Verifying the Validity of a Conformant Plan is co-NP-Complete

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The purpose of this document is to show the complexity of verifying the validity of a deterministic conformant plan. We concentrate on a simple version of the conformant planning problem (i.e., one where there is no precondition on the actions and where all conditions are defined as sets of positive or negative facts) in order to show that the complexity does not come from solving a single such formula.

Conformant Plan Validity

Let F be a set of *facts*; a fact is denoted by $f \in F$. The *extension* $Ext(F)$ of F is the set of facts and their negations: $Ext(F) = \bigcup_{f \in F} \{f, \neg f\}$. To simplify notations we use $\neg(\neg f)$ to denote f.

A *state* $q \subseteq F$ is represented by a set of facts assumed to be true. We write $ext(q) \subset Ext(F)$ the *extension* of the state q defined as the set of facts in q and the negation of the facts not in q. Formally $ext(q) = \{f \mid f \in q\} \cup \{\neg f \mid f \in F \setminus q\}$.

An *action* is a set of pairs $pre_i \rhd eff_i$ called *conditional effects* where $pre_i \subset Ext(F)$ and $eff_i \subset Ext(F)$ are subsets of extended facts. The conditional effect $pre_i \rhd eff_i$ indicates that if pre_i is satisfied in the current state then eff_i will be true in the next state. Formally applying an action $a = \{pre_1 \rhd eff_1, \dots, pre_k \rhd eff_k\}$ in state q leads to the state $q' = a(q)$ that can be computed as follows:

- Let $E = \bigcup_{pre_i \subseteq ext(q)} eff_i$ be the set of effects of the action;
- If $\{f, \neg f\} \subseteq E$ for some fact $f \in F$, then q' is undefined (no contradiction);
- $q' = (q \cup E) \setminus \{\neg f \mid f \in E\}$.

The *conformant planning problem* is the tuple $P = \langle F, A, I, G\rangle$ where F is the set of facts, A is the set of actions, $I \subset Ext(F)$ is the initial belief defined as a set of literals, $G \subset Ext(F)$ is the goal.

A state q is an *initial state* if $I \subseteq ext(q)$; notice that the facts are initially partitioned into two categories: the facts whose value is known ($\{f \in F \mid (f \in I) \vee (\neg f \in I)\}$), and the facts whose value is unknown and independent from that of other facts ($\{f \in F \mid \{f, \neg f\} \cap I = \emptyset$). A state q is a *goal state* if $G \subseteq ext(q)$.

A *plan* π is a sequence of actions. A plan is *valid* for a state q if successively applying all actions in the plan from q leads to a goal state $G \subseteq ext(\pi(q))$. A plan is a *valid conformant plan* for problem P if it is a valid for every initial state of P.

Definition 1 *The* conformant plan validity problem *is a conformant planning problem P together with a plan π. The decisional problem asks whether π is a valid conformant plan.*

In the rest of this document we show that this problem is co-NP-complete by first proving membership in the co-NP-hard class, and hardness by reduction from SAT.

Membership

Membership of conformant planning validity to CO-NP-HARD class should be obvious: if the plan is not valid, guess an initial state where the plan does not apply (checking that this state is initial is polytime); then simulate the execution of the plan (in polytime); finally check that the reached state does not belong to the goal state (in polytime again).

Reduction

A *SAT problem* S is a set V of n propositional variables together with a set C of m clauses $\{c_1, \ldots, c_m\}$. Each clause c_i is a subset of k_i literals, where a literal is a variable $v \in V$ or its negation $\neg v$. An assignment $\alpha : V \to \{\bot, \top\}$ *satisfies* a clause c if there exists a variable $v \in V$ such that either $\alpha(v) = \top$ and $v \in c$, or $\alpha(v) = \bot$ and $(\neg v) \in c$. A SAT problem is *satisfiable* if there exists an assignment that satisfies all the clauses. Verifying the satisfiability of a SAT problem is an NP-HARD problem.

Here is an informal explanation of the reduction. We define a conformant planning problem that "simulates" the SAT problem. The set of initial states represents all the possible assignments of variables V. It plays the role of the assignment function α. For an initial state, the plan checks the validity for each clause and it fails (does not reach the goal) if all clauses are satisfied. Consequently the SAT formula is satisfiable iff the plan is invalid.

The set of facts of the planning problem is partitioned in three subsets: the first subset models the assignment (unspecified in the initial state, so that all assignments are considered); the second subset records whether each clause has been proved satisfied (initially false, set by actions a_i): this models the role of the disjunction in each clause; the last subset records whether all clauses have been proved satisfied (initially false, set by action a): this models the role of the conjunction in the SAT formula.

Given a SAT problem S the conformant planning problem P_S is defined as follows:

- $F = \{f_v \mid v \in V\} \cup \{f_{c_i} \mid c_i \in C\} \cup \{f_{\mathrm{g}}\}$,
- $A = \{a_1, \ldots, a_m\} \cup \{a\}$ where for all i,

$$a_i = \left\{ \begin{array}{cl} \{f_v\} \rhd \{f_{c_i}\} & \text{for all } v \in c_i \\ \{\neg f_v\} \rhd \{f_{c_i}\} & \text{for all } \neg v \in c_i \end{array} \right\}$$

  and

$$a = \{\ \{f_{c_1}, \ldots, f_{c_m}\} \rhd \{f_{\mathrm{g}}\}\ \},$$

- $I = \{\neg f_{c_i} \mid c_i \in C\} \cup \{\neg f_{\mathrm{g}}\}$, and
- $G = \{\neg f_{\mathrm{g}}\}$.

The plan π_S is defined as: $a_1, \ldots, a_m, a$.

Properties of the Reduced Problem

Each initial state q represents implicitly an assignment α_q, where variable v is assigned to $\top$ iff f_v belongs to the state. We say that "clause c is satisfied in state q" if α_q satisfies c. This is denoted $q \models c_i$.

Lemma 1 *The state reached from an initial state by applying π_S includes the fact f_{c_i} iff the clause c_i was satisfied in the initial state. Formally:*

$$\forall c_i \in C.\ \forall q \subseteq F. \quad I \subseteq ext(q) \Rightarrow (f_{c_i} \in \pi_S(q) \Leftrightarrow q \models c_i)\,.$$

Proof sketch: The fact f_{c_i} is initially false and the only action that changes its value is action a_i (which is part of π_P). This action sets f_{c_i} iff $q' \models c_i$, where q' is the state before the application of action a_i. Given that the facts f_v are not modified by the actions in the plan, the action sets f_{c_i} iff $q \models c_i$, where q is the initial state.

Lemma 2 *The state reached from an initial state by applying plan π_S includes fact f_{g} iff all clauses are satisfied in the initial state. Formally:*

$$\forall q \subseteq F. \quad I \subseteq ext(q) \Rightarrow (f_{\mathrm{g}} \in \pi_S(q) \Leftrightarrow \forall c_i \in C.\ q \models c_i)\,.$$

Proof sketch: The fact f_{g} is initially false and the only action that changes its value is action a (which is part of π_P). This action sets f_{g} iff f_{c_i} is true for all clause c_i. From Lemma 1, this implies $q \models c_i$ for all clauses c_i.

Corollary 1 *The plan π_S is valid for P_S iff S is not satisfiable.*

Proof sketch: The plan π_S is not valid for the planning problem iff it is not valid for some initial state q, i.e. f_{g} holds in $\pi_s(q)$. From Lemma 2 this implies $q \models c_i$ for all clause c_i, i.e., it implies that S is satisfiable. Thus, the plan π_S is valid iff S is not satisfiable as in that case f_{g} does not hold in $\pi_s(q)$.

Theorem 1 *Conformant plan validity is* CO-NP-HARD.

Proof sketch: Checking whether S is satisfiable is NP-HARD. From Corollary 1, verifying whether S is satisfiable can be reduced to verify whether π_S is not valid for P_S. Furthermore P_S can be built in polytime from S. Therefore conformant plan validity is CO-NP-HARD.

Additional Results

We discuss some extensions of the results. The proofs in the previous sections can be easily adapted for these results, but we decided against, in order to keep the original proofs as simple as possible.

Non-deterministic conformant planning is a variant of the conformant planning problem where actions can have non-deterministic effects, i.e., some predefined "optional" conditional effects may or may not trigger.

Theorem 2 *Non-deterministic conformant plan validity is* CO-NP-COMPLETE

Proof sketch: Hardness is consequence of the fact that non-deterministic conformant planning is more general than the deterministic variant. Membership is proved as follows: guess not only the initial state but also all the optional conditional effects that trigger.

Theorem 3 *Conformant plan validity is* CO-NP-COMPLETE *even with a single conditional effect per action.*

Proof sketch: This can be shown by splitting each action a_i into k_i actions $a_{i,\ell}$ where $\ell \in c_i$ is one of the literals of c_i.

Theorem 4 *Conformant plan validity is* CO-NP-COMPLETE *even when the plan has only two actions.*

Proof sketch: This can be shown by merging all actions a_i into a single action. The resulting action has the same semantics as all m original actions as all conditional effects are independent.

Notice that we cannot have both the property that there is a single conditional effect per action and the property that the plan has only two actions.

Theorem 5 *Conformant plan validity is* CO-NP-COMPLETE *even when we have the guarantee that if the plan is valid, then it is optimal.*

Proof sketch: This can be shown by adding facts that guarantee that the plan is π_S and nothing else.

Notice that we cannot have both the optimality property above and the property that there is a single conditional effect per action, but we can enforce the property that there are only thee conditional effects per action.