Decomposition of exponential function into derivative ring. Derivative ring: suitable basis for derivative-matching approximations.

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Abstract

A set of functions which allows easy derivative-matching is proposed. Several examples of approximations are shown.

1 Introduction

In my previous text [1] I made an effort to find elegant examples of approximations based on derivative-matching. An additional interesting idea came to me some time later and lead to the creation of this text. If the order of approximation (the number of derivatives to match) is given in advance, a nice set of functions can be found with the derivative-matching procedure being very straightforward (as easy as for polynomials). Actually, these functions can be regarded as a generalization of the Taylor polynomials: their derivatives are periodically repeating and the Taylor polynomials can be seen a the limit where the period length goes to infinity.

2 Derivative ring

Let g be a function which fulfills the differential equation

$$g^{(N)} = g. \tag{1}$$

I call the sequence of its derivatives $S_N^g = \{g, g', g'', g^{(3)}, \ldots, g^{(N-1)}\}$ a derivative ring. Clearly, the ring is invariant under the derivative operator: each function becomes its neighbor, the ring rotating itself to a ring with the same elements. The sum of ring elements is thus necessarily also derivative-invariant and therefore it has to fulfill

$$\sum_{i=0}^{N-1} g^{(i)} = \alpha e^x, \tag{2}$$

with α some number. If one wants to use the functions $g^{(i)}$ as series elements, one wants them to be linearly independent. However, this needs not to be true in general. Consider the derivative ring $S_4^{\sin}$:

$$\begin{array}{ccc} \sin(x) & \rightarrow & \cos(x) \\ \uparrow & & \downarrow \\ -\cos(x) & \leftarrow & -\sin(x) \end{array}.$$

The element functions are linearly dependent and thus not suitable for building a series. The condition (2) still holds, but becomes trivial with $\alpha = 0$.

The theory of linear differential equations [2] tells us that the equation (1) allows for N independent solutions. It is therefore always possible to construct a linearly independent derivative ring.

3 Standard derivative ring functions

Let me define the "standard" derivative-ring functions and let me start by introducing the notation

$$dex_{[N,n]}(x),$$

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where N corresponds to the size (length) of the derivative ring (number of its elements) and n ($0 \leq n < N$) points to a given element in the ring[1]. The definition then is

$$dex_{[N,n]}\left(x\right) \equiv \sum_{k=0}^{\infty} \frac{1}{(n+kN)!} x^{n+kN}. \tag{3}$$

This definition literally corresponds to the decomposition of the exponential function: its series is re-grouped into N sub-series, each sub-series containing each N-th element of the original series with a different shift at the origin. One should also notice that the index n increases in the anti-derivative direction:

$$\frac{d}{dx} dex_{[N,n+1]}\left(x\right) = dex_{[N,n]}\left(x\right).$$

The immediate consequences are:

$$dex_{[1,0]}\left(x\right) = \exp\left(x\right),$$

$$dex_{[2,0]}\left(x\right) = \cosh\left(x\right),$$

$$dex_{[2,1]}\left(x\right) = \sinh\left(x\right),$$

$$\sum_{n=0}^{N-1} dex_{[N,n]}\left(x\right) = \exp\left(x\right),$$

$$dex_{[N,0]}\left(0\right) = 1,$$

$$dex_{[N,0<j]}\left(0\right) = 0,$$

where the two last properties are of a particular interest, because they allow for an easy derivative matching at $x = 0$. They mean that each term in the series is in the one-to-one correspondence with a given derivative (at zero), i.e. if the n-th derivative should take the value d_n than the series needs to contain the term

$$d_n \times dex_{[N,n]}\left(x\right).$$

In other words, if the numbers $d_0, d_1, d_2, \ldots, d_{N-1}$ represent the (function and the) derivative values to be matched at $x = 0$, then the "derivative-ring" series takes the form

$$Drs\left(x\right) = \sum_{n=0}^{N-1} d_n \times dex_{[N,n]}\left(x\right).$$

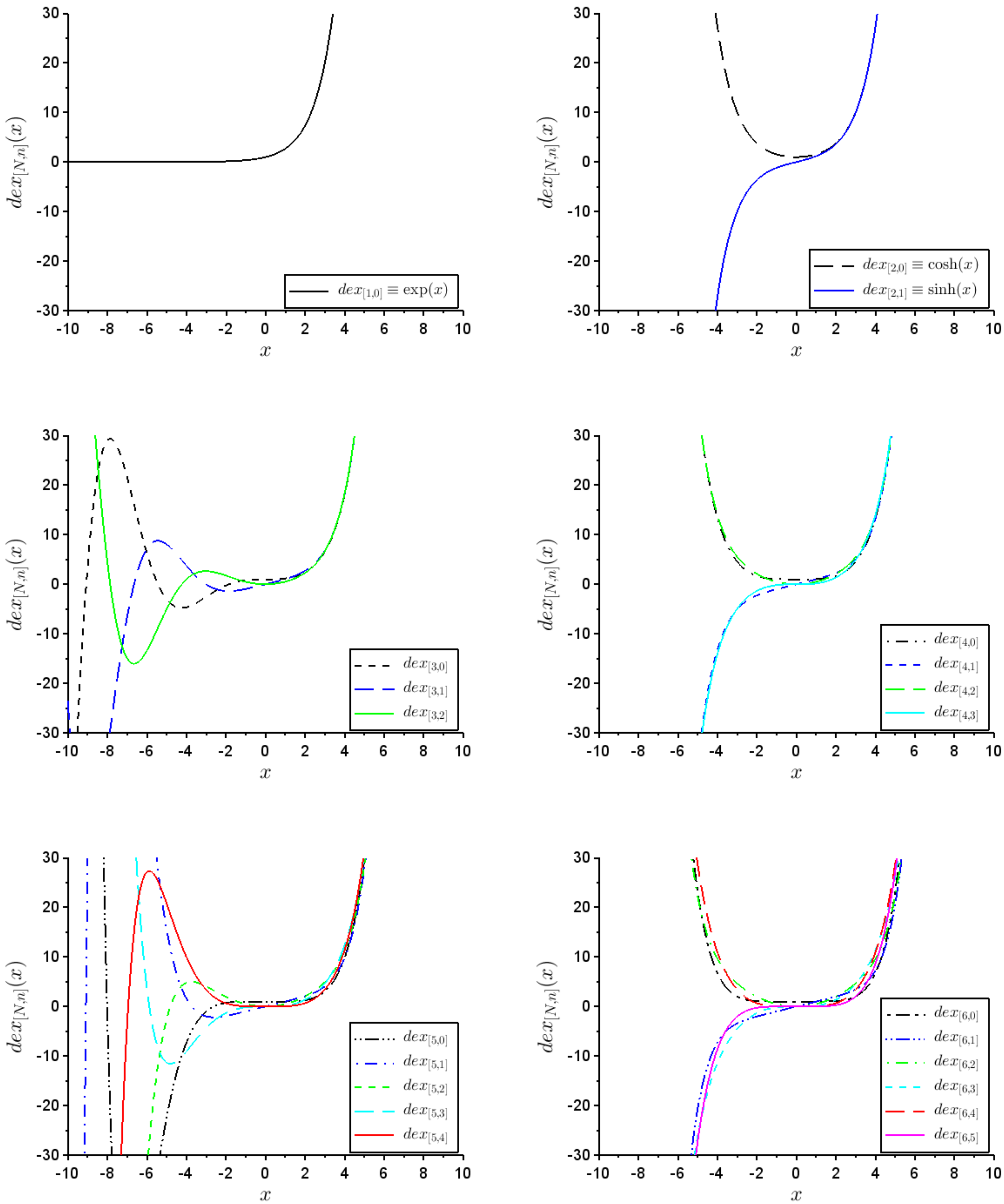
An arbitrary number of differentiation leads always to only one non-zero term at $x = 0$. It is worthless to say that the expansion can be used at an arbitrary point x_0 by shifting the whole series

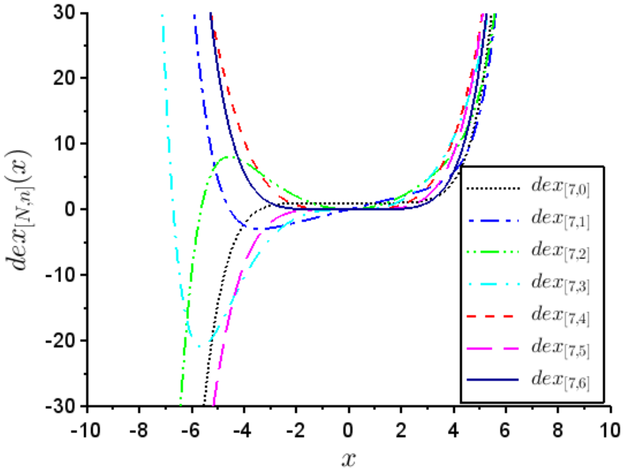
$$Drs\left(x\right) = \sum_{n=0}^{N-1} d_n \times dex_{[N,n]}(x - x_0), \quad d_n = \frac{d^n}{dx^n} f\left(x\right)|_{x=x_0}.$$

The convergence of the expression (3) can be questioned. It can be easily argued that the series is convergent for any real (and also complex) number. If $x > 0$ then the series is positive, growing and majorated by $\exp(x)$. Therefore it is necessarily convergent. Because the convergence is symmetrical with respect to the point of expansion (one talks about convergence *radius*) the series needs to converge for any real (complex) number.

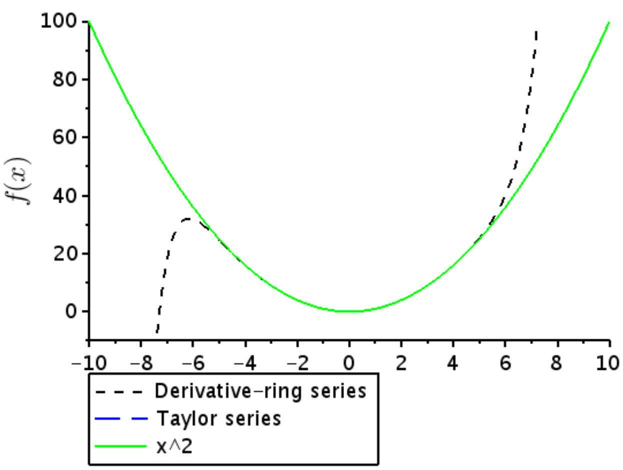
[1]The "*dex*" name was chosen as abbreviation from "decomposition of exponential".

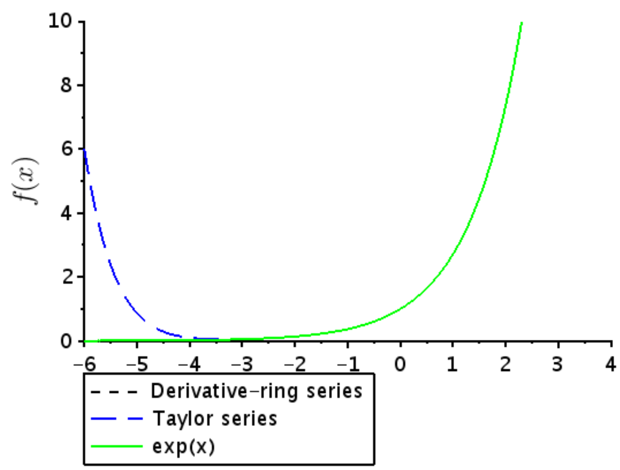
4 Plots of *dex* functions and examples

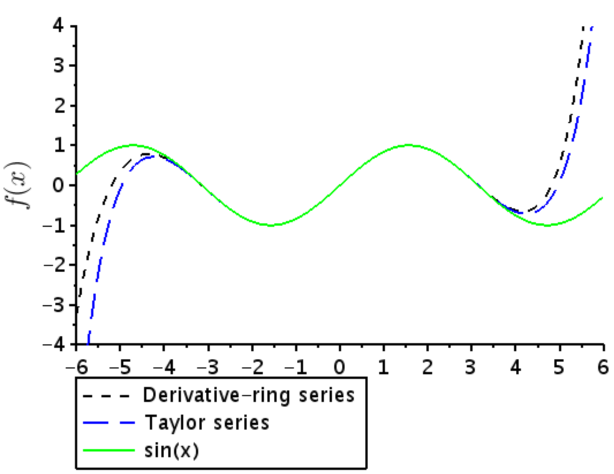


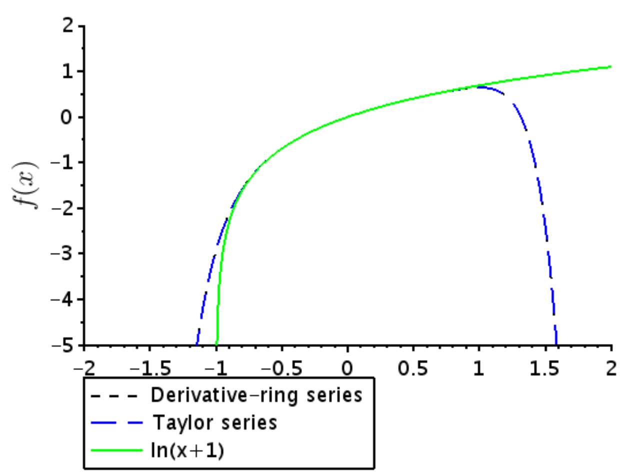


The approximation power was tested with 11 terms (value and 10 derivatives matched) and test functions are x^2, $\exp(x)$, $\sin(x)$, $\ln(x+1)$, the same settings as I have chosen in [1]. Two approximations for each test function were done: derivative-ring functions and Taylor polynomials.









For $y = x^2$ the Taylor polynomial gives, of course, an exact result and the derivative-ring method does a decent job. The situation is opposite for $\exp(x)$, where the derivative-ring series provides, by construction, the exact result. Both methods provide very comparable behavior for $\sin(x)$ and $\ln(x+1)$, in the latter case their graphs are actually overlaid.

5 Relation to prime numbers

We saw that the summation of the $dex_{[N,n]}$ functions in the second index gives exponential function. The summation over the first index corresponds to a summation over period lengths and makes us think about the Sieve of Eratosthenes. An interesting function can be constructed as follows:

- One defines one-subtracted ("reduced") $dex_{[N,n]}$ functions

$$dex^{R}_{[N,n]}(x) = dex_{[N,n]}(x) - 1.$$

  All derivatives of these functions are, of course, identical to those of standard $dex_{[N,n]}$ functions.

$$\frac{d^k}{dx^k} dex^{R}_{[N,n]}(x) \equiv \frac{d^k}{dx^k} dex_{[N,n]}(x) \text{ for any } k > 0.$$

- An infinite summation is performed in the first index, keeping the second one equal to zero

$$sDex(x) = \sum_{i=2}^{\infty} dex^{R}_{[i,0]}(x).$$

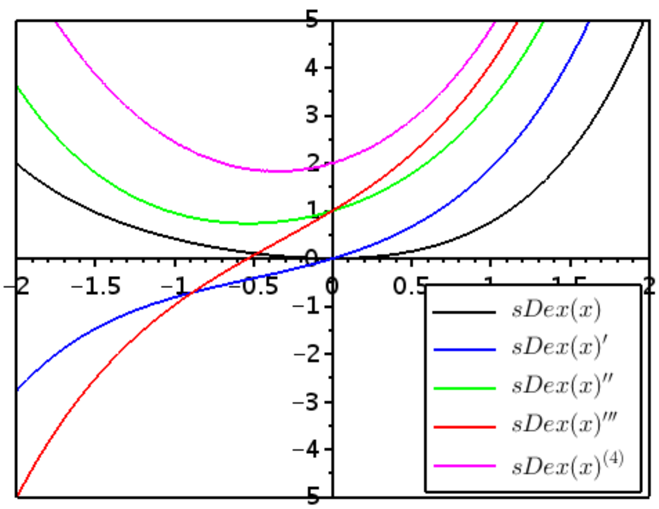
This construction has many similarities to the one I presented in [3]. The relation to prime numbers then stands:

$$k \text{ is prime } \Leftrightarrow \frac{d^k}{dx^k} sDex(x)_{|x=0} = 1.$$

Clearly

$$\frac{d^k}{dx^k} sDex(x)_{|x=0} = (\text{number of } \textit{all} \text{ divisors of } k) - 1.$$

I can hardly think of any profit one could get from this construction. The graph of the $sDex(x)$ function together with its first four derivatives looks as



References

[1] A. Liptaj, "Examples of approximations by series based on derivatives matching ("Taylor-like" approximations)", https://www.scribd.com/document/228743662/Examples-of-approximations-by-series-based-on-derivatives-matching-Taylor-like-approximations (2014).
http://vixra.org/abs/1704.0030

[2] https://en.wikipedia.org/wiki/Linear_differential_equation#Homogeneous_equations_with_constant_coefficients

[3] A. Liptaj, “Mathematical beauty with prime numbers:Elegant sieve-based primality test formulaconstructed with periodic functions ”,
https://www.scribd.com/doc/295973151/Mathematical-beauty-with-prime-numbers-Elegant-sieve-based-primality-test-formula-constructed-with-periodic-functions (2016).
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