2}\Rightarrow \lim_{n\to\infty} w_n=z \tag{10}$$

$$w_{n+1}=\frac{2w_n-2w_n\ln w_n+\pi i w_n}{2\left(1+w_n\right)}\quad ,w_1=\frac{1+i}{2}\Rightarrow \lim_{n\to\infty} w_n=z \tag{11}$$

$$z=\frac{1}{2\pi}\int_0^{2\pi}\frac{\left(1+e^{ix}\right)e^{2ix}}{e^{ix}-ie^{-e^{ix}}}dx \tag{12}$$

$$u=e^{-u}\sin v \tag{13}$$

$$v=e^{-u}\cos v \tag{14}$$

$$u=v\tan v \tag{15}$$

$$v^2+u^2=e^{-2u} \tag{16}$$

$$v^2-u^2=e^{-2u}\cos\left(2v\right) \tag{17}$$

$$2uv=e^{-2u}\sin\left(2v\right) \tag{18}$$

$$v^2-u^2=e^{-2u}\left(1-2\sin^2 v\right) \tag{19}$$

$$v^2-u^2=e^{-2u}\left(-1+2\cos^2 v\right) \tag{20}$$

$$v=e^{-v\tan v}\cos v \tag{21}$$

$$\begin{cases} x_{n+1}=e^{-x_n}\sin y_n & ,x_1=0 \\ y_{n+1}=e^{-x_n}\cos y_n & ,y_1=0 \end{cases}\Rightarrow \begin{cases} \lim_{n\to\infty} x_n=u \\ \lim_{n\to\infty} y_n=v \end{cases} \tag{22}$$

$$x_{n+1}=\left(\sin v\right)e^{-x_n}\quad ,x_1=0\Rightarrow \lim_{n\to\infty} x_n=u \tag{23}$$

$$u=\sin v\, e^{-\sin v\, e^{-\sin v\, e^{-\sin v \ldots}}}=\sin v\exp\left(-\sin v\exp\left(-\sin v\exp\left(-\sin v \ldots\right)\right)\right) \tag{24}$$

$$y_{n+1}=e^{-u}\cos y_n\quad ,y_1=0\Rightarrow \lim_{n\to\infty} y_n=v \tag{25}$$

$$v=e^{-u}\cos\left(e^{-u}\cos\left(e^{-u}\cos\left(e^{-u}\ldots\right)\right)\right) \tag{26}$$

$$\pi = 4v + 4\tan^{-1}\left(\frac{v-u}{v+u}\right) \tag{27}$$

$$\pi = 4v + 2\tan^{-1}\left(\frac{v^2-u^2}{2uv}\right) \tag{28}$$

$$\pi = 6v - 2\tan^{-1}\left(\frac{3u^2v-v^3}{3uv^2-u^3}\right) \tag{29}$$

$$\pi = 2v + 2\sin^{-1}\left(\frac{v}{\sqrt{u^2+v^2}}\right) \tag{30}$$

$$\pi = -2i + 2i\left(1-z\right) + 2i\sum_{n=1}^{\infty}\frac{1}{n}\left(1-z\right)^n \tag{31}$$

$$\pi = 2v + 2\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Re}\left(i\left(1-z\right)^n\right) \tag{32}$$

$$\pi = 2v + 2\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=0}^{\left[(n-1)/2\right]}\binom{n}{2k+1}\left(-1\right)^k\left(1-u\right)^{n-2k-1}v^{2k+1} \tag{33}$$

$$\pi = 4v + 4\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n-1}}{n}\left(\frac{v-u}{2u}\right)^n\operatorname{Im}\left(\left(1+i\right)^n\right) \tag{34}$$

$$\pi = 4v - 4\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{v-u}{2v}\right)^n\operatorname{Im}\left(\left(1-i\right)^n\right) \tag{35}$$

$$\pi = 4v - 4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(\frac{u^2+v^2-2uv-i\left(v^2-u^2\right)}{2\left(u^2+v^2\right)}\right)^n\right) \tag{36}$$

$$\pi = 3 + 2\tan^{-1}\left(\frac{v-u\tan\left(\left(3/2\right)-v\right)}{u+v\tan\left(\left(3/2\right)-v\right)}\right) \tag{37}$$

$$z = \cfrac{1}{1-2i+\cfrac{1}{-1-i+\cfrac{1}{1+\cfrac{1}{1-2i+\ldots}}}} \tag{38}$$

$$z = \left[0;1-2i,-1-i,1,1-2i,1-2i,-1-i,2+i,-2i,-i,1+i,1+i,1,-i,3+2i,4-6i,\ldots\right] \tag{39}$$

$$z = re^{i\theta} \Rightarrow r = 0.687496\ldots \wedge \theta = 0.994383\ldots \tag{40}$$

$$r = |z| = \sqrt{u^2 + v^2} \quad , \quad \theta = \tan^{-1}\left(\frac{v}{u}\right) \tag{41}$$

$$r = e^{-r\cos\theta} \tag{42}$$

$$\theta = \frac{\pi}{2} - r\sin\theta \tag{43}$$

$$\begin{cases} r_{n+1} = e^{-r_n\cos\theta_n} \, , r_1 = 0 \\ \theta_{n+1} = \frac{\pi}{2} - r_n\sin\theta_n \, , \theta_1 = 0 \end{cases} \Rightarrow \begin{cases} \lim_{n\to\infty} r_n = r \\ \lim_{n\to\infty} \theta_n = \theta \end{cases} \tag{44}$$

$$\sin\theta = \frac{v}{\sqrt{u^2 + v^2}} \quad , \quad \cos\theta = \frac{u}{\sqrt{u^2 + v^2}} \tag{45}$$

$$\sin\theta = \cos\left(r\sin\theta\right) \tag{46}$$

$$\cos\theta = \sin\left(r\sin\theta\right) \tag{47}$$

$$r_{n+1} = e^{-(\cos\theta)r_n} \, , r_1 = 0 \Rightarrow \lim_{n\to\infty} r_n = r \tag{48}$$

$$r = e^{-\cos\theta e^{-\cos\theta e^{-\cos\theta\ldots}}} = \exp\left(-\cos\theta \cdot \exp\left(-\cos\theta \cdot \exp\left(-\cos\theta\ldots\right)\right)\right) \tag{49}$$

$$\theta_{n+1} = \frac{\pi}{2} - r\sin\theta_n \, , \theta_1 = 0 \Rightarrow \lim_{n\to\infty} \theta_n = \theta \tag{50}$$

$$\theta_{n+1} = \theta_n \frac{\cos\left(r\sin\theta_n\right)}{\sin\theta_n} \, , \theta_1 = 1 \Rightarrow \lim_{n\to\infty} \theta_n = \theta \tag{51}$$

$$\theta = \frac{\pi}{2} - r\sin\left(\frac{\pi}{2} - r\sin\left(\frac{\pi}{2} - r\sin\left(\frac{\pi}{2}\ldots\right)\right)\right) \tag{52}$$

$$\sin\theta = \cos\left(r\cos\left(r\cos\left(r\ldots\right)\right)\right) \tag{53}$$

$$x_{n+1} = \cos\left(r\,x_n\right), x_1 = 0 \Rightarrow \lim_{n\to\infty} x_n = \sin\theta \tag{54}$$

$$\pi = 2r\sin\theta + 2\sum_{n=1}^{\infty} \frac{1}{n} R^n \sin\left(n\phi\right) \tag{55}$$

$$R = \sqrt{1 + r^2 - 2r\cos\theta} = \sqrt{1 + u^2 + v^2 - 2u} \tag{56}$$

$$\phi = \tan^{-1}\left(\frac{r\sin\theta}{1 - r\cos\theta}\right) = \tan^{-1}\left(\frac{v}{1-u}\right), \sin\phi = \frac{v}{R} \tag{57}$$

Graphics

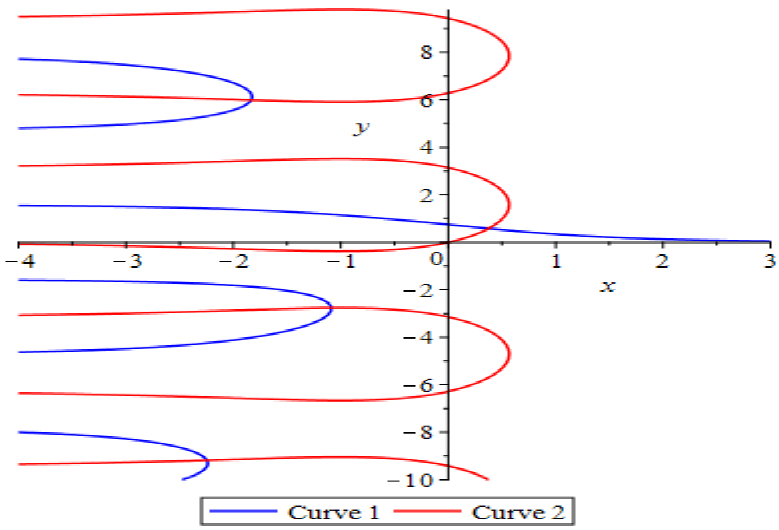


Figure 1. Curve 1: $y = e^{-x}\cos y$, Curve 2: $x = e^{-x}\sin y$.

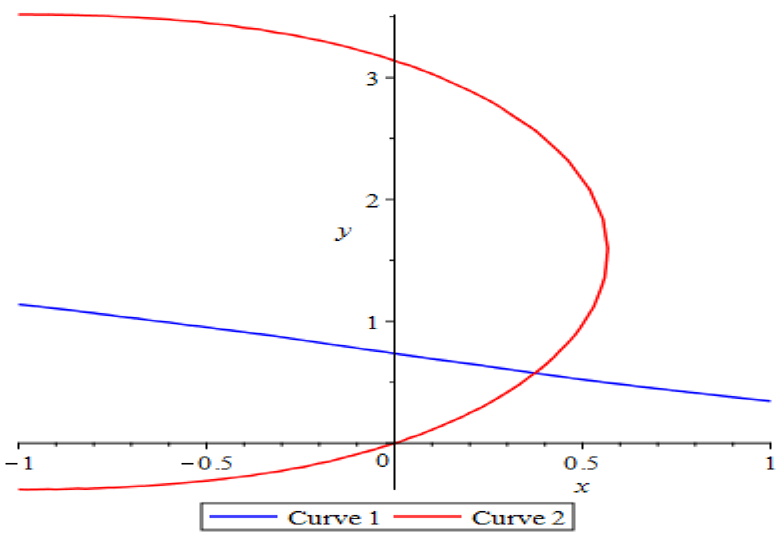


Figure 2. Curve 1: $y = e^{-x}\cos y$, Curve 2: $x = e^{-x}\sin y$.

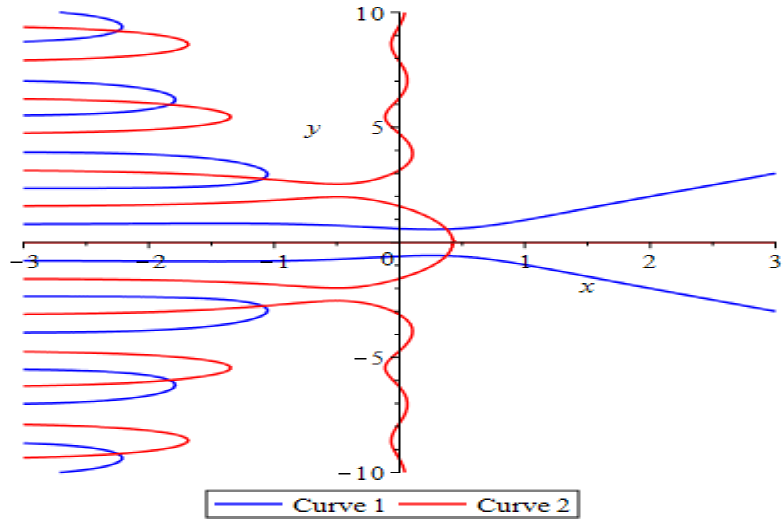


Figure 3. Curve 1: $y^2 - x^2 = e^{-2x}\cos(2y)$, Curve 2: $2xy = e^{-2x}\sin(2y)$

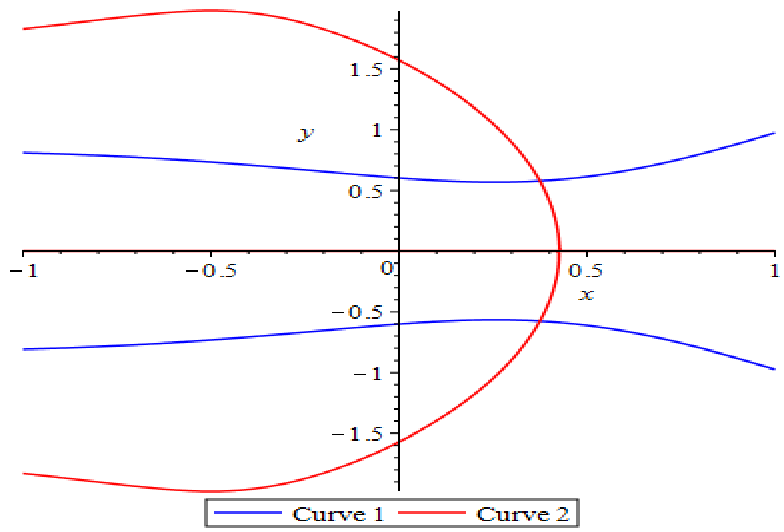


Figure 4. Curve 1: $y^2 - x^2 = e^{-2x}\cos(2y)$, Curve 2: $2xy = e^{-2x}\sin(2y)$

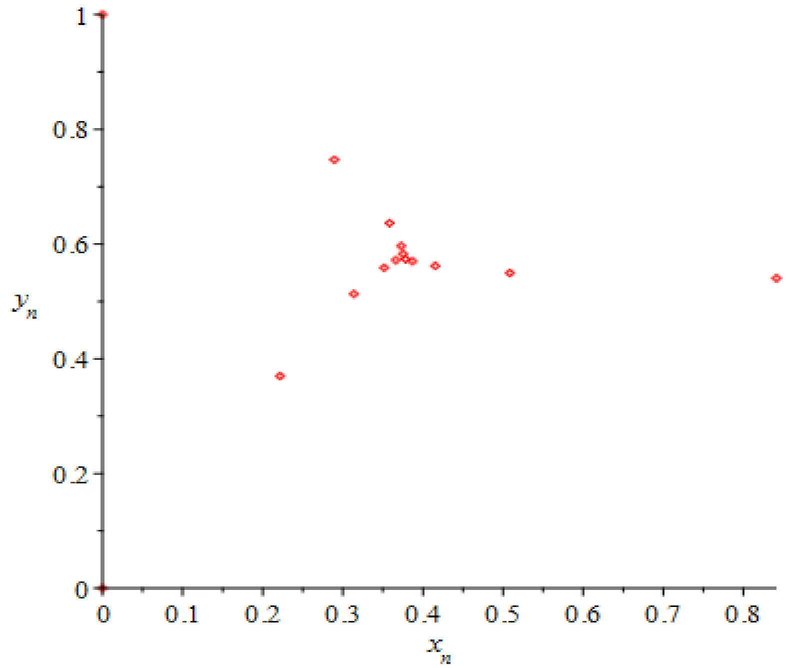


Figure 5. $x_{n+1} = e^{-x_n} \sin y_n$, $y_{n+1} = e^{-x_n} \cos y_n$, $x_1 = y_1 = 0$

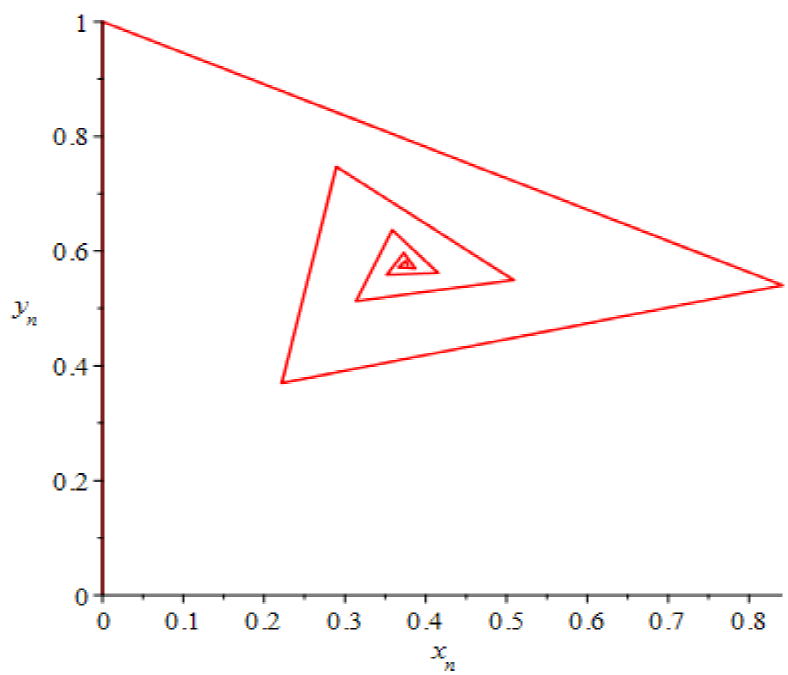
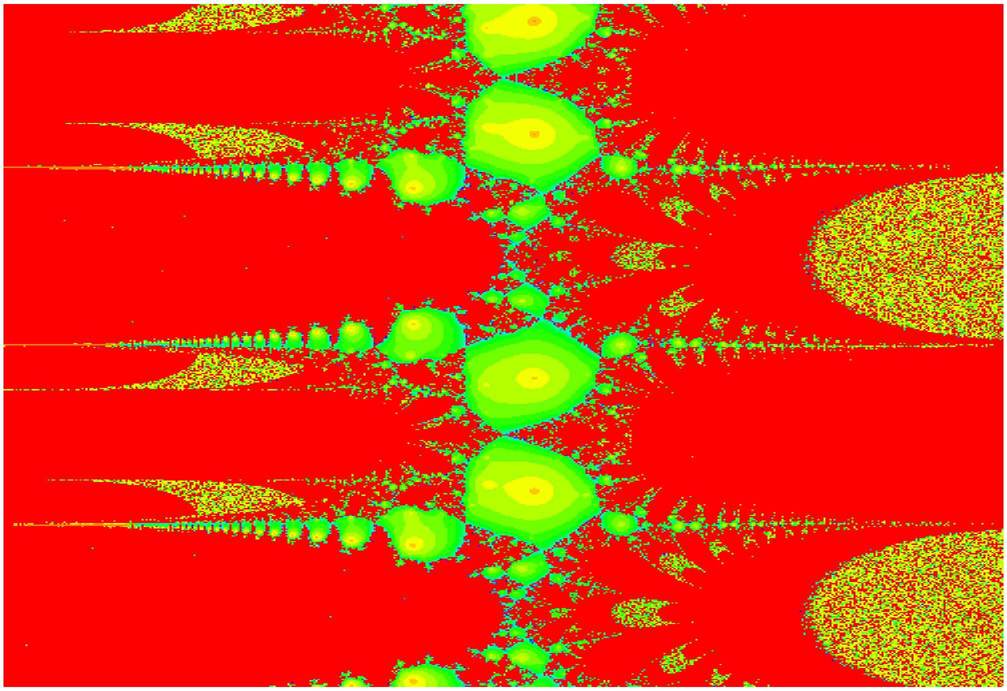
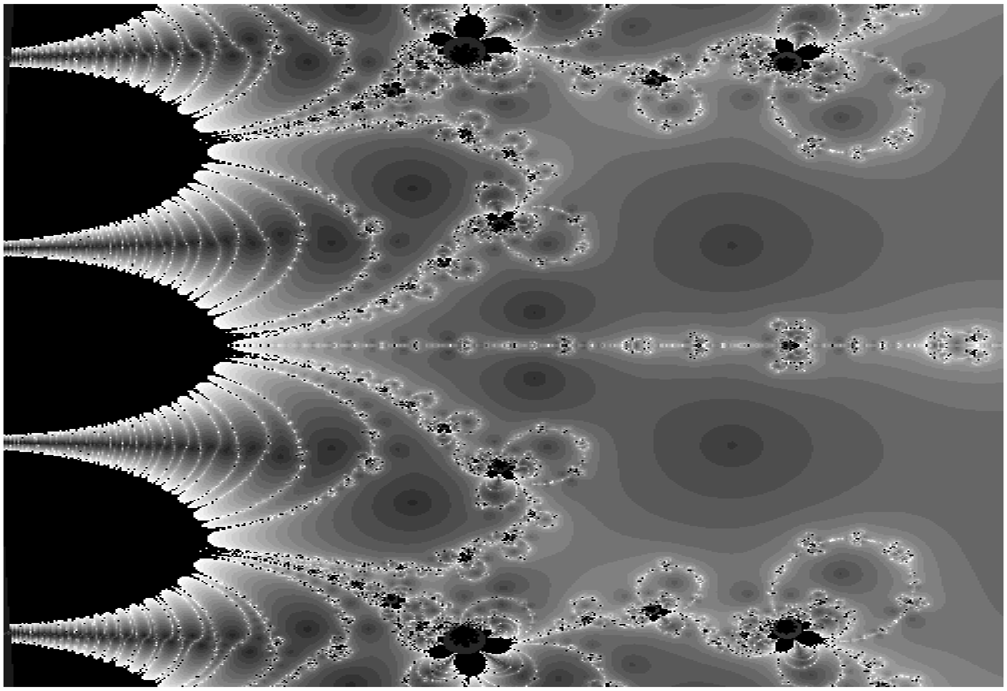
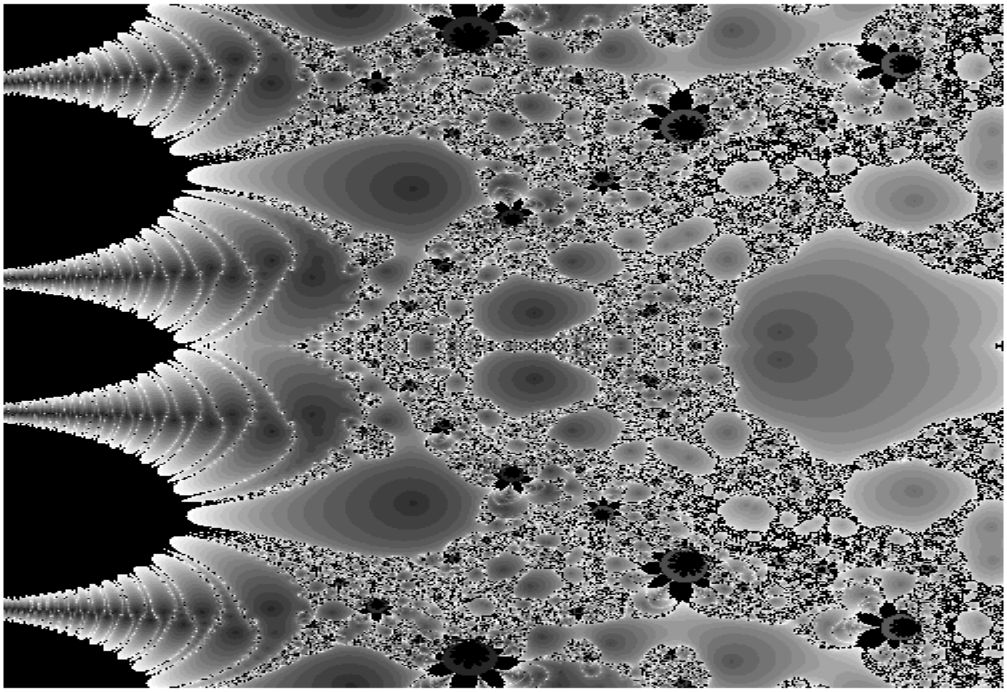


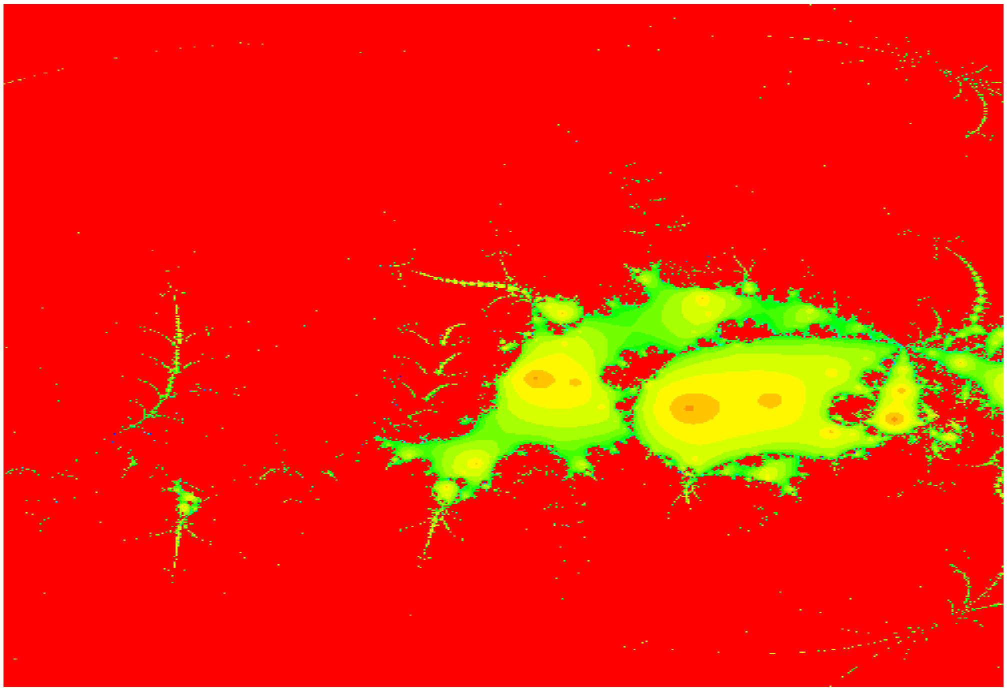
Figure 6. $x_{n+1} = e^{-x_n} \sin y_n$, $y_{n+1} = e^{-x_n} \cos y_n$, $x_1 = y_1 = 0$

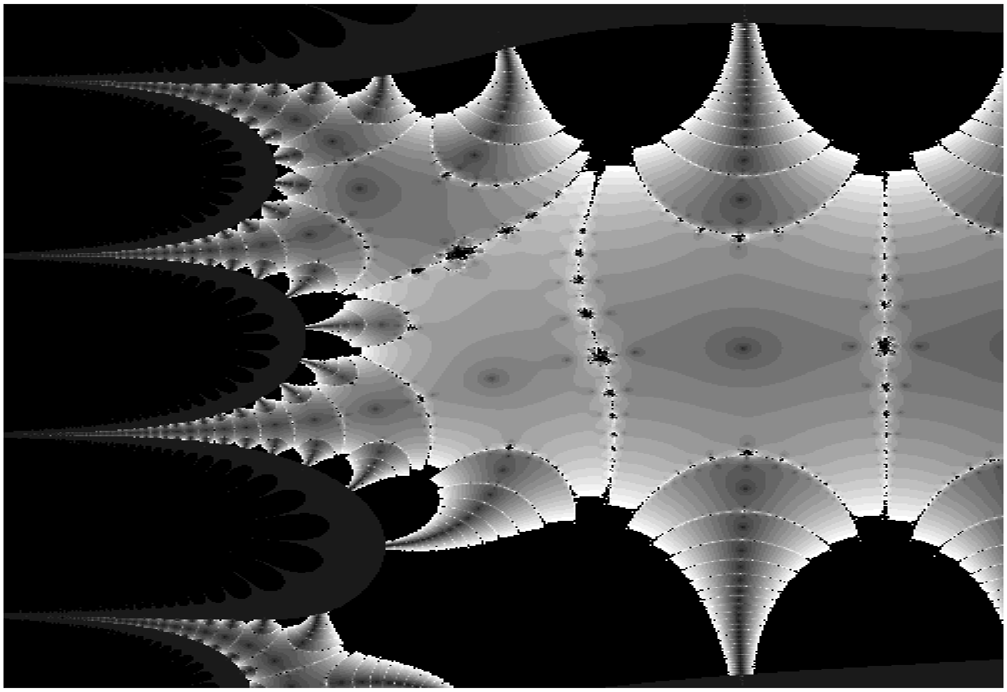
Related Fractals

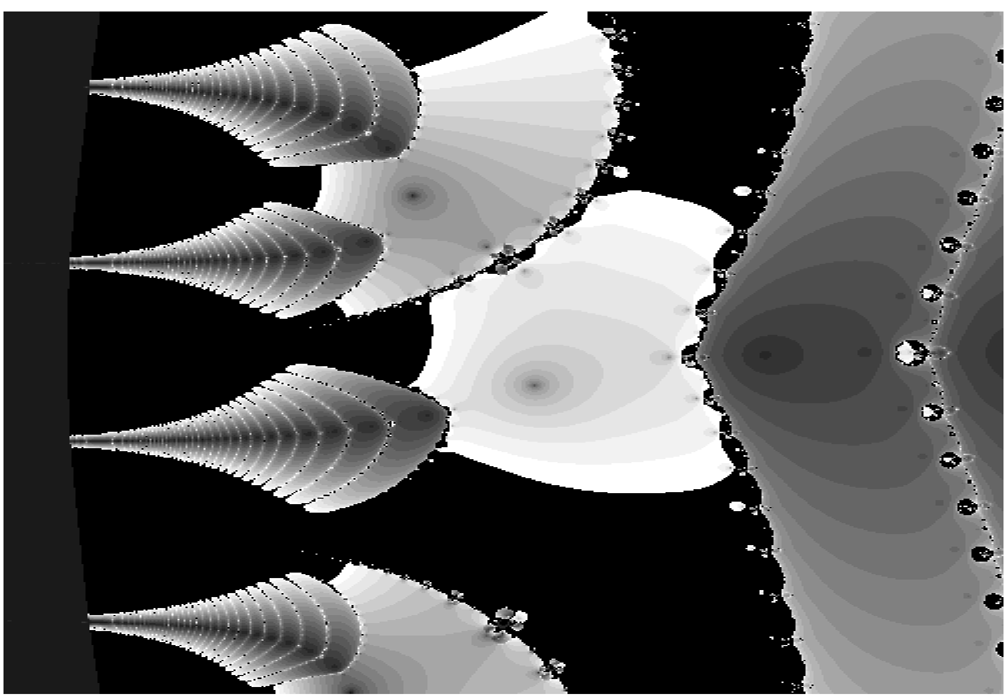


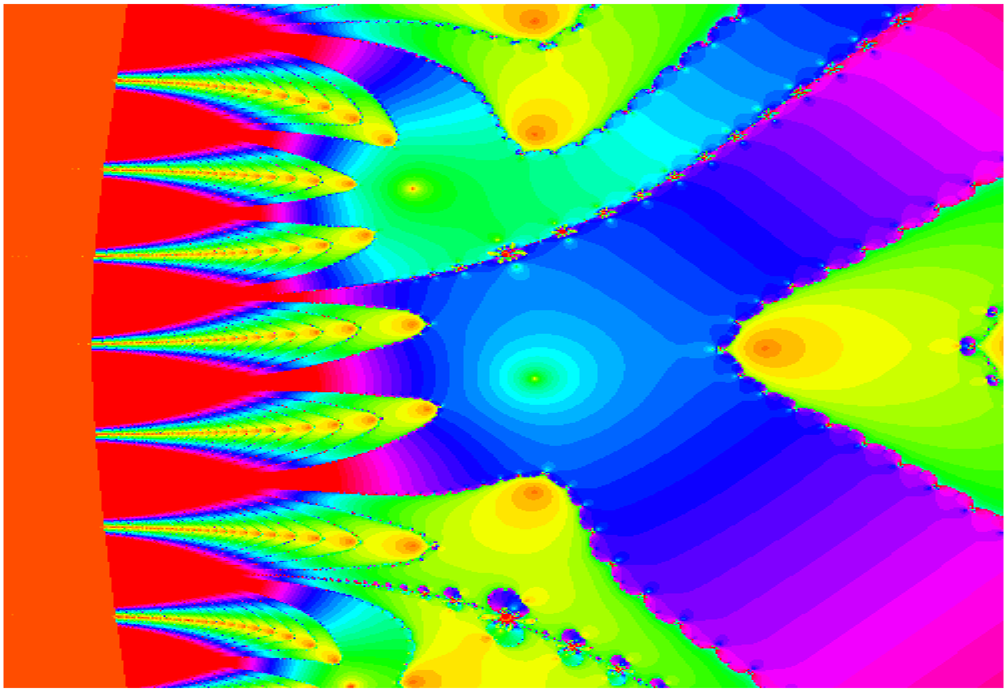


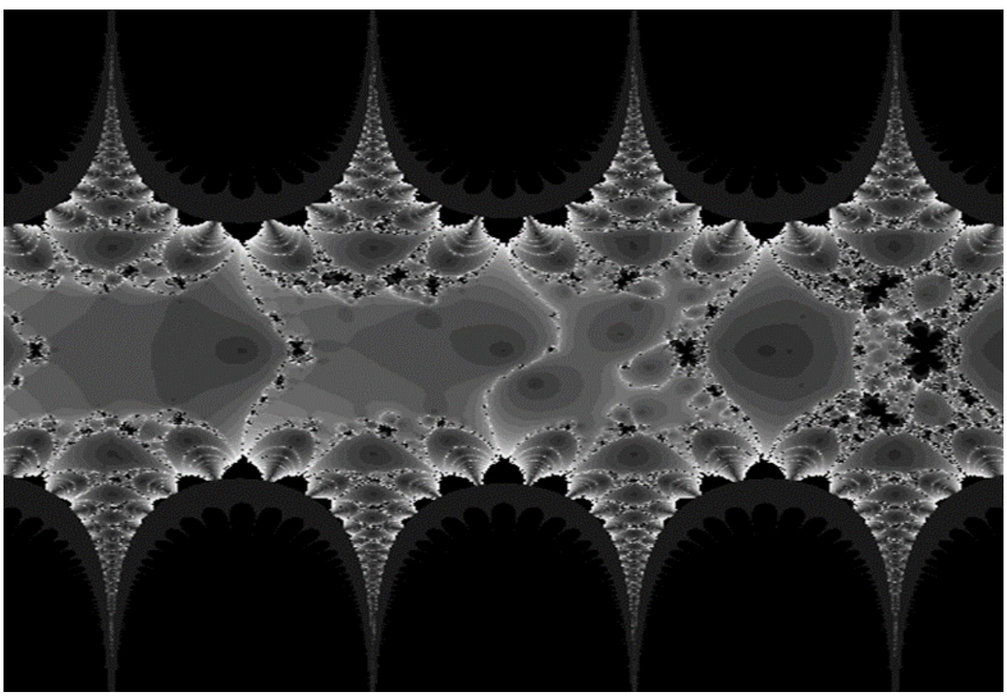


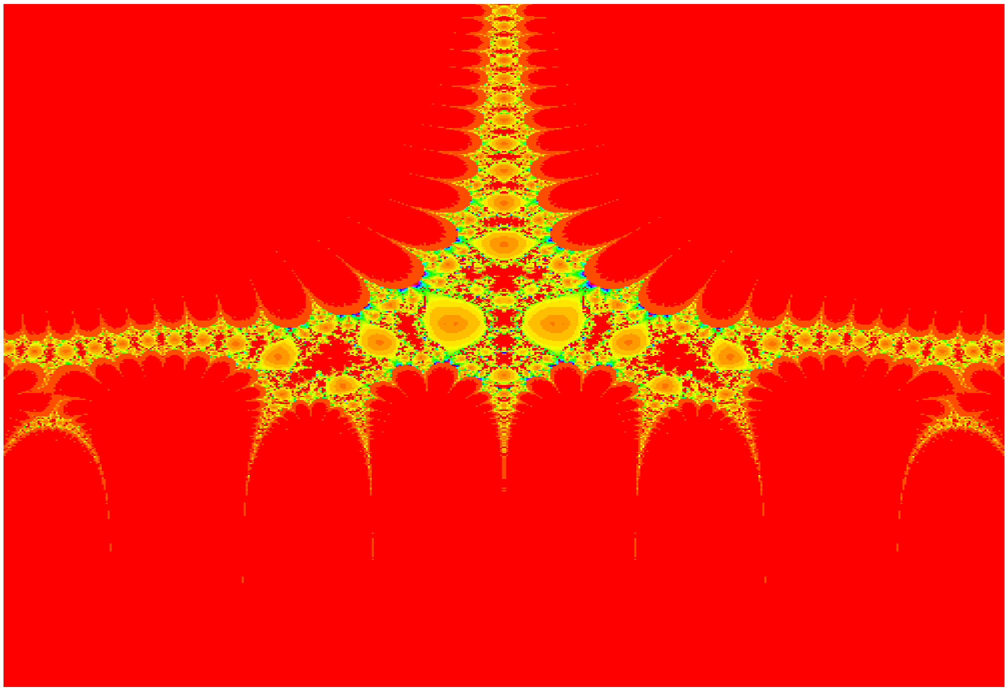


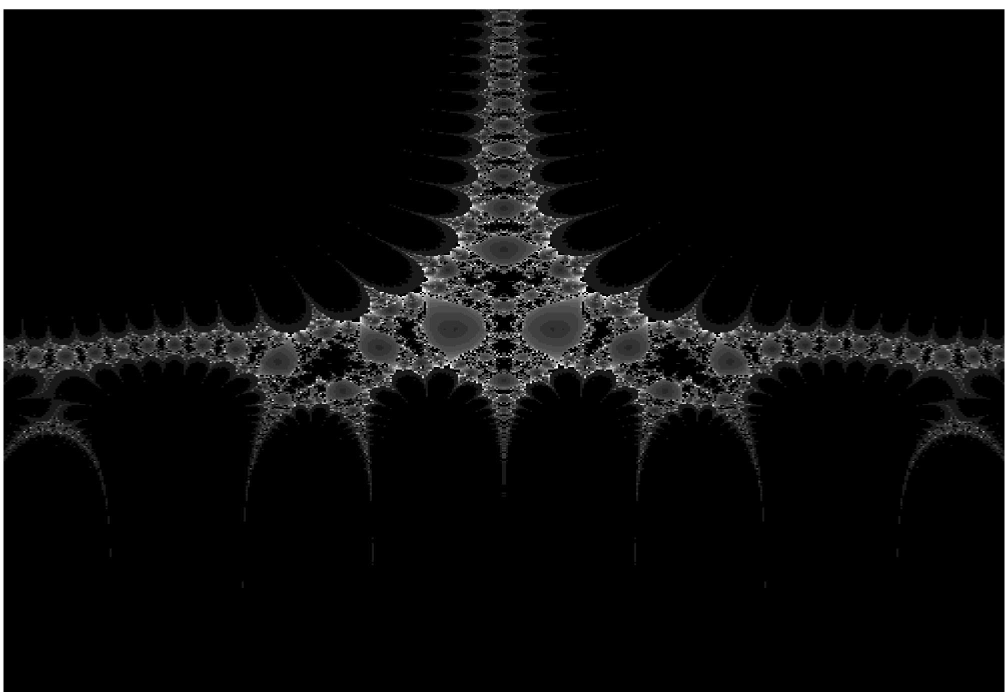


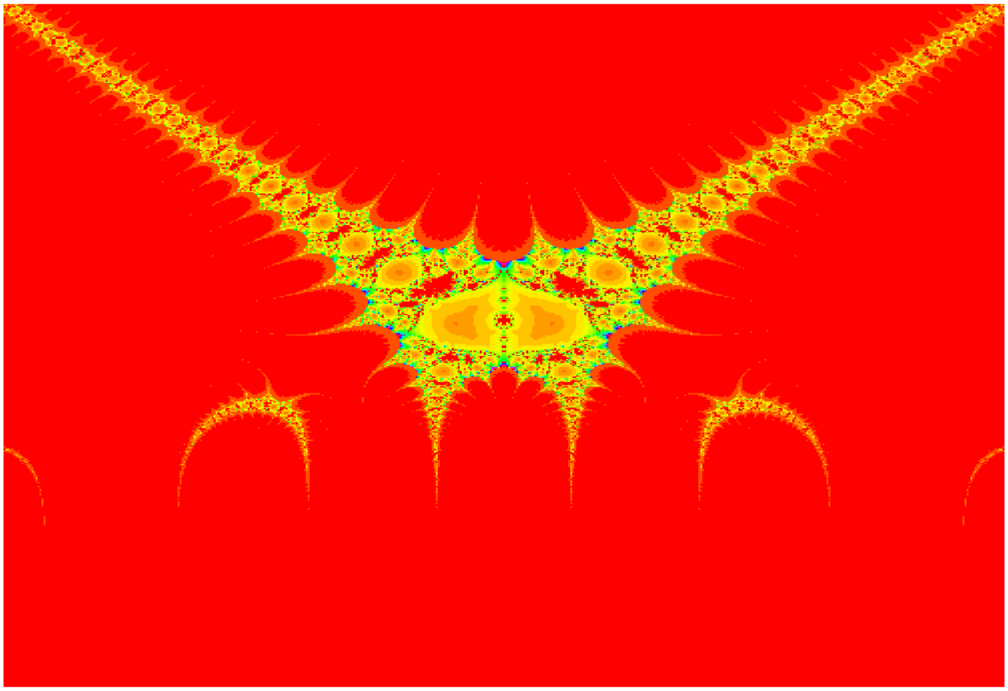


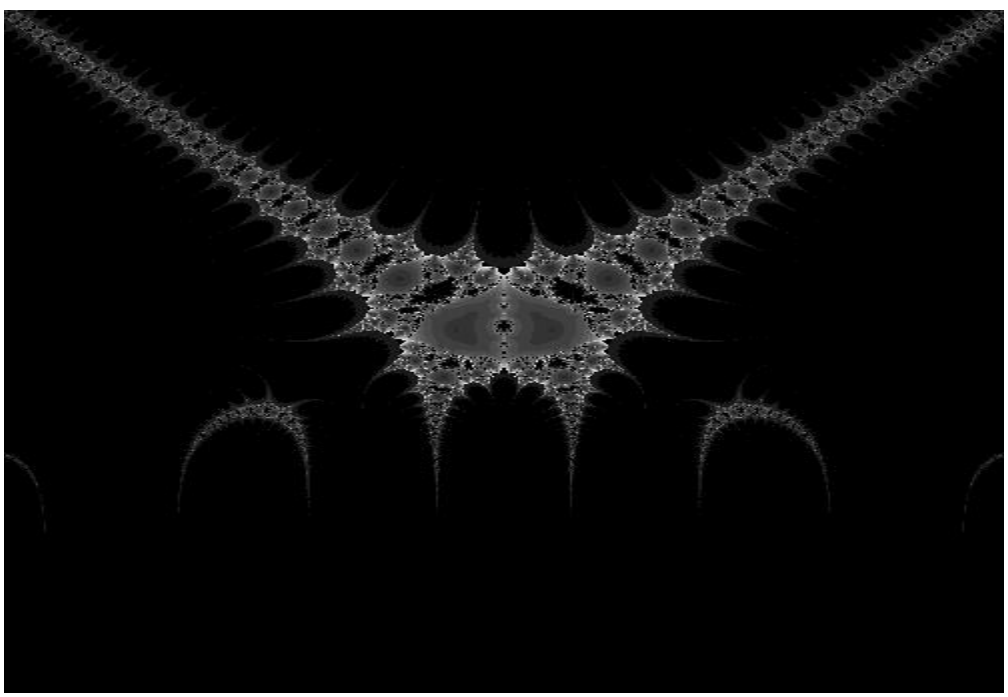


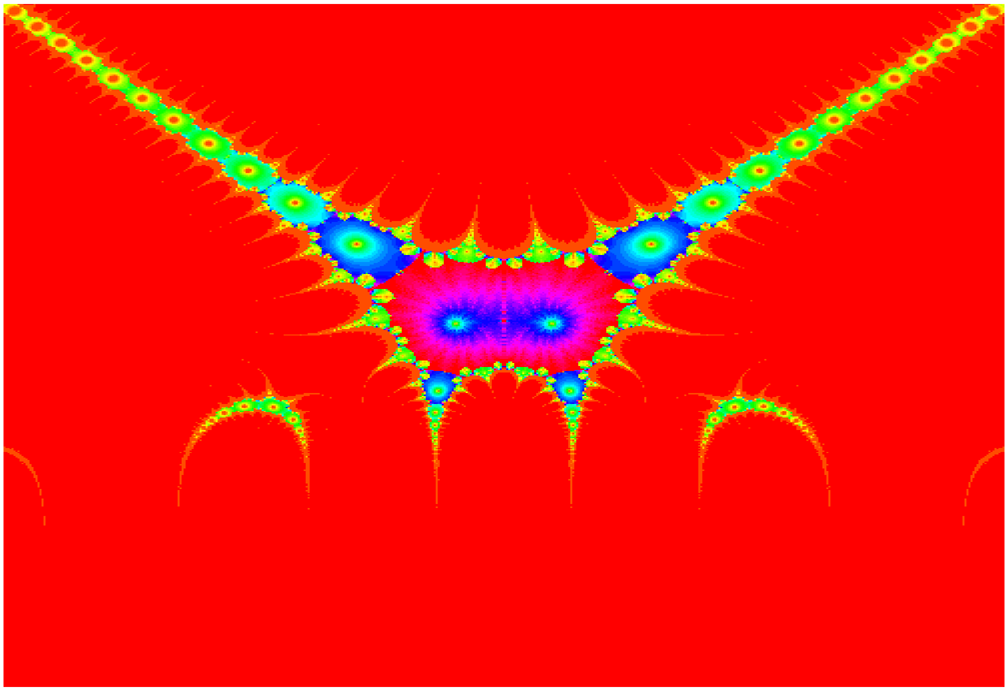


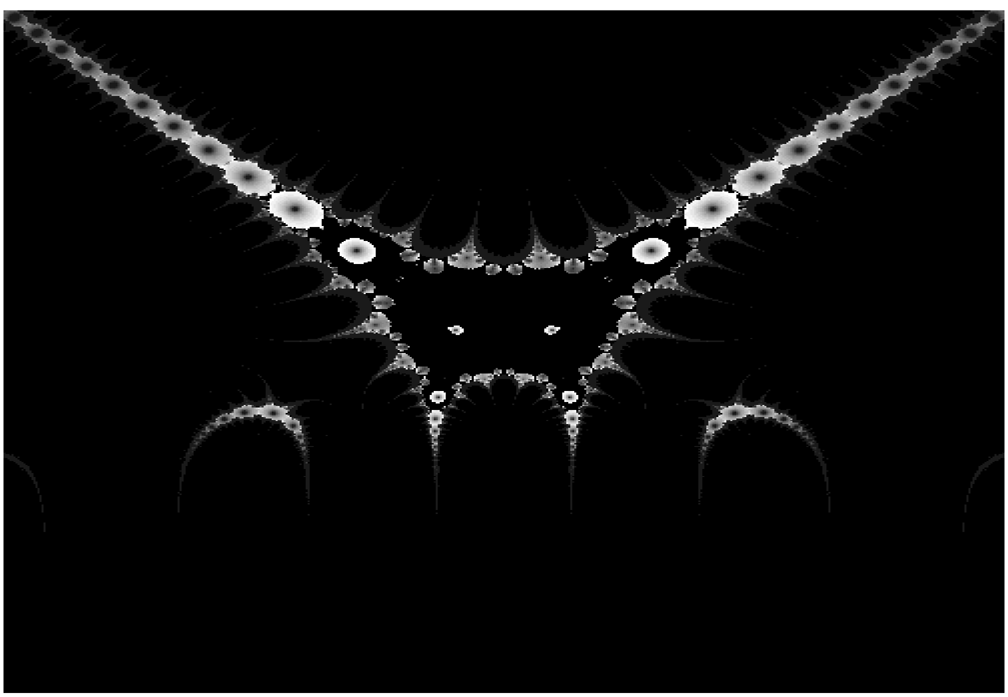


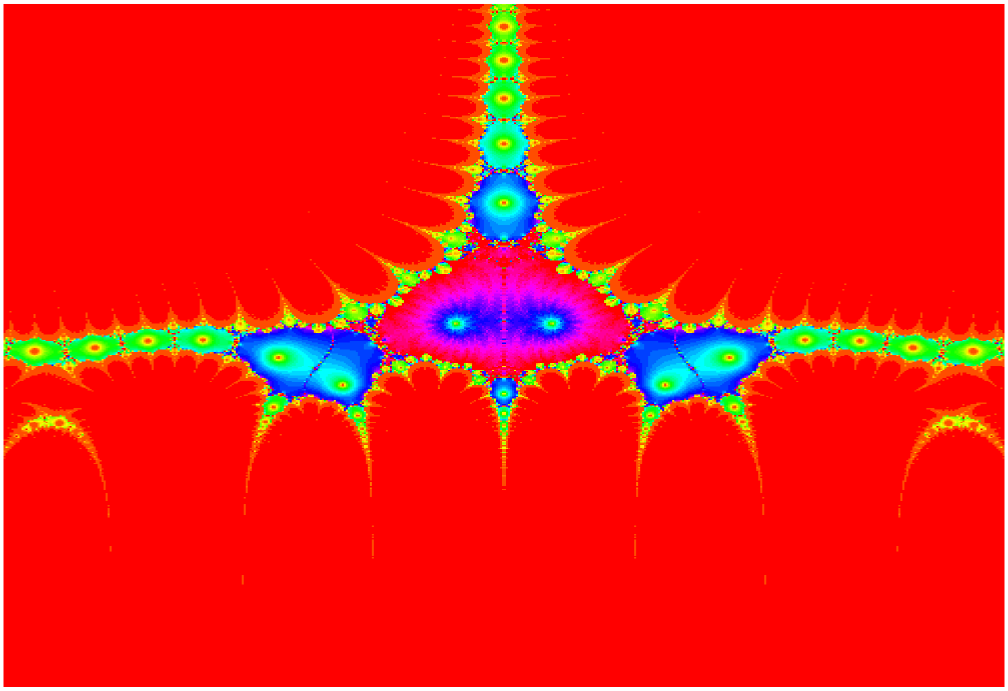


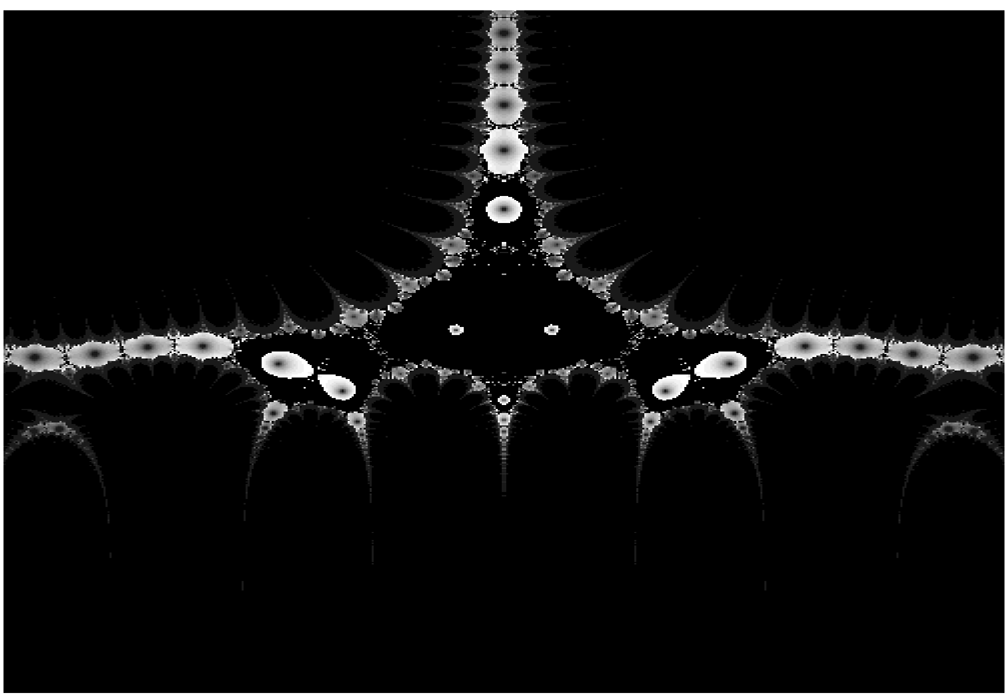


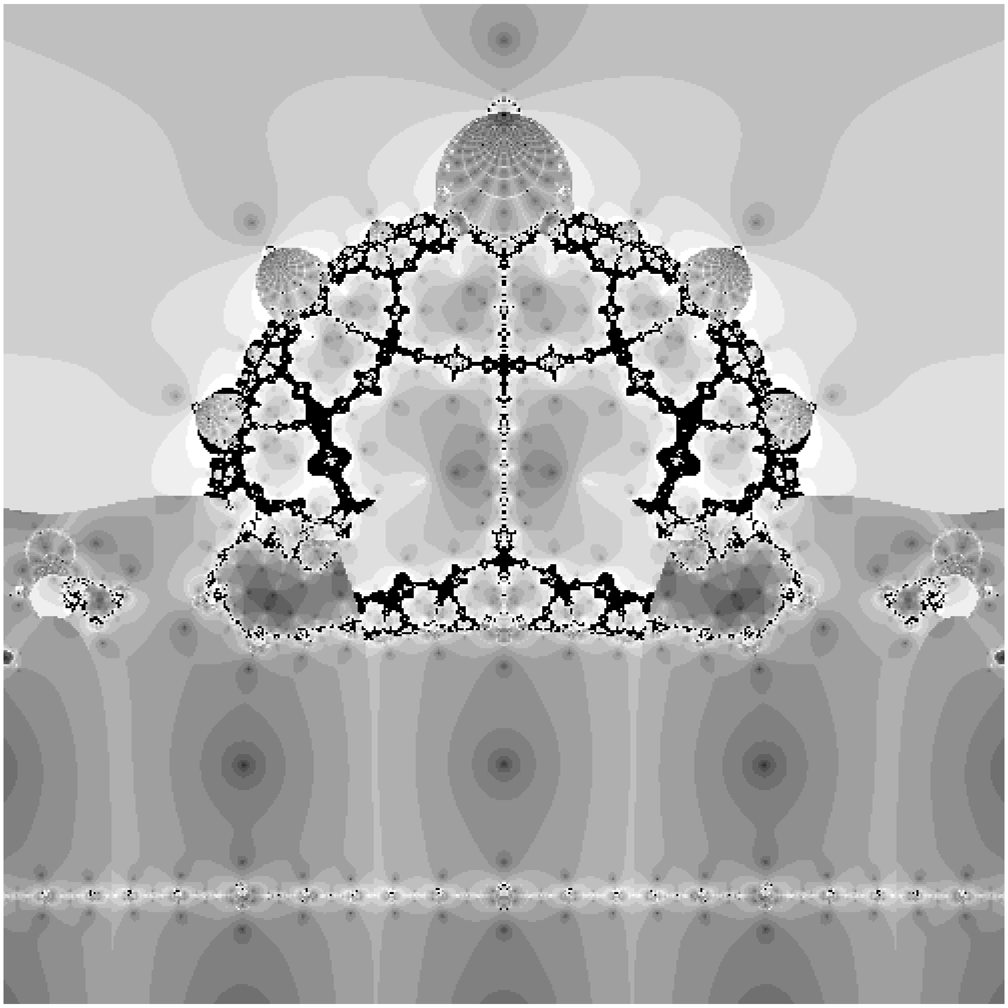












References

1. Corless, R.M.,Gonnet, G.H.,Hare,D.E.G.,Jeffrey,D.J., and Knuth, D.E.: On the Lambert W Function. Advances in Computational Mathematics, Vol. 5,(1996):329-359.