Derivation of the Planck Force from Heisenberg Uncertainty Relations

In this paper I derive the expression for the Planck force from the Heisenberg uncertainty relations.

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1. Derivation of the Planck Force

Let us consider a real particle that moves along the *x* axis of certain reference system *S* under the action of a net force *F*. Let us suppose that at certain time, *t*, the particle has an uncertainty in momentum along the *x* axis of Δp_x , an uncertainty in position of Δx , an uncertainty in energy of ΔE and an uncertainty in time of Δt (we shall use ΔE and Δt later in this section).Then according to Heisenberg, the relation between Δp_x and Δx is given by

Momentum-position uncertainty relation in the x direction

$$\Delta p_x \, \Delta x \geq \frac{\hbar}{2} \tag{1.1}$$

Let's also assume that the speed of the particle is much less than the speed of light. This is

$$v \ll c \tag{1.2}$$

Because of the definition of momentum we may write

$$m \, \Delta v_x \, \Delta x \geq \frac{\hbar}{2} \tag{1.3}$$

Where Δv_x is the uncertainty in the velocity of the particle. Because of the uncertainty in velocity, this velocity could be anywhere, let's say, between v_{x1} and v_{x2} . This means that

$$\Delta v_x = v_{x2} - v_{x1} \tag{1.4}$$

According to Heisenberg, the uncertainty, Δv_x , in the velocity of the particle yields and uncertainty, ΔE , in its energy. Let us assume that Δt is the uncertainty in the time for which the particle has an uncertainty in energy of ΔE . Thus, according to Heisenberg ΔE and Δt are related by

Energy-time uncertainty relation

$$\Delta E \, \Delta t \geq \frac{\hbar}{2} \tag{1.5}$$

The time for which the particle has an energy equal to *E* (with an uncertainty in energy of ΔE) could be anywhere between, let's say, t_1 and t_2 , so that we write

$$\Delta t = t_2 - t_1 \tag{1.6}$$

Because we have both an uncertainty in the velocity of the particle and an uncertainty in the time for which this uncertainty occurs, we must have an uncertainty, Δa_x , in the acceleration of the particle (we assume that there is a net force, *F,* acting on the particle). Therefore, the uncertainty in acceleration is given by

$$\Delta a_x = \frac{v_{x2} - v_{x1}}{t_2 - t_1} = \frac{\Delta v_x}{\Delta t} \tag{1.7}$$

Now let us divide eq. (1.3) by Δt . This yields

$$m \frac{\Delta v_x}{\Delta t} \Delta x \geq \frac{\hbar}{2} \frac{1}{\Delta t} \tag{1.8}$$

According to eq. (1.7) we may write

$$m \, \Delta a_x \, \Delta x \geq \frac{\hbar}{2} \frac{1}{\Delta t} \tag{1.9}$$

But, according to Newton's second law of motion, the quantity $m \, \Delta a_x$ has to be the uncertainty, ΔF , in the net force, *F*, that is exerted on the particle. This lead us to write

$$\Delta F \, \Delta x \geq \frac{\hbar}{2} \frac{1}{\Delta t} \tag{1.10}$$

But the first side of this inequation is the product of the uncertainty in force times the uncertainty in position: $\Delta F \, \Delta x$. This must represent the uncertainty in the energy of the particle, ΔE , as given by relation (1.5). Now one may ask: what is the maximum uncertainty in force?. To answer this question let us write the above inequation as follows

$$\Delta F \geq \frac{\hbar}{2\, \Delta x\, \Delta t} \tag{1.11}$$

The uncertainty in force will be maximum when both, the uncertainty in position and the unceratinty in time are minimum. So if we denote Δx_{min} the minimum uncertainty in position and Δt_{min} the minimum uncertainty in time, we may write the maximum uncertainty in force, ΔF_{max} , as follows

$$\Delta F_{max} \geq \frac{\hbar}{2\, \Delta x_{min}\, \Delta t_{min}} \tag{1.12}$$

Now let us assume that the minimum length with physical meaning is the Planck length, L_P . This length is defined as

Planck length

$$L_P = \sqrt{\frac{hG}{2\pi c^3}} \tag{1.13}$$

And let us also assume that the minimum time with physical meaning is the Planck time, T_P . This time is defined as

Planck time

$$T_P = \sqrt{\frac{hG}{2\pi c^5}} \tag{1.14}$$

Thus, replacing the values of Δx_{min} and Δt_{min} by L_P and T_P , respectively, we get

$$\Delta F_{max} \geq \frac{\hbar}{2\sqrt{\frac{hG}{2\pi c^3}}\sqrt{\frac{hG}{2\pi c^5}}} \tag{1.15}$$

Which turns out to be

$$\Delta F_{max} \geq \frac{c^4}{2G} \tag{1.16}$$

Then, to make the definition of the Planck force compatible with previous definitions based on (a) Newton's second law of motion and (b) Coulomb's law, we are compelled to define the Planck force as

Planck force

$$F_P \equiv \frac{c^4}{G} \tag{1.17}$$

Thus, the maximum uncertainty in force is

$$\Delta F_{max} \geq \frac{F_P}{2} \tag{1.18}$$

The minimum value of this uncertainty occurs for the equal sign

minimum value of the uncertainty in force

$$\left(\Delta F_{max}\right)_{\text{min\_unceratainty}} = \frac{F_P}{2} \tag{1.19}$$

thus, the minimum uncertainty in the maximum force is equal to the Planck force divided by 2. Hence, according to Heisenberg, the Planck force is twice the minimum uncertainty for the maximum uncertainty in force.

$$F_P = 2\left(\Delta F_{max}\right)_{\text{minimum\_unceratainty}} \tag{1.20}$$

2. Conclusions

I have presented, in the foregoing pages, a derivation of the Planck force based on the Heisenberg uncertainty relations:

Momentum-position uncertainty relation in the x direction

$$\Delta p_x \, \Delta x \geq \frac{\hbar}{2} \qquad (2.1) = (1.1)$$

and

Energy-time uncertainty relation

$$\Delta E \, \Delta t \geq \frac{\hbar}{2} \qquad (2.2) = (1.5)$$

The conclusions that we extract from the previous mathematical approach are

(a) the definition of the Planck force:

Planck force

$$F_P \equiv \frac{c^4}{G} \qquad (2.3) = (1.17)$$

(b) the interpretation of the Planck force as a force that is twice the minimum uncertainty for the maximum uncertainty in force:

$$F_P = 2\left(\Delta F_{max}\right)_{\text{minimum\_unceratainty}} \qquad (2.4) = (1.20)$$

The Plank force is an extremely important composite constant (or unit if you like) because allow us to derive Newton's law of universal gravitation from first principles [1] and because it has been silently incorporated by Einstein in his general relativity's field equations [1].

REFERENCES

[1] R. A. Frino, *Why Does Gravity Obey an Inverse Square Law?*, viXra: 1704.0052, (2017).