Question 2990 : Euler's Constant , Fractals

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Abstract. This note presents formulas related with Euler-Mascheroni constant and fractals.

Introduction." Euler-Mascheroni constant γ ":

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}-\ln n\right)=0.577215... \tag{1}$$

Formulas and fractals:

$$\int_{-\infty}^{\infty} x e^{x-e^{x-\gamma}}\,dx = 0 \tag{2}$$

$$\gamma = \int_{-\infty}^{\infty} x e^{-x-e^{-x}}\,dx \tag{3}$$

$$\int_{0}^{\infty} e^{-xe^{-\gamma}} \ln x\,dx = 0 \tag{4}$$

$$\int_{0}^{\infty} x e^{-xe^{1-\gamma}} \ln x\,dx = 0 \tag{5}$$

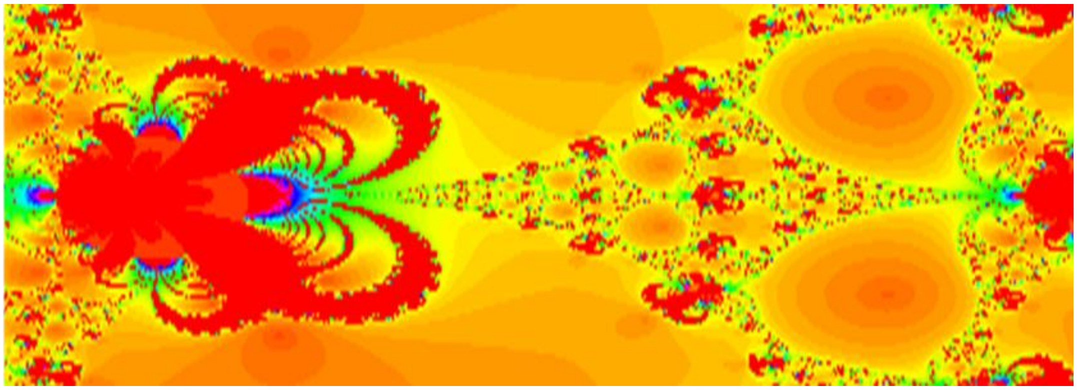

Figure 1.

$$\gamma = \int_0^\infty \frac{\ln x}{x^2} e^{-1/x} dx \tag{6}$$

$$\gamma = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)^2} - \int_0^\infty x e^{x-e^x} dx \tag{7}$$

$$\gamma = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n e^{n+1}}{n!(n+1)^2} - \int_1^\infty x e^{x-e^x} dx \tag{8}$$

$$\gamma = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)^2} - \sum_{n=0}^{\infty} e^{n-e^n} I(n) \tag{9}$$

$$\gamma = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n e^{n+1}}{n!(n+1)^2} - \sum_{n=1}^{\infty} e^{n-e^n} I(n) \tag{10}$$

$$I(n) = \int_0^1 (x+n) e^{x-e^n\left(e^x-1\right)} dx \quad , n \in \mathbb{N} \cup \{0\} \tag{11}$$

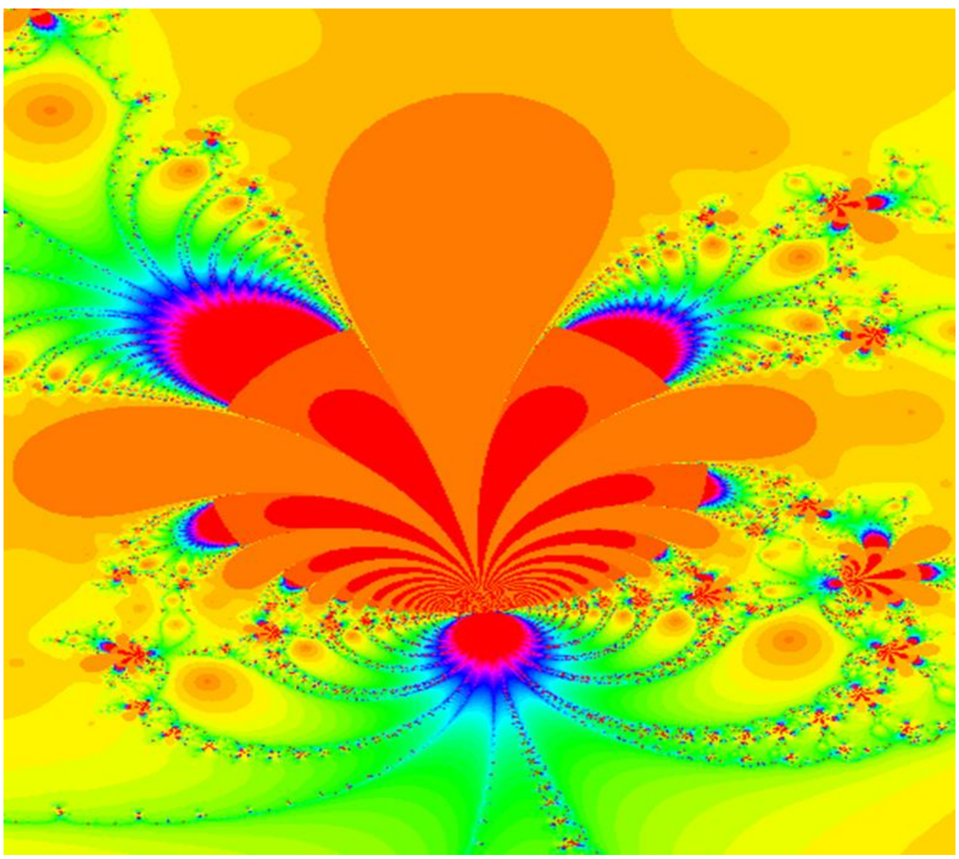

Figure 2.

$$\gamma = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)^2} - \cfrac{e^{-1}}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{1+\cfrac{2}{1+\cfrac{3}{1+\cfrac{3}{1+\ldots}}}}}}} \tag{12}$$

$$\gamma = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n e^{n+1}}{n!(n+1)^2} - e^{-e} - \cfrac{e^{-e}}{e+\cfrac{1}{1+\cfrac{1}{e+\cfrac{2}{1+\cfrac{2}{e+\cfrac{3}{1+\cfrac{3}{e+\ldots}}}}}}} \tag{13}$$

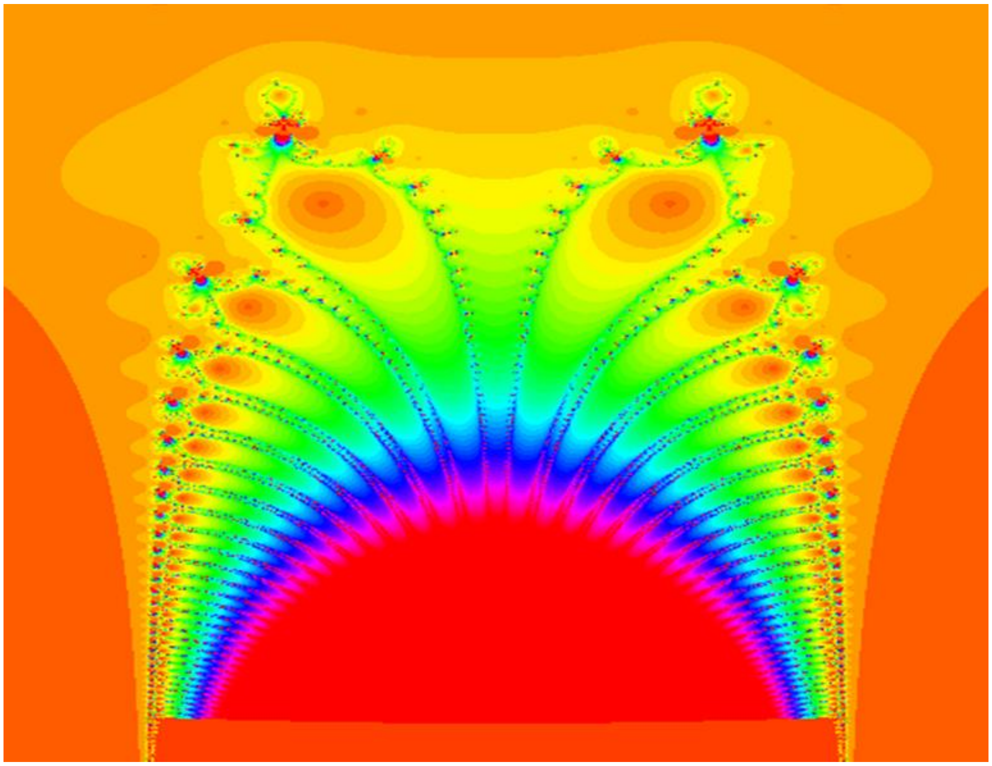

Figure 3.

$$\gamma = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \alpha^n}{n!n} = \int_0^{\alpha} \frac{1-e^{-x}}{x} dx \tag{14}$$

$$\alpha = 0.676355077886536... \tag{15}$$

$$\alpha = \exp\left(-\int_{\alpha}^{\infty} \frac{e^{-x}}{x} dx \right) = \exp\left(-Ei\left(1,\alpha\right)\right) \tag{16}$$

$$x_1 = 1/2, x_{n+1} = \exp\left(-Ei\left(1, x_n\right)\right) \Rightarrow \lim_{n\to\infty} x_n = \alpha \tag{17}$$

$$x_1 = 1/2, x_{n+1} = \frac{x_n\left(1-e^{-x_n}\right)}{x_n \exp\left(Ei\left(1, x_n\right)\right) - e^{-x_n}} \Rightarrow \lim_{n\to\infty} x_n = \alpha \tag{18}$$

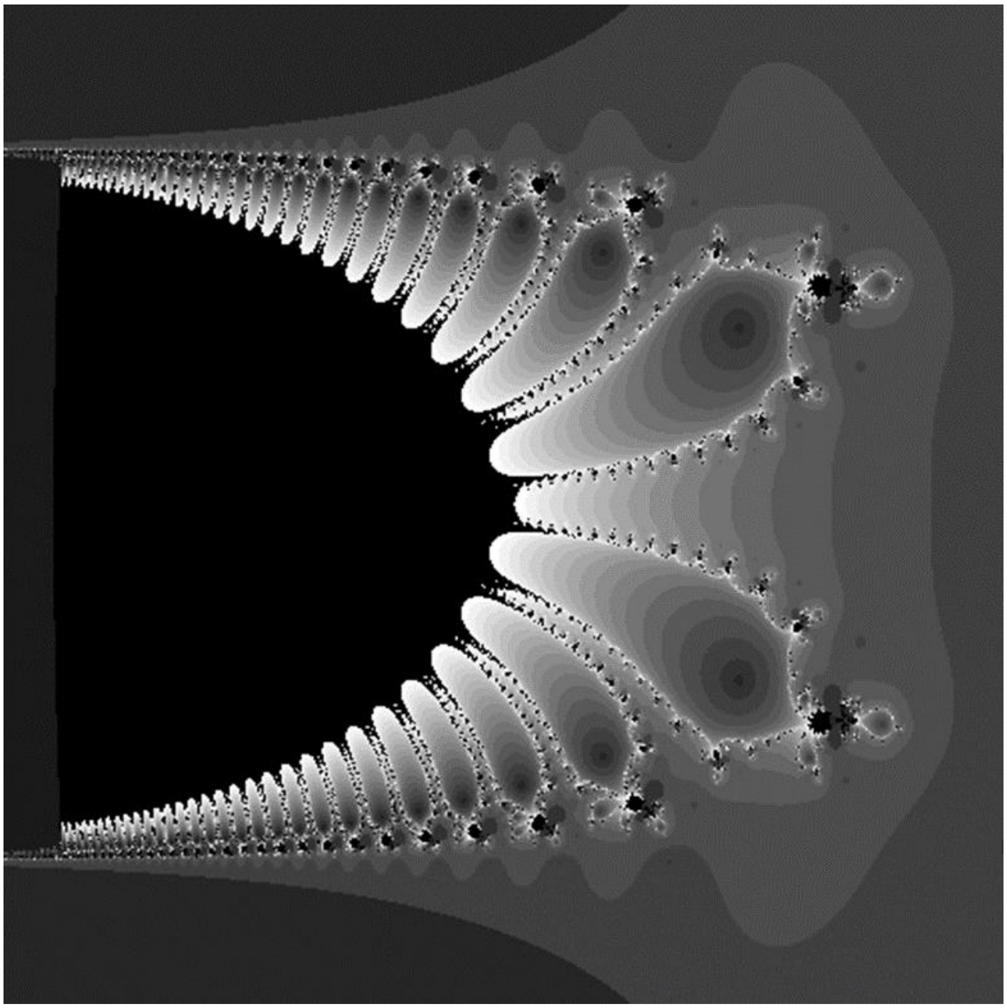

Figure 4.

$$\gamma = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \beta^{2n}}{(2n)!(2n)} = \int_0^{\beta} \frac{1-\cos x}{x} dx \tag{19}$$

$$\beta = 1.602686491719712... \tag{20}$$

$$\beta = \exp\left(-\int_{\beta}^{\infty} \frac{\cos x}{x} dx \right) = \exp\left(Ci(\beta) \right) \tag{21}$$

$$x_1 = 3/2, x_{n+1} = \exp\left(Ci\left(x_n \right) \right) \Rightarrow \lim_{n\to\infty} x_n = \beta \tag{22}$$

$$x_1 = 3/2, x_{n+1} = \frac{x_n \left(1-\cos x_n \right)}{x_n \exp\left(-Ci\left(x_n \right) \right) - \cos x_n} \Rightarrow \lim_{n\to\infty} x_n = \beta \tag{23}$$

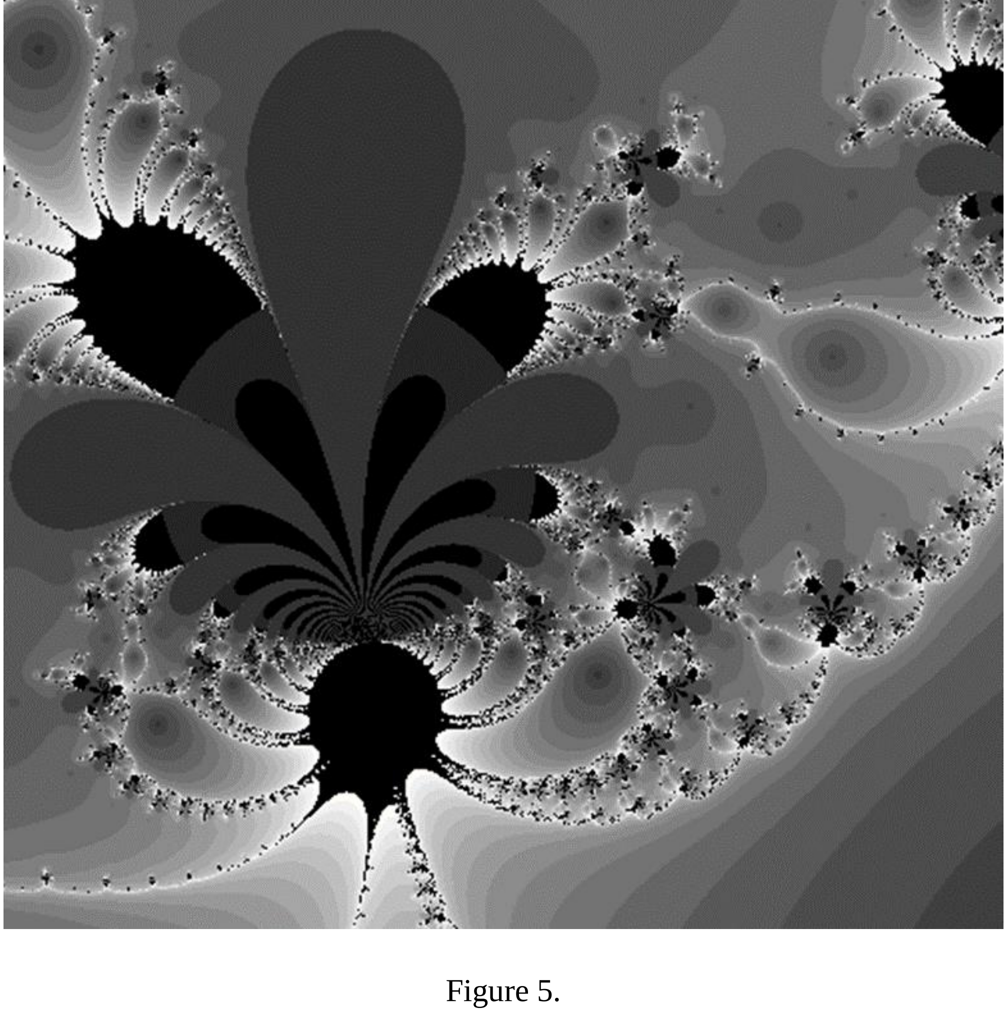

Figure 5.

$$\gamma = H_n - \ln n + n\,e^{-H_n} \int_{-\infty}^{\infty} x\,e^{-x-n\,e^{-x-H_n}}\,dx\,,\; n \in \mathbb{N} \tag{24}$$

$$\gamma = H_n - \ln n - n e^{-H_n} \int_{0}^{\infty} x\,e^{x-n\,e^{x-H_n}}\,dx + n e^{-H_n} \sum_{k=0}^{\infty} \frac{(-1)^k \left(n e^{-H_n}\right)^k}{k!\,(k+1)^2} \;\;,\, n \in \mathbb{N} \tag{25}$$

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \tag{26}$$

$$\gamma = 1 - e^{-1} \int_{-\infty}^{\infty} x\,e^{x-e^{x-1}}\,dx \tag{27}$$

$$\gamma = \frac{1}{2} + \frac{1}{\sqrt{e}} \int_{0}^{\infty} e^{-x/\sqrt{e}} \ln\left(\frac{1}{x}\right) dx \tag{28}$$

$$\gamma = \frac{1}{2} + \frac{1}{2\sqrt{e}} \int_{0}^{\infty} \frac{e^{-\sqrt{x/e}}}{\sqrt{x}} \ln\left(\frac{1}{x}\right) dx \tag{29}$$

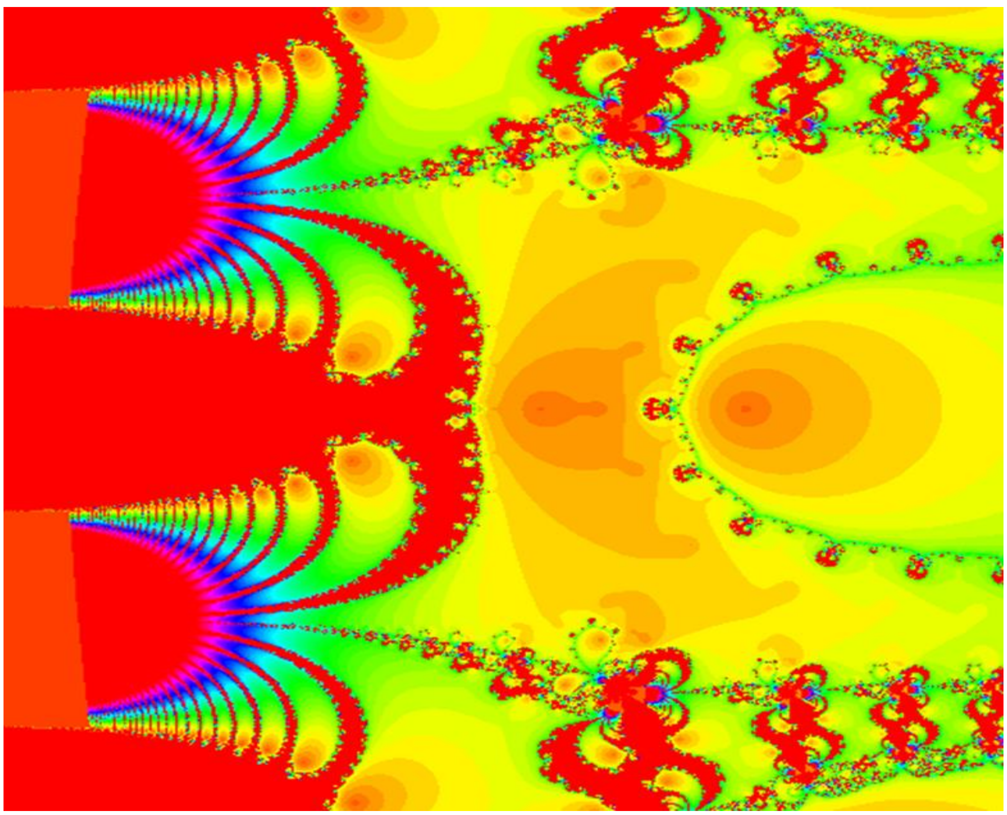

Figure 6.

$$\gamma + \ln \pi - \ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \pi^{2n}}{(2n)!(2n) 2^{2n}} + \sum_{n=0}^{\infty} 2^{-n-1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} c_k \tag{30}$$

$$c_k = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{x + (k+1)\pi} dx = \int_0^{\pi} \frac{\sin x}{x + \left(k + \frac{1}{2}\right)\pi} dx \quad , k = 0,1,\ldots,n \tag{31}$$

$$c_k = (-1)^k \left(Ci\left(\frac{\pi}{2} + k\pi\right) - Ci\left(\frac{3\pi}{2} + k\pi\right)\right) \quad , k = 0,1,\ldots,n \tag{32}$$

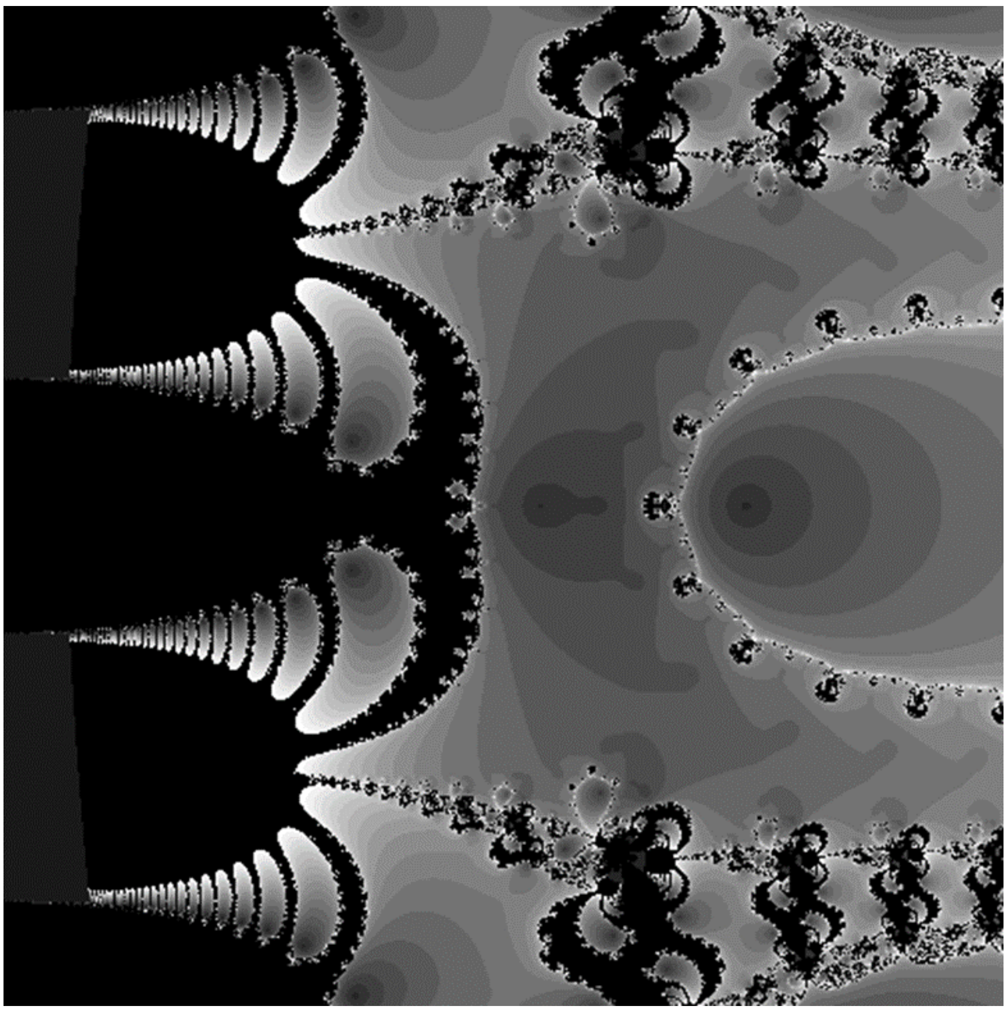

Figure 7.

$$\gamma = -\int_0^\delta e^{-x} \ln x dx \tag{33}$$

$$\delta = 0.285014262249661... \tag{34}$$

$$\gamma = \sum_{n=0}^{\infty} \frac{(-1)^n \delta^{n+1}}{n!(n+1)^2} - \ln\delta \sum_{n=0}^{\infty} \frac{(-1)^n \delta^{n+1}}{n!(n+1)} \tag{35}$$

$$x_1 = 1/3, x_{n+1} = \exp\left(-Ei(1, x_n)\exp(x_n)\right) \Rightarrow \lim_{n\to\infty} x_n = \delta \tag{36}$$

$$\ln\delta + e^\delta Ei(1,\delta) = 0 \tag{37}$$

$$\gamma = \int_{-\ln\delta}^{\infty} x e^{-x-e^{-x}} dx \tag{38}$$

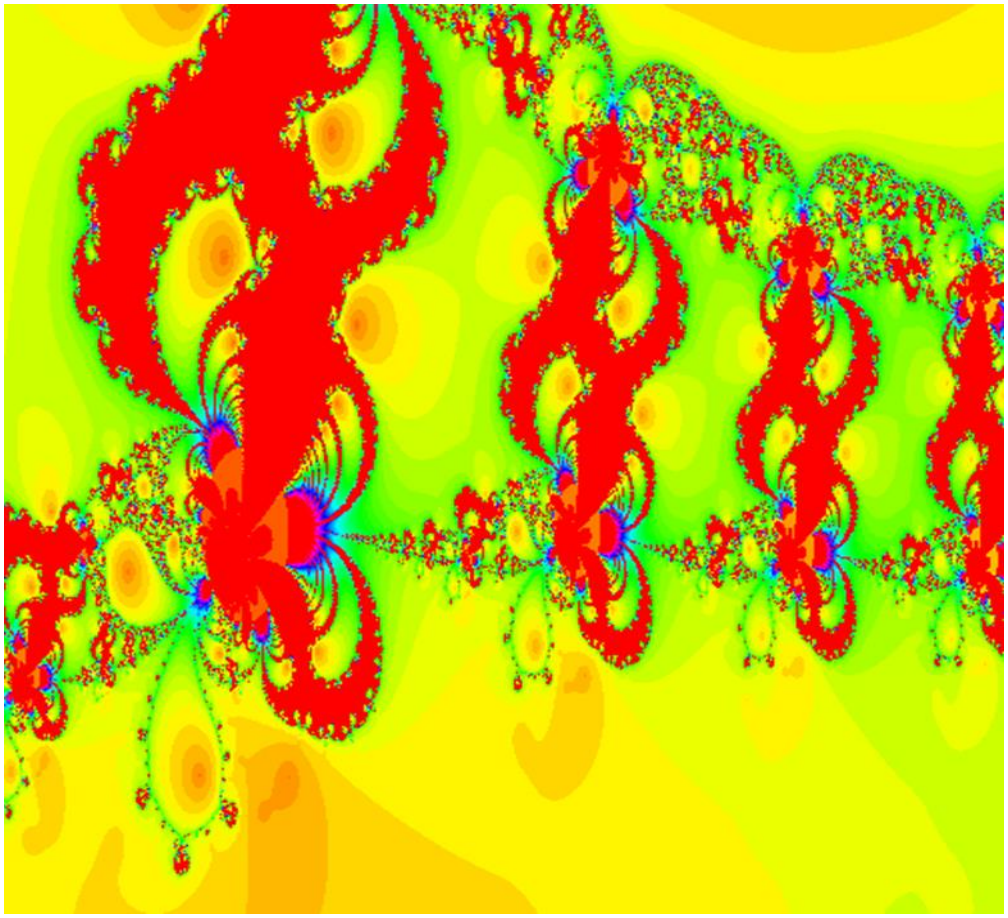

Figure 8.

$$\gamma = \left(1 - e^{-\alpha}\right)\ln\alpha - \int_0^{\alpha} e^{-x}\ln x\, dx \tag{39}$$

$$\gamma = \left(1 - \cos\beta\right)\ln\beta - \int_0^{\beta} \ln x \sin x\, dx \tag{40}$$

$$\gamma = 1 - \frac{\sin\beta}{\beta} + \int_0^{\beta} \frac{x - \sin x}{x^2}\, dx \tag{41}$$

$$\gamma = -\beta \int_0^{1} \ln x \sin\left(\beta x\right) dx \tag{42}$$

$$\gamma = 2\int_0^{\beta} \frac{1}{x}\left(\sin\left(\frac{x}{2}\right)\right)^2 dx \tag{43}$$

$$\gamma = \int_{1/\delta}^{\infty} \frac{\ln x}{x^2} e^{-1/x} dx \tag{44}$$

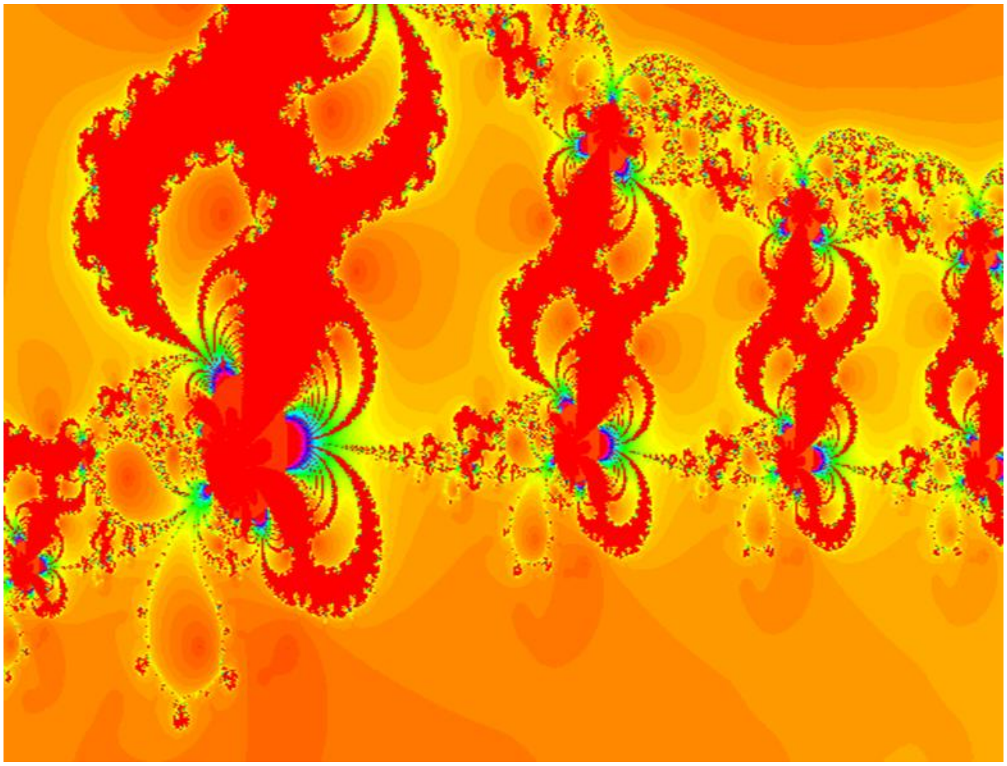

Figure 9.

$$\gamma = 2 - \sqrt{\pi} - \sum_{n=1}^{\infty} 2^n \left(-1 - \Gamma\left(1 + 2^{-n}\right) + 2\Gamma\left(1 + 2^{-n-1}\right) \right) \tag{45}$$

$$\gamma = 2 - \sqrt{\pi} - \sum_{n=2}^{\infty} n! \left(-n - \Gamma\left(1 + \frac{1}{n!}\right) + (n+1)\Gamma\left(1 + \frac{1}{(n+1)!}\right) \right) \tag{46}$$

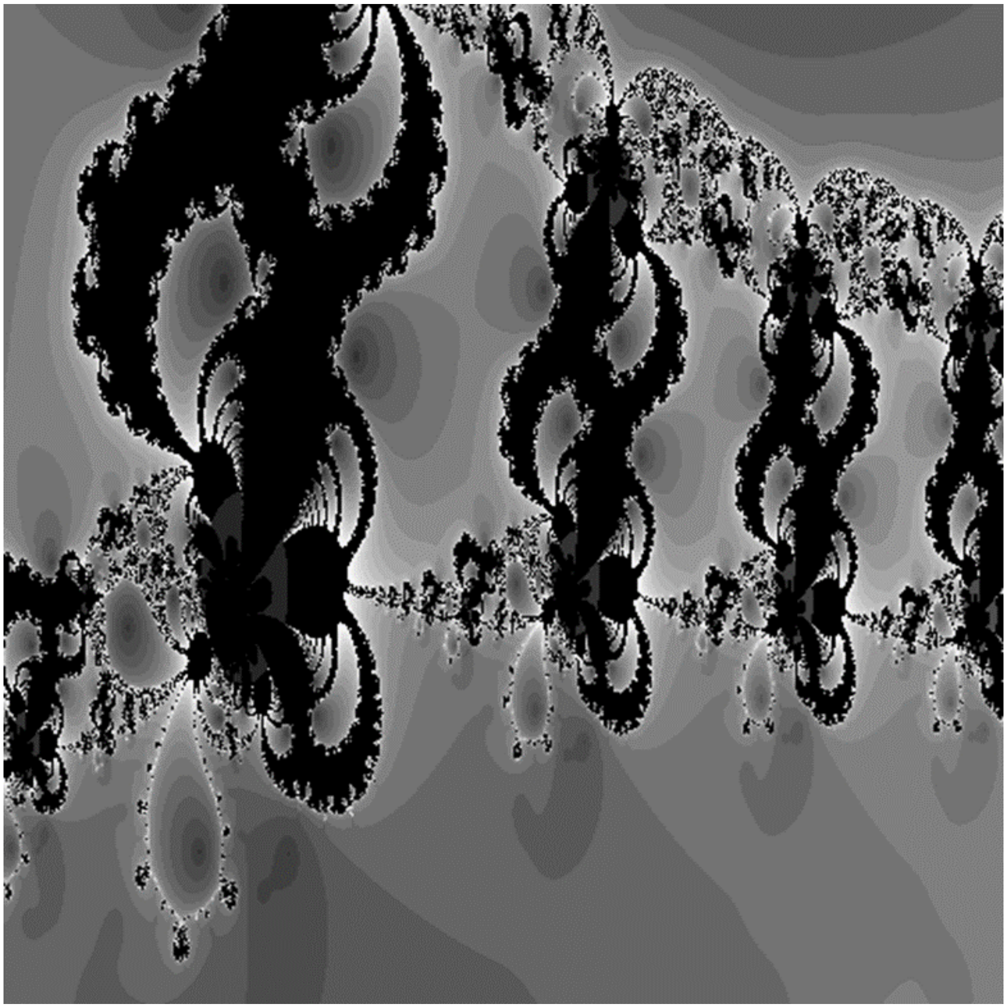

Figure 10.

$$\gamma = H_n - \ln n - \int_0^{\infty} \left(e^{-n\,e^{-H_n+x}} + e^{-n\,e^{-H_n-x}} - 1 \right) dx \quad , n \in \mathbb{N} \tag{47}$$

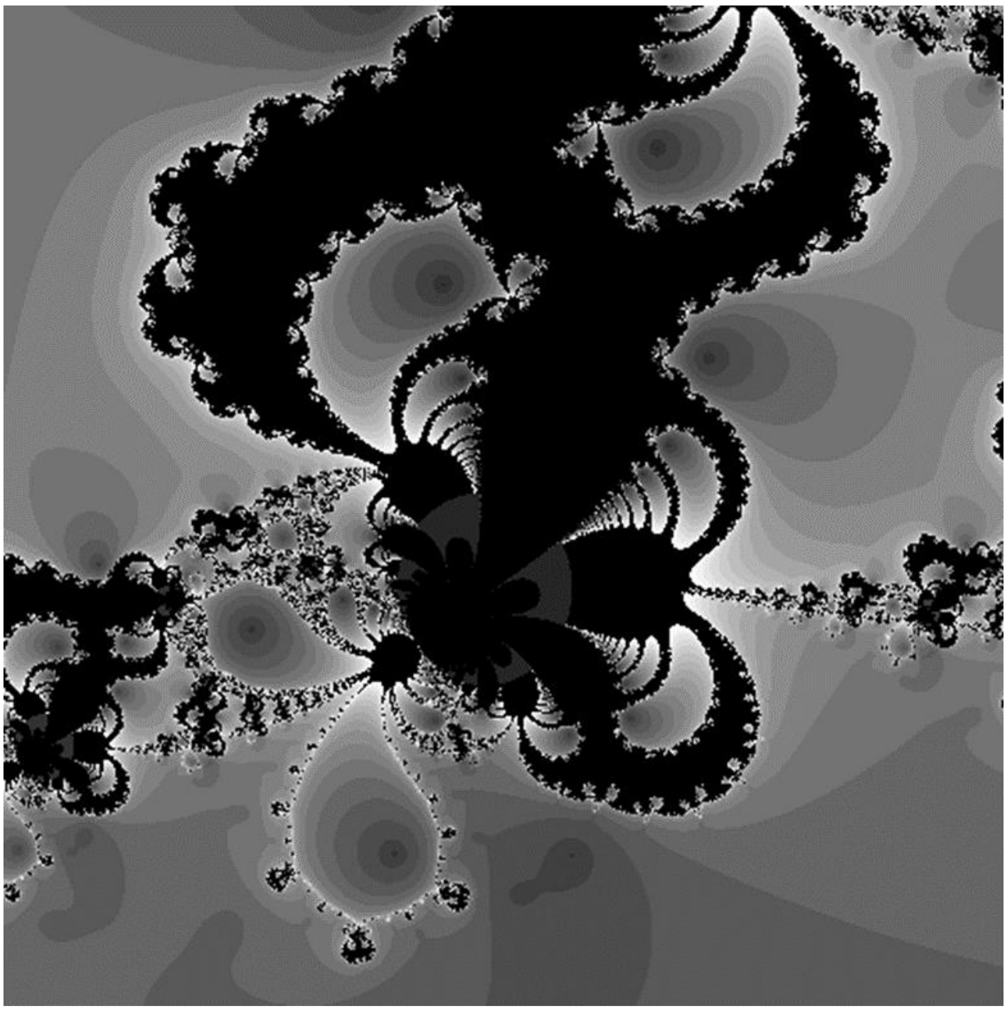

Figure 11.

$$\gamma = H_n - \ln n - \int_0^\infty \left(e^{-n\, e^{-H_n x}} - \frac{1}{1+x} \right) \frac{1}{x}\, dx \quad , n \in \mathbb{N} \tag{48}$$

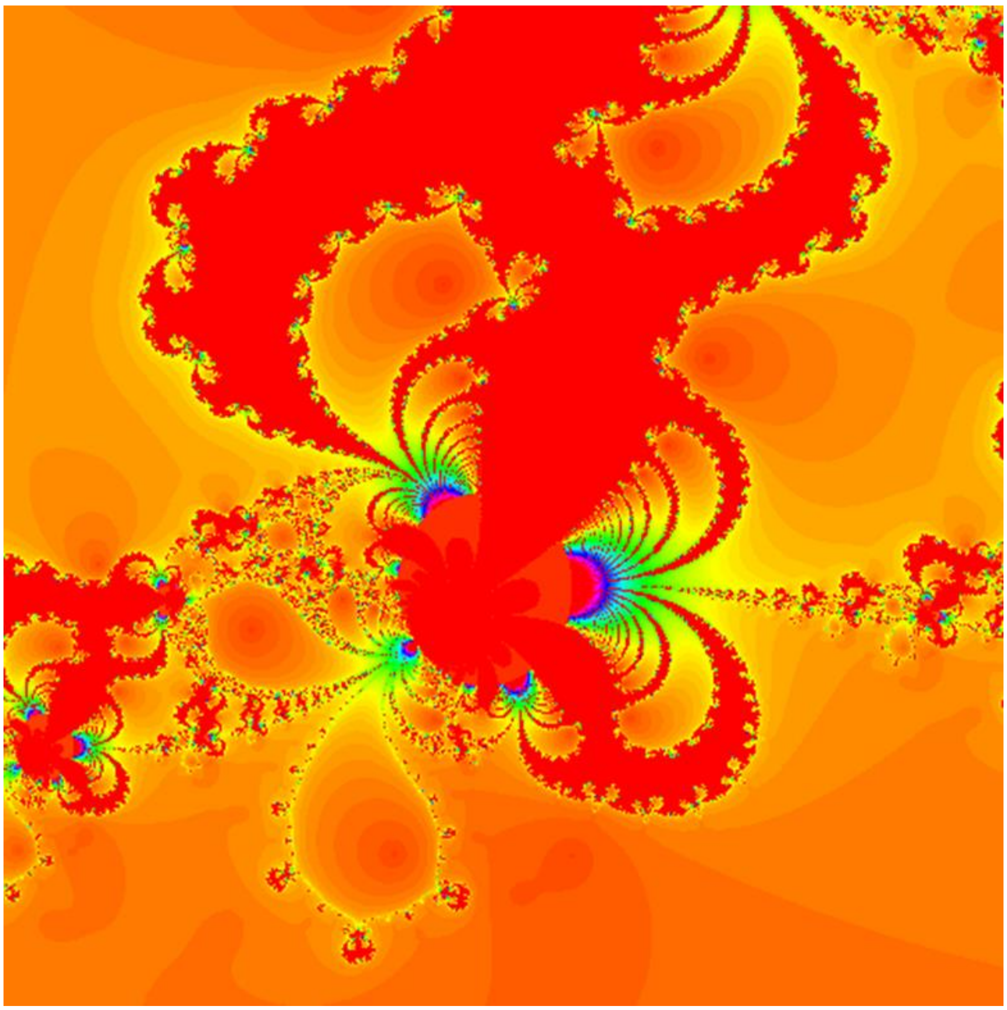

Figure 12.

$$\gamma = H_n - \ln n - \frac{2}{\pi}\int_0^\infty \frac{\ln x}{x}\sin\left(ne^{-H_n}x\right)dx \quad , n\in\mathbb{N} \tag{49}$$

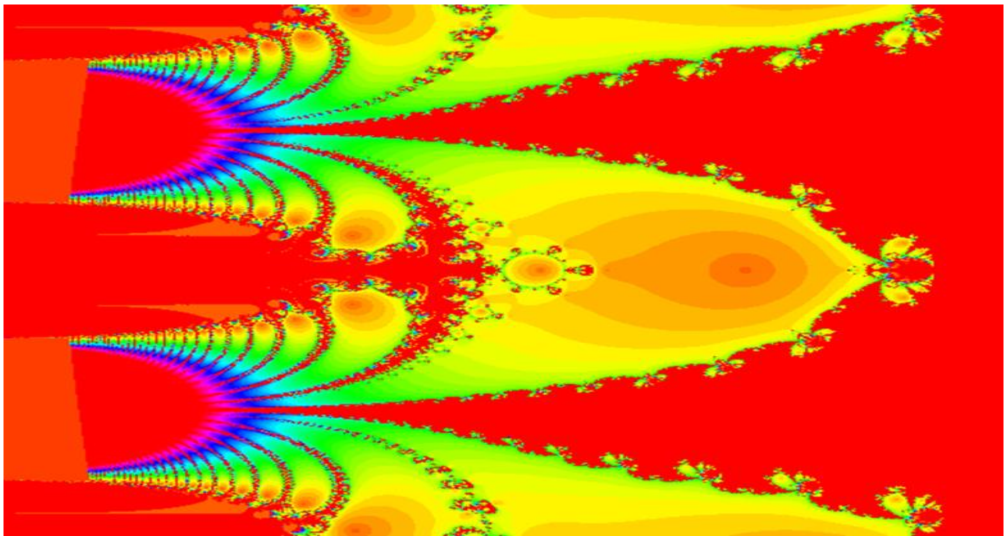

Figure 13.

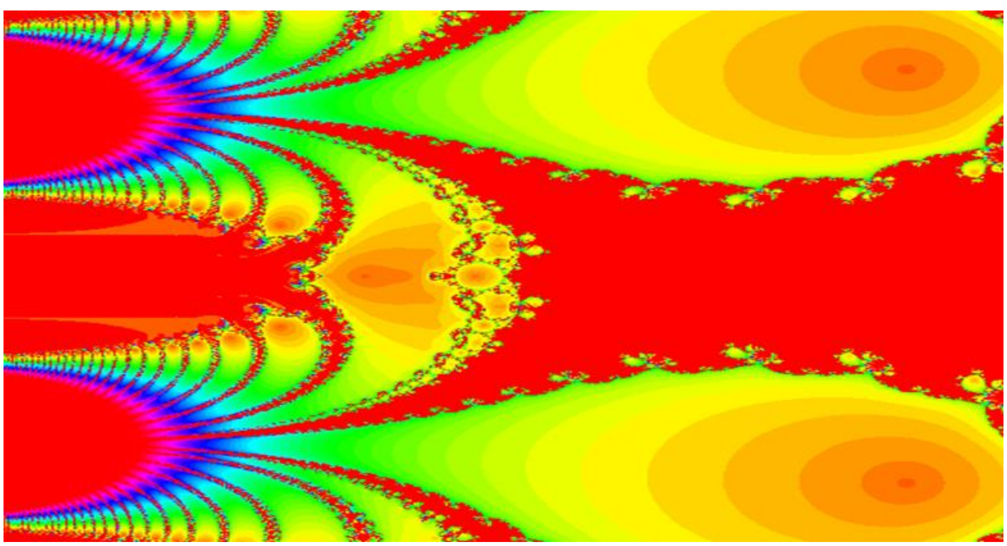

Figure 14.

$$\gamma = H_n - \ln n + Ci\left(ne^{-H_n}\right) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\left(ne^{-H_n}\right)^{2k}}{(2k)!(2k)} \quad , n \in \mathbb{N} \tag{50}$$

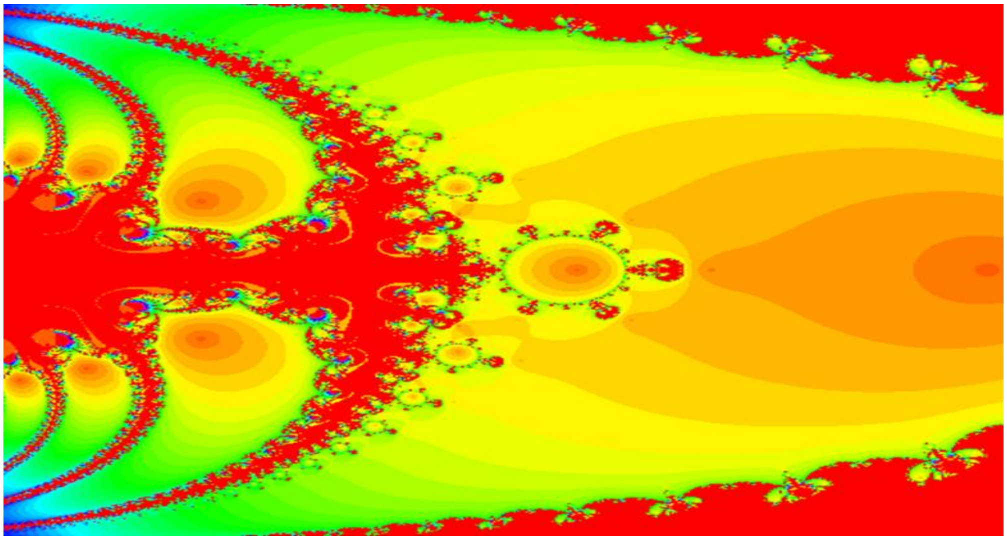

Figure 15.

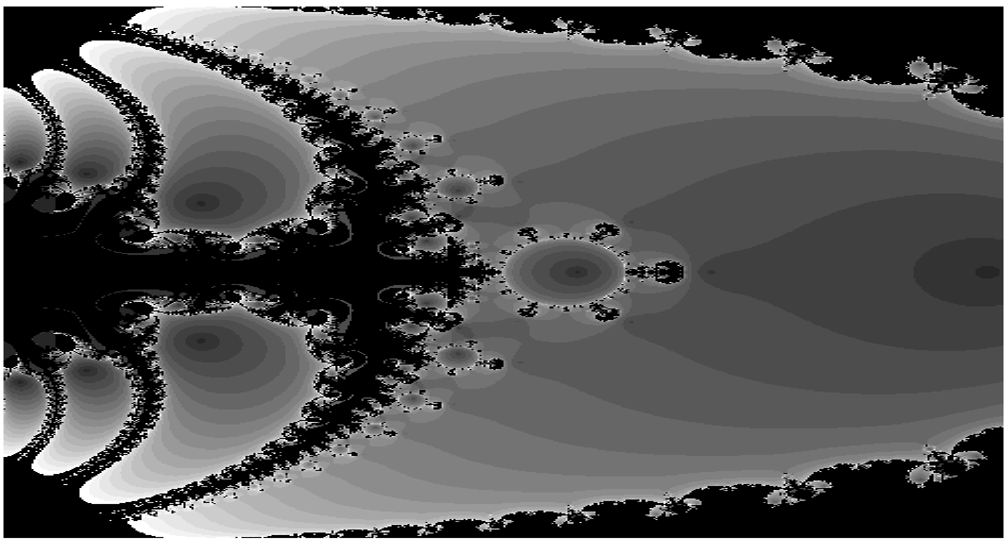

Figure 16.

$$\gamma = 1 - \int_0^1 \frac{1}{1+x} \left(\sum_{k=1}^{\infty} x^{2^k} \right) dx \quad , Catalan \tag{51}$$

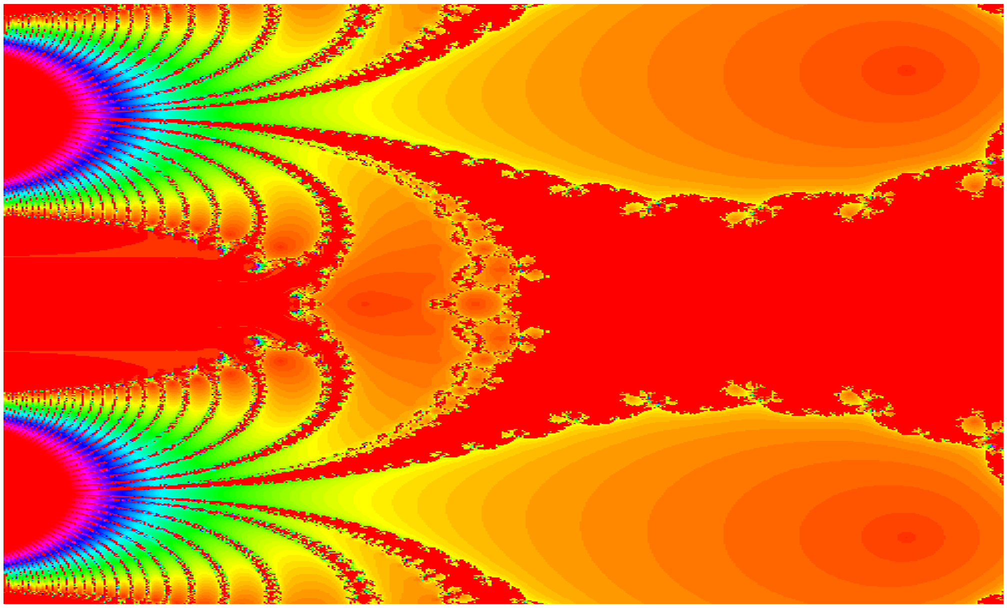

Figure 17.

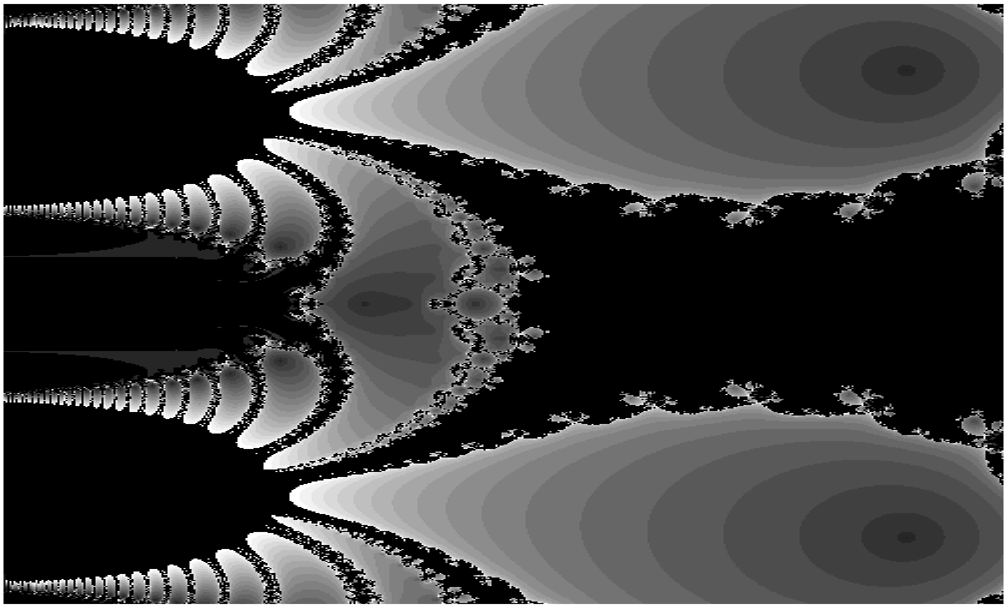

Figure 18.

$$\gamma = 1 - \int_0^1 \frac{1+2x}{1+x+x^2}\left(\sum_{k=1}^{\infty} x^{3^k}\right)dx \quad , Ramanujan \tag{52}$$

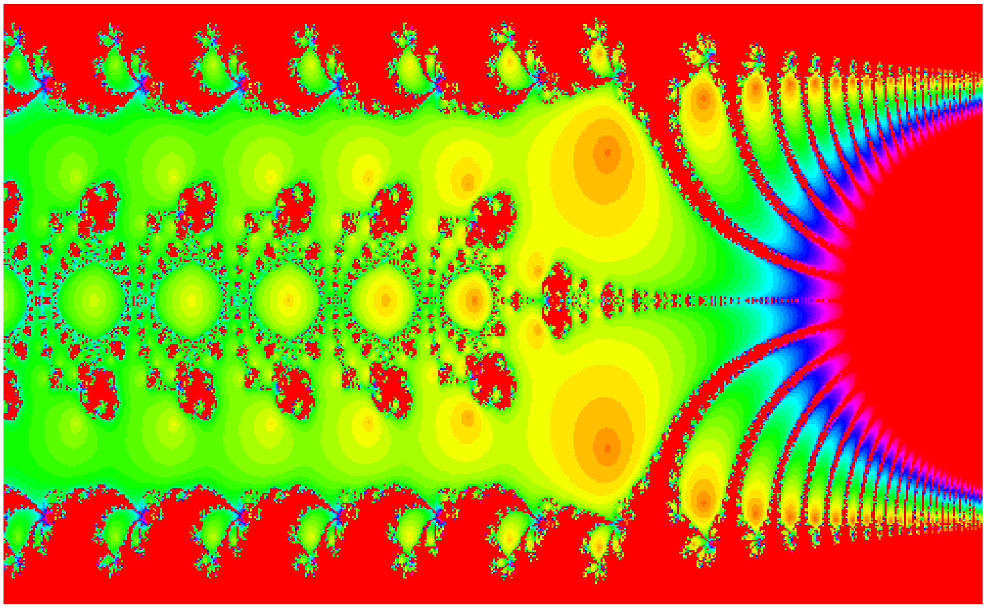

Figure 19.

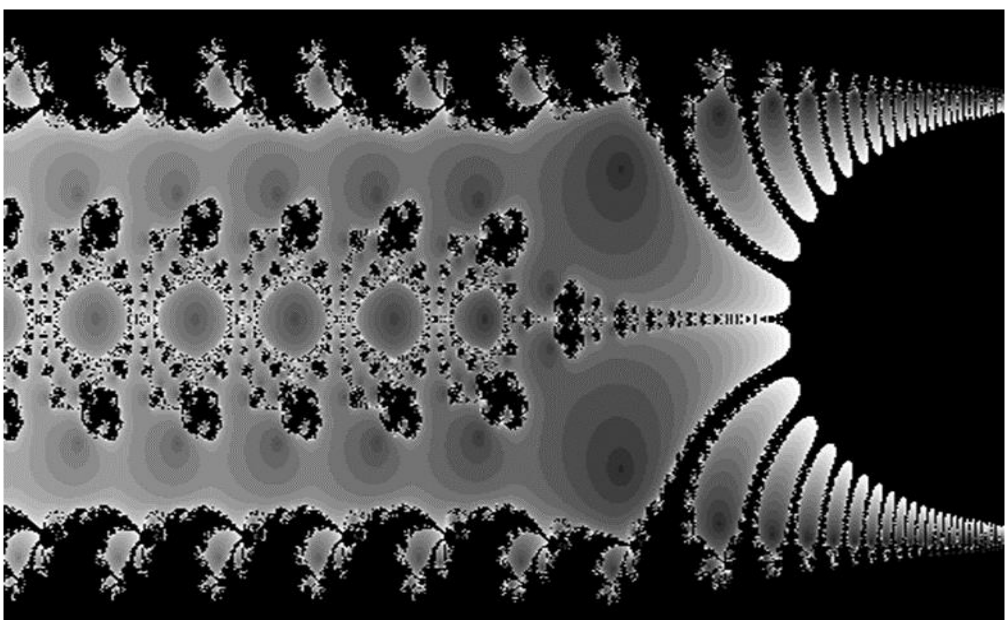

Figure 20.

$$\gamma = -\int_0^1\int_0^1 \frac{1-x}{(1-xy)\ln(xy)}dxdy \quad , Sondow \tag{53}$$

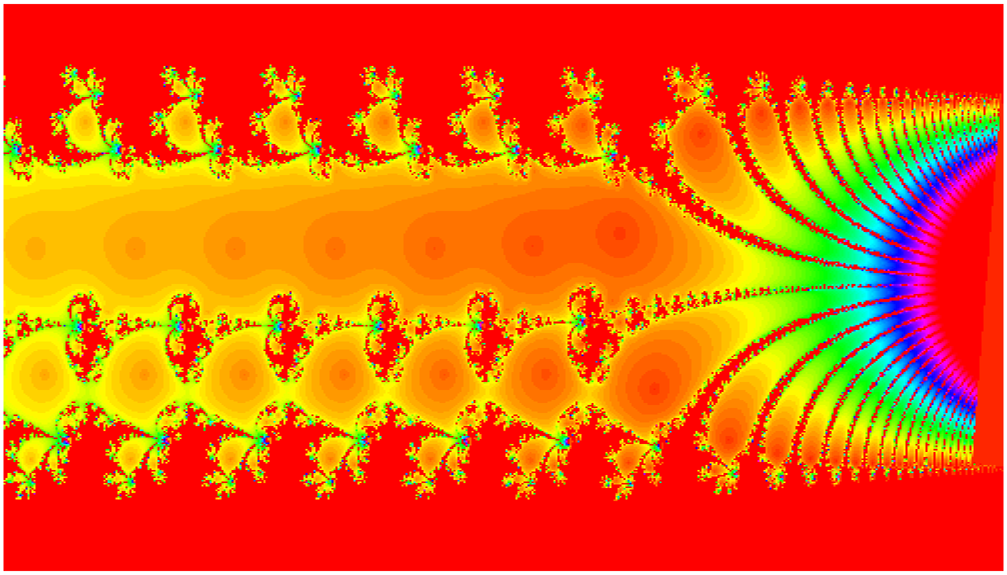

Figure 21.

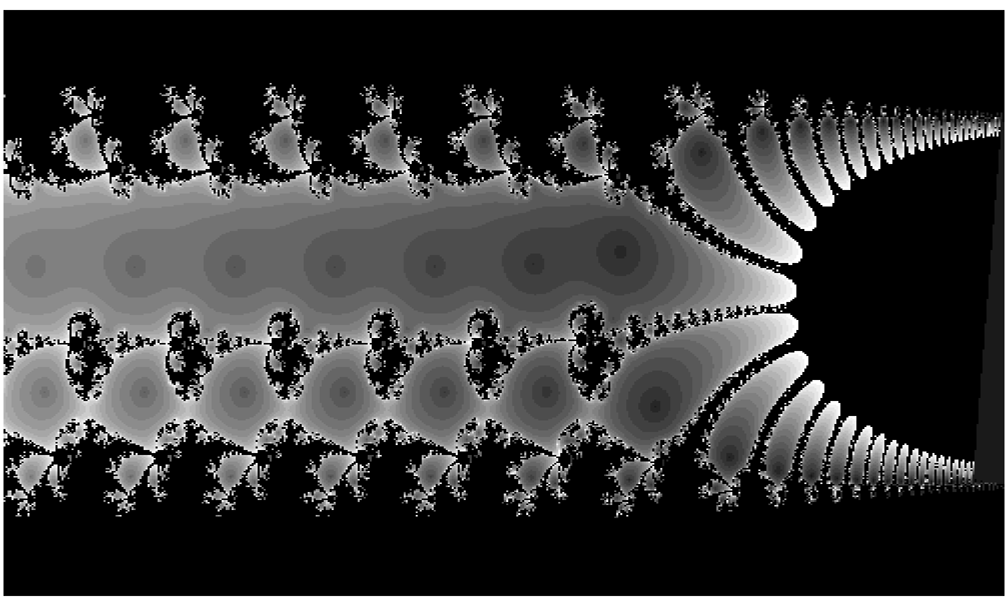

Figure 22.

$$\gamma = \int_0^1 \int_0^1 \frac{\ln\left(-\ln\left(xy\right)\right)}{\ln\left(xy\right)} dxdy \tag{54}$$

$$\gamma = -\int_0^\infty \int_0^\infty \frac{\ln\left(x+y\right)}{x+y} e^{-x-y} dxdy \tag{55}$$

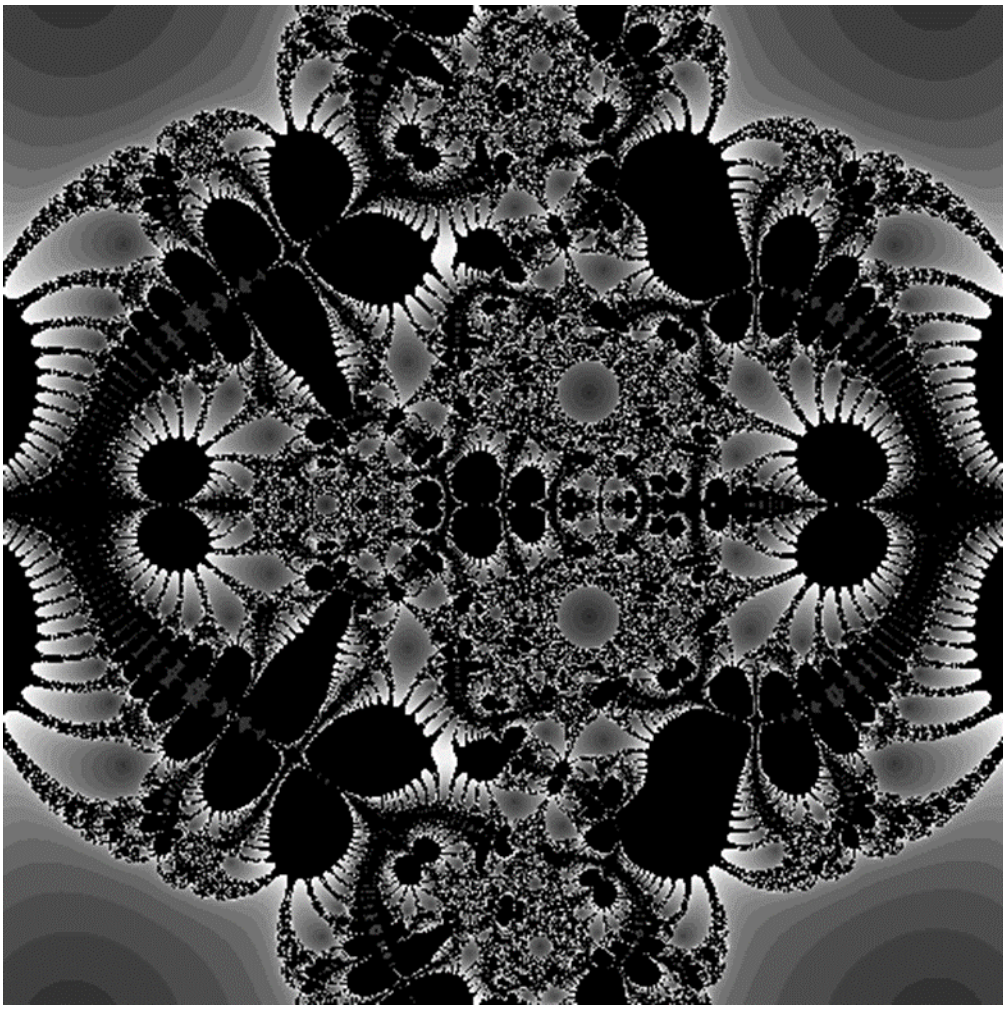

Figure 23.

Remarks:

$$Ei(1,x)=\int_x^{\infty}\frac{e^{-t}}{t}dt \quad ,x>0 \tag{56}$$

$$Ci(x)=-\int_x^{\infty}\frac{\cos t}{t}dt \quad ,x>0 \tag{57}$$

$$\Gamma(x)=\int_0^{\infty}t^{x-1}e^{-t}dt \quad ,x>0 \tag{58}$$

$$\Gamma(x+1)=x\Gamma(x) \tag{59}$$

References

1. Abramowitz, M. e I.A. Stegun, Handbook of Mathematical Functions, Nueva York: Dover , 1965.
2. Erdélyi, A., W. Magnus, F. Oberhettinger y F.G. Tricomi, Higher Transcendeltal Functions, 3 vols. Nueva York: McGraw-Hill, 1953, 1955.
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