On the Scale Factors of Energy Formulas

This paper explores the scale factors of three laws: (a) the Einstein's relativistic energy law, (b) Newton's law of universal gravitation and (c) the special universal uncertainty principle. Two new concepts are defined: complete energy laws and incomplete energy laws. This investigation shows that the first two laws have scale factors of 1 while the third one has a scale factor of -1. These results could be useful in the future to predict scale factors of new laws of nature.

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Contents

1. Complete and Incomplete Energy Laws

I shall begin by defining two new concepts: (a) complete energy laws and (b) incomplete energy laws. An energy law is a physical law that may be expressed either as

Definition of energy law
(two energy ratios)

$$\frac{E_a}{E_b} \; operator \; S \frac{E_c}{E_d} \tag{D.1}$$

or

Definition of energy law (one energy ratio) $$\frac{E_a}{E_b}\ operator\ S \tag{D.2}$$

The *operator* (relational operators) can be any of the following: equal to, $=$; greater than, $>$; less than, $<$; greater than or equal to $\geq$; less than or equal to $\leq$; approximately, $\approx$, etc. Most of the time we shall see the equal sign of an equation, such as

Definition of an energy equation $$\frac{E_a}{E_b} = S\frac{E_c}{E_d} \tag{D.2}$$

Where E_a , E_b , E_c , E_d are any variables whose units are units of energy. *S* is the scale factor (or scaling factor if you like). *S* is always a dimensionless real number. (Strictly speaking *S* can also be an imaginary number. For instance, the Schrödinger equation is a law with an imaginary scale factor. However I have never found an energy law with an imaginary scale factor). Also the scale factor could have more than one value in the same relationship (for example the formula for the energy levels of the hydrogen atom). The reader may want to refer to my papers on the scale law [1, 2] for more information.

A complete energy law is an energy law that satisfies either the following conditions

$$E_a \neq E_d \quad \text{and} \quad E_b = E_c \tag{1.1a}$$

or

$$E_a = E_d \quad \text{and} \quad E_b \neq E_c \tag{1.1b}$$

or

$$E_a \neq E_b \neq E_c \neq E_d \tag{1.1c}$$

Therefore energy, work and products like $F(x, y, z, t) \times r(x, y, z, t)$, where *F* is a variable force and *r* is a distance, are all quantities that may appear in an energy law. This means, for example, that if we have a law such as the Newton's gravity law, and we were able to derive it from first principles from a formula that satisfies equation (D.1) and conditions (1.1a) or (1.1b) or (1.c), then this law will be regarded as a complete energy law. It is important to observe that the product of a variable force times a distance like: $F(x, y, z, t) \times r(x, y, z, t)$ do not represent the work done by the force *F* along the distance *r*. This is so because the force, in this case, is variable. Despite this fact, the units of this product are units of work/energy. In order to get the work done by this force through a given distance, we should integrate. However, we do not need to use integration in this paper.

On the other hand, an incomplete energy law is an energy law that satisfies the following two conditions

$$E_a = E_d$$
$$\text{and} \qquad (1.2)$$
$$E_b = E_c$$

The result is $E_a^2 = S' E_b^2$, that may be written as $E_a = \sqrt{S'} E_b = S E_b$. Finally we write

$$\frac{E_a}{E_b} = S \qquad (1.3)$$

If we use expression (1.3) from the beginning we avoid the squares that, in the end, are eliminated by taking square root on both sides of the expression. To make it more clear to the reader section 5 includes examples of incomplete energy laws.

2. The Scale Factor of Einstein's Relativistic Energy Law

I shall begin this section with the famous Einstein's relativistic energy law

$$E^2 = p^2 c^2 + m_0^2 c^4 \qquad (2.1)$$

Now, we want to express this law in the form of the scale law. In order to do that let us draw a scale table as shown below

Energy	Energy	Energy	Energy
E_1	E_2	pc	pc

Table 2.1: *scale table for the formula:* $E^2 = p^2 c^2 + m_0^2 c^4$

Where

$E_1 =$ energy (the meaning is given below)
$E_2 =$ energy (the meaning is given below)
$p =$ momentum of the particle
$c =$ speed of light in vacuum

According to the above scale table we write

$$E_1 E_2 = S\, pc\, pc \qquad (2.2)$$

This is: the product of the first two columns of the table is proportional to the product of the last two columns. The proportionality constant, *S*, is the so called: scale factor or scaling factor. You may have noticed that I haven't included the scale factor in the table.

This was deliberately done for simplicity reasons. However we must include it in the equation (unless $S = 1$) to be able to get a correct relationship (of course the other quantities must be correct too). Thus, the scale factor is a dimensionless number that equates the two sides making a meaningful mathematical relation.

Equation (2.2) may be rewritten in the form of the scale law, as follows

$$\frac{E_1}{pc} = S\frac{pc}{E_2} \tag{2.3}$$

The meaning of E_1 and E_2 is as follows

$$E_1 = E + m_0 c^2 \tag{2.4}$$

$$E_2 = E - m_0 c^2 \tag{2.5}$$

Where

$E_1 =$ is the total relativistic energy of the particle *plus* its rest energy, and
$E_2 =$ is the total relativistic energy of the particle *less* its rest energy.
$E =$ total relativistic energy of the particle
$m_0 =$ rest mass of the particle

Substituting E_1 and E_2 in equation (2.3) with the second side of equations (2.4) and (2.5), respectively, we get

$$\frac{E + m_0 c^2}{pc} = S\frac{pc}{E - m_0 c^2} \tag{2.6}$$

A scale factor of 1 yields the total relativistic energy in the form of the scale law

The scale law for
Einstein's relativistic energy

$$\frac{E + m_0 c^2}{pc} = \frac{pc}{E - m_0 c^2} \tag{2.7}$$

Now, let us rewrite equation (2.7) as follows

$$\left(E + m_0 c^2\right)\left(E - m_0 c^2\right) = \left(pc\right)^2 \tag{2.8}$$

$$E^2 - \left(m_0 c^2\right)^2 = \left(pc\right)^2 \tag{2.9}$$

Finally

$$E^2 = p^2 c^2 + m_0^2 c^4 \tag{2.10}$$

Thus we have derived Einstein's formula for the total relativistic energy of a particle from the scale law. The scale factor of this law turned out to be $S = 1$.

3. The Scale Factor of Newton's Law of Universal Gravitation

First of all let me say that the contents of this section is included in a previous paper I wrote [3]. I shall begin the analysis by drawing a scale table for gravity

Work	Work	Energy	Energy
W_G	W	E_1	E_2

Table 3.1: *scale table for Newton's law of universal gravitation*

As done before, the scale factor is not shown in the table but will be incorporated into the equation later if necessary (meaning if it's different from 1). The meaning of the quantities shown in this table is as follows

$W_G = F_G\, r$ Unknown quantity which has units of work (The units of work and energy are the same: joules). $F_G = F_G(r)$ is the gravity force and r is the distance between the centers of the two attracting bodies or particles.

$W = F_P\, r$ Work done by the Planck force F_P (constant force) through a displacement equal to r

$E_1 = m_1\, c^2$ Rest energy of body 1

$E_2 = m_2\, c^2$ Rest energy of body 2

I shall assume that the speed of the bodies with respect to a given reference system, is much less than the velocity of light and that the bodies are spherical. The bodies can, of course, be particles instead.

For Table 3.1 to work, the quantities shown there must be defined as follows

$$W_G(r) = F_G\, r \tag{3.1}$$

$$W(r) = W = F_P\, r = M_P\, a_P\, r \tag{3.2}$$

$$E_1 = m_1\, c^2 \tag{3.3}$$

$$E_2 = m_2\, c^2 \tag{3.4}$$

From the table we establish the following relationship

The scale law for gravity as a product, with a scale factor to be determined	$W_G W = S\, E_1 E_2$	(3.5 a)

or, equivalently

The scale law for gravity as a ratio, with a scale factor to be determined	$\frac{W_G}{E_1} = S \frac{E_2}{W}$	(3.5 b)

Now I assume that the scale factor, *S*, is 1. Then, we write the previous two equations as follows

The scale law for gravity with S = 1	$W_G W = E_1 E_2$ or, equivalently $\frac{W_G}{E_1} = \frac{E_2}{W}$	(3.5 c) (3.5 d)

The are two ways of putting this law into words; and they are:

The Scale law as a product

The product of the force of gravity, $F_G(r)$, times the distance *r*, times the work done by the Planck force (which is a constant force) along that distance; equals the product of the rest energy of body 1 times the rest energy of body 2.

Or, equivalently

The Scale law as a ratio

The ratio between the force of gravity, $F_G(r)$, times the distance *r*, to the rest energy of one of the bodies; equals the ratio between the rest energy of the other body to the work done by the Planck force along the same distance.

Because gravity is a force that varies with the distance between the two bodies, it is important to observe that the product: $F_G(r)\, r$ is not the work done by gravity. However, defining this unknown quantity, W_G , as $F_G r$ (whatever the physical meaning turns out to be) is correct because gravity has the property of satisfying equations (3.5 a)/(3.5 b)/(3.5 c)/(3.5 d) if and only if W_G is defined this way and W is defined as shown above. Now it should be clear that I have denoted this quantity W_G simply because its units are units of work (or, equivalently, energy).

Replacing the variables: W_G , W by equations (3.1) and (3.2), respectively, we get

$$F_G r F_P r = E_1 E_2 \tag{3.6}$$

or

Relativistic gravity law (form 1)

$$F_G = \frac{1}{F_P} \frac{E_1 E_2}{r^2} \tag{3.7}$$

But the Planck force is

Planck force

$$F_P = \frac{c^4}{G} \tag{3.8}$$

Therefore, using this result we may write

Relativistic gravity law (form 2)

$$F_G = \frac{G}{c^4} \frac{E_1 E_2}{r^2} \tag{3.9}$$

Equations (3.8) and (3.9) are two relativistic forms of Newton's gravity law. Thus, this law is relativistic because it contains the product of the rest energy of particle 1 $\left(E_1 = m_1 c^2\right)$ times the rest energy of particle 2 $\left(E_2 = m_2 c^2\right)$. The formula is not quantum mechanical because the expression for the Planck force does not contain the Planck's constant.

Now we replace the quantities E_1 and E_2 by the second side of equations (3.3) and (3.4), respectively, and we get

$$F_G = \frac{G}{c^4} \frac{m_1 c^2 m_2 c^2}{r^2} \tag{3.10}$$

Which, after simplification, turns out to be identical to the Newton's gravity law

Newton's law of universal gravitation

$$F_G = \frac{G m_1 m_2}{r^2} \tag{3.11}$$

The way Newton formulated his law of gravitation does not allow us to appreciate the relativistic nature of this law simply because c^4 in the numerator cancels out with c^4 in the denominator as we have seen. Now, let us consider equation (3.5 d)

The work-energy ratio law of universal gravitation

$$\frac{W_G}{E_1} = \frac{E_2}{W} \qquad (3.5\ d) = (3.12)$$

Because equation (3.12) is another form of expressing the Newton's law of gravitation we need to differentiate it from the original Newton's form given by eq. (3.11). This can be achieved by using a different name for equation (3.12): the *'work'-energy ratio law of universal gravitation* (or simply *work-energy ratio law*). This doesn't mean that we have two different laws, it simply means that we have two different forms of expressing the same law. However there is an important difference. The *work-energy ratio law of* *universal gravitation* is more fundamental that the Newton's gravity law because it contains the work done by the Planck force: $W = F_P r$.

Thus we have proved that Newton's law of universal gravitation obeys the scale law with a scale factor of 1. Moreover, equations (3.5 c)/(3.5 d) show that Newton's law is a complete energy law.

4. The Scale Factor of the Special Energy-Time Universal Uncertainty Relation

I shall begin this section by writing the universal energy-time uncertainty principle (or universal time-energy uncertainty principle if you like) [4] as follows

Universal energy-time uncertainty relation

$$\Delta E\,\Delta t \geq \sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4}\Delta E\, T_P} \tag{4.1}$$

Taking the square of both sides

$$\Delta E^2 \Delta t^2 \geq \frac{\hbar^2}{4} - \frac{\hbar}{4}\Delta E\, T_P \tag{4.2}$$

Where I would like to clarify that: $\Delta E^2 = (\Delta E)^2$ and that: $\Delta t^2 = (\Delta t)^2$, to avoid confusions. Now, I perform a number of simple algebraic steps to get this law in the form of the scale law

$$\Delta E^2 \Delta t^2 - \left(\frac{\hbar}{2}\right)^2 \geq -\frac{\hbar}{4}\Delta E\, T_P \tag{4.3}$$

$$\left(\Delta E\,\Delta t + \frac{\hbar}{2}\right)\left(\Delta E\,\Delta t - \frac{\hbar}{2}\right) \geq -\frac{\hbar}{4}\Delta E\, T_P \tag{4.4}$$

$$\Delta t\left(\Delta E + \frac{\hbar}{2\Delta t}\right)\Delta t\left(\Delta E - \frac{\hbar}{2\Delta t}\right) \geq -\frac{\hbar}{4}\Delta E\, T_P \tag{4.5}$$

$$\left(\Delta E + \frac{\hbar}{2\Delta t}\right)\left(\Delta E - \frac{\hbar}{2\Delta t}\right) \geq -\frac{\hbar}{4}\frac{\Delta E\, T_P}{\Delta t^2} \tag{4.6}$$

$$\left(\Delta E + \frac{\hbar}{2\Delta t}\right)\left(\Delta E - \frac{\hbar}{2\Delta t}\right) \geq -\frac{\hbar}{2\Delta t}\frac{\Delta E\, T_P}{2\Delta t} \tag{4.7}$$

Finally, dividing both sides by $\left(\Delta E + \frac{\hbar}{2\Delta t}\right)\left(\frac{\hbar}{2\Delta t}\right)$ we get

The scale law for the energy-time universal uncertainty principle

$$\frac{\Delta E - \frac{\hbar}{2\,\Delta t}}{\frac{\hbar}{2\,\Delta t}} \geq -\frac{\frac{\Delta E\, T_P}{2\,\Delta t}}{\Delta E + \frac{\hbar}{2\,\Delta t}} \tag{4.8}$$

Inspecting this inequation we find that the scale factor is $S = -1$. Thus, we have proved that the special universal uncertainty principle is a complete energy law. By the way, it is interesting to observe that all the ratios have identical denominators: $2\,\Delta t$. Let us check this formula by replacing the Planck time, T_P , by zero. This yields

$$\Delta E - \frac{\hbar}{2\,\Delta t} \geq 0 \tag{4.9}$$

Which written in a more familiar way turns out to be the famous energy-time Heisenberg uncertainty principle as it should be

$$\Delta E\,\Delta t \geq \frac{\hbar}{2} \tag{4.10}$$

5. Incomplete Energy Laws

An incomplete energy law is a law that is not a complete energy law. This does not mean that there is something wrong with the law, it simply means that the law does not comply with the definition given in the first section for complete energy laws. The following is a list of incomplete energy laws (by the way, the list is also incomplete)

a) Einstein's formula for the energy of a photon

$$E = h\,f \tag{5.1 a}$$

This law can be expressed in the form of the scale law as follows

Incomplete energy law

$$\frac{E}{h\,f} = 1 \tag{5.1 b}$$

b) The quantum harmonic oscillator

$$E_n = \left(n+1\right)\,h\,f \tag{5.2 a}$$

$$n = 0,\ 1,\ 2,\ 3,\ 4,\ \ldots$$

This law can be expressed in the form of the scale law as follows

Incomplete energy law $$\frac{E_n}{hf} = (n+1) \qquad (5.2\text{ b})$$

$$n = 0, 1, 2, 3, 4, \ldots$$

It is important to observe that the scale factor for the quantum harmonic oscillator is $S = n+1$ with $n = 0, 1, 2, 3, 4, \ldots$. This means that this scale factor depends on the quantum number n, and consequently, has infinite values.

c) The photoelectric effect

$$\frac{1}{2} m v_{max}^2 = hf - W \qquad (5.3\text{ a})$$

Where $(1/2) m v_{max}^2$ is the maximum kinetic energy of the emitted electrons, m is the rest mass of the electron and W is the work function which is the minimum energy required to liberate an electron from the surface of the metal. This law can be expressed in the form of the scale law as follows

Incomplete energy law $$\frac{\frac{1}{2} m v_{max}^2}{hf - W} = 1 \qquad (5.3\text{ b})$$

As an example I shall analyze the Einstein's formula for the energy of a photon

Example
Einstein's Formula for the Energy of a Photon

A photon is characterized by certain wavelength, λ, and frequency, f. The energy carried by a single photon is given by the following Einstein's equation (also known as Planck-Einstein equation)

$$E = h\,f \qquad (\text{E.1})$$

We know that this formula is a special case of the Einstein's relativistic energy law given in the previous section when the rest mass is zero.

Rest mass of a photon $$m_0 = 0 \qquad (\text{E.2})$$

Taking this into account we modify the scale table of the previous section by replacing the total energies E_1 and E_2 by E. This yields

Energy	Energy	Energy	Energy
E	E	pc	pc

Table 5.1: *Scale table for* $E = h\,f$

From this table we write

Scale law as a product $$E\,E = S\;pc\;pc \tag{E.3}$$

Where S is the scale factor used to make an equation from the two energy products of eq. (E.3). Another way of writing this relation is as follows

Scale law as a ratio $$\frac{E}{pc} = S\,\frac{pc}{E} \tag{E.4}$$

This is the scale law as a ratio. Adopting a scale factor of 1

$$S = 1 \tag{E.5}$$

We have

The scale law for the photon energy (not very nice!) $$\frac{E}{pc} = \frac{pc}{E} \tag{E.6}$$

$$E^2 = (pc)^2 \tag{E.7}$$

If we take the square root of both sides we get

$$E = pc \tag{E.8}$$

But according to de Broglie

$$p = \frac{h}{\lambda} \tag{E.9}$$

and considering that the speed of light can be written as the product of the wavelength times the frequency, we may write

$$c = \frac{\lambda}{T} = \lambda\; f \tag{E.10}$$

Now we may combine the last two equations with eq. (E.8) to get

$$E = \frac{h}{\lambda} \lambda f \tag{E.11}$$

Finally

$$E = h f \tag{E.12}$$

Thus, I have proved that choosing a scale factor of 1 was the correct choice. I have also proved that the formula for the total relativistic energy for photons obeys the scale law. (Since the rest mass of a photon is zero, the total relativistic energy for a photon is equal to its kinetic energy). However looking at equation (E.6), we see that this is not a complete energy law because it does not comply with the condition imposed by the definition. The reason is this

$$E_a = E \tag{E.13a}$$

$$E_b = pc \tag{E.13b}$$

$$E_c = pc \tag{E.13c}$$

$$E_d = E \tag{E.13d}$$

Therefore

$$\begin{gathered} E_a = E_d \\ \text{and} \\ E_b = E_c \end{gathered} \tag{E.14}$$

Because of this result the formula for the energy of a photon is an incomplete energy law. This law may also be written as

The scale law for
the photon energy
(preferred form)

$$\frac{E}{hf} = 1 \tag{E.15}$$

This should be the preferred form. We should avoid using form (E.6) which was included for didactic reasons.

6. Conclusions

This paper is concerned with the scale factors of energy formulas. The results I have obtained are summarized in the following table

COMPLETE ENERGY LAWS	SCALE FACTOR (*S*)
Einstein's Relativistic Energy Formula	1
Newton's Law of Universal Gravitation	1
Special Universal Uncertainty Relations	-1

In summary, as far as I know, all complete energy laws (at least the ones we have seen here) have scale factors whose absolute values are 1. This is an interesting result that, if true, will allow us to predict scale factors of new energy laws that we may discover in the future.

Appendix 1
Nomenclature

I shall use the following nomenclature for the constants and variables used in this paper

$E_a =$ variable whose units are units of energy
$E_b =$ variable whose units are units of energy
$E_c =$ variable whose units are units of energy
$E_d =$ variable whose units are units of energy
$F(x, y, z, t) =$ variable force
$r(x, y, z, t) =$ distance
$S =$ scale factor or scaling factor
$c =$ speed of light in vacuum
$E_1 =$ is the total relativistic energy of particle 1 *plus* its rest energy. Also, in a different context, the rest energy of particle 1
$E_2 =$ is the total relativistic energy of particle 2 *less* its rest energy. Also, in a different context, the rest energy of particle 2
$E =$ total relativistic energy of the particle
$p =$ momentum of the particle
$m_0 =$ rest mass of the particle
$h =$ Planck's constant
$\hbar =$ reduced Planck's constant $(\hbar = h/2\pi)$
$G =$ Newton's gravitational constant
$F_G =$ Gravitational force between two bodies of masses m_1 and m_2 (this force is also known as force of universal gravitation, gravity, gravity force, force of gravitational attraction, force of gravity, Newtonian force of gravity, force of universal mutual gravitation, etc.)
$F =$ force
$n =$ power of a hypothetical gravity law. Also, in a different context, a quantum number.

$a =$ acceleration
$m_1 =$ mass of body or particle 1
$m_2 =$ mass of body or particle 2
$r =$ distance between the centers of body 1 and body 2. Also displacement.
$dr =$ infinitesimal distance
$L_P =$ Planck length
$T_P =$ Planck time
$F_P =$ Planck force
$a_P =$ Planck acceleration
$M_P =$ Planck mass
$W_G =$ unknown quantity which has units of work
$W =$ work done by F_P (this is a constant force). Also, work function of the surface
$dW =$ infinitesimal work
$K =$ relativistic kinetic energy
$v =$ velocity
$t =$ time
$\Delta E =$ uncertainty in energy
$\Delta t =$ uncertainty in time
$T_P =$ Planck time
$f =$ frequency of a photon
$T =$ period of a photon
$\lambda =$ wavelength of a photon
$v_{max} =$ maximum speed of an electron

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