Coordinate free non abelian geometry I: the quantum case of simplicial manifolds.

Johan Noldus*

May 6, 2017

Abstract

We study the geometry of a simplicial complexes from an algebraic point of view and devise general quantization rules; the rules emerging in spin foam theory are shown to comprise a particular subcase.

1 Introduction.

It is currently unknown what a quantum geometry really is starting from a classical one. The answer relies, as we shall see, upon the concept of algebrization as is usually the case and we indicate how spin foam theory fits nicely into this picture. Ours is however much larger as different rules are allowed for regarding lower dimensional objects. The simplicity of the construction invites for further investigations.

2 Classical configuration space $\mathcal{C}$ and determination of the phase space of gravity.

We shall work out in full detail the configuration space and phase space of Euclidean and Lorentzian simplicial geometry; those are generated by simplices $(v_1, \ldots, v_k, A_k)$ where A_k is a symmetric matrix of Euclidean or Lorentzian signature meaning the points are embedded in a Euclidean or Lorentzian space, supplemented with a time orientation τ_k in the Lorentzian case on the timelike edges so that a partial ordering exists locally on the simplex. The field for the matrixalgebra chosen is the one of the real numbers. One can define the configuration space of one single, $n-1$ dimensional universe, as a subspace of the linear space

$$\mathcal{V}^n = \{\sum_{\alpha} r_\alpha(v_1, \ldots, v_k, A_k, \tau_k) | k \in \mathbb{N}_0\}$$

where r_α is part of a discrete commutative ring or field which is a subset of the integer numbers, determining the statistics of the geometric building blocks. More precisely, $\mathcal{C}^n$ is the subspace where each $(v_1, \ldots, v_k)$ appears only once or not and

*email: johan.noldus@gmail.com, Relativity group, departement of mathematical analysis, University of Gent, Belgium.

moreover, if $(v_1, \ldots, v_k)$ and $(w_1, \ldots, w_l)$ share a subsimplex $(u_1, \ldots, u_r)$ then both matrices A_k and B_l as well as possibly the orientations need to coincide on these subsets. The plus operatation can be interpreted as gluing (possibly with opposite orientation), in case the coefficient ring $\mathcal{R}$ equals $\mathbb{Z}_2$ then each simplex equals its inverse and the translation or gluing

$$a_{(v_1,\ldots,v_k,A_k,\tau_k)} : \mathcal{V}^n \to \mathcal{V}^n : z \to z + (v_1, \ldots, v_k, A_k, \tau_k)$$

with $k \leq n$ satisfies $(a_{(v_1,\ldots,v_k,A_k,\tau_k)})^2 = 0$ giving rise to Fermionic statistics whereas in case $\mathcal{R} = \mathbb{Z}$ the statistics is Bose; everything in between gives rise to parastatistics.

There is some interesting algebra to be performed here, clearly, the rules for the *topological* complex $(v_1, \ldots, v_k)$ are antisymmetric giving rise to a wedge product of the point space $\mathcal{V}^1$ where the point measures have been erased. In either $(v_1, \ldots, v_k) \in \mathcal{V}^1 \wedge \ldots \wedge \mathcal{V}^1$ where the $\wedge$ product has been taken $k-1$ times. Therefore, in general, we can write

$$(v_1, \ldots, v_k, A_k, \tau_k).(w_1 \ldots, w_l, B_l, \sigma_l) =$$

$$\sum_{i_1 < \ldots < i_s; j_1 < \ldots < j_r; s \leq k, r \leq l} \text{Sign shift } G_{k-s,l-r}(v_i, w_h; i \neq i_p, j \neq j_q; p : 1 \ldots s, q : 1 \ldots r, A_k, B_l) b_c \alpha^c$$

$$(v_{i_1}, \ldots, v_{i_s}, w_{j_1}, \ldots w_{j_r}, (A_k \cup B_l)_{i_1 \ldots i_s j_1 \ldots j_r}, (\tau_k \cup \sigma_l)_{i_1 \ldots i_s j_1 \ldots j_r})$$

where identical points are identified as one using the appropriate antisymmetrical rules and $c = |\{v_{i_p}; p : 1 \ldots s\} \cap \{w_{j_q}; q : 1 \ldots r\}|$. The number α has dimension of length and b_c is a dimensionless constant; finally, it is assumed that A_k and B_k are the same on the common vertices $\{v_{i_p}; p : 1 \ldots s\} \cap \{w_{j_q}; q : 1 \ldots r\}$ and $(A_k \cup B_l)_{i_1 \ldots i_s j_1 \ldots j_r}$ is the zero extension of A_k, B_l to $\{i_1 \ldots i_s j_1 \ldots j_r\}$ where the new cross matrix elements are all put to zero. A similar definition holds for $(\tau_k \cup \sigma_l)_{i_1 \ldots i_s j_1 \ldots j_r}$ which is the transitive closure and zero extension of both. $G_{t,r}$ is a geometrical invariant of the t and r vector arguments defined by means of the matrices A_k, B_l. One can make this definition continuous or smooth be relaxing the definition of $A_k \cup B_l$ on the common vertices whereas now a total match of the weights is assumed. The general product between elements of $\mathcal{V}^n$ is henceforth defined by linearity from the above. The "Sign shift" is a function of the permutation needed to bring (v_i, w_j) to $(v_i, w_h; i \neq i_p, j \neq j_q; p : 1 \ldots s, q : 1 \ldots r)(v_{i_1}, \ldots, v_{i_s}, w_{j_1}, \ldots w_{j_r})$ and ignoring it results in a mixed Bose-Fermi rule. This product is in general not associative as the $G_{t,r}$ functions have to satisfy quadratic identities. Moreover, these functions are assumed to depend upon relative properties $w_i - w_j$ of their arguments w_i, w_j so that no masses enter the product formula.

The latter formula is a very general one and to get down to the quantization rules akin of spin foam, we need further restrictions originating from the fact that the $n-1$ simplices are embedded in n dimensional Euclidean or Minkowski spacetime. In particular, this gives us the Hodge dual which we will use now. As an example, let us treat the Euclidean case of a three dimensional simplex; its boundary consists out of four three simplices $(v_1 v_2 v_3, A_3)$, each spanning a two dimensional surface given by $(v_1 v_2) \wedge (v_1 v_3)$ and surface given by the absolute value of

$$\frac{1}{2}\mathrm{Det}\left(\begin{array}{cc} -2a_{12}+a_{11}+a_{22} & a_{23}+a_{11}-a_{13}-a_{21} \\ a_{23}+a_{11}-a_{13}-a_{21} & -2a_{13}+a_{11}+a_{33} \end{array}\right)$$

which is nothing but the length of the Hodge dual vector

$$\frac{1}{2}\epsilon^{i}{}_{jk}(v_2^j - v_1^j)(v_3^k - v_1^k) \equiv \frac{1}{2}\epsilon(\cdot, v_2 - v_1, v_3 - v_1).$$

The algebra of adjacent edges $(v_1 v_2)$ and $(v_1 v_3)$ then becomes

$$(v_1v_2).(v_1v_3) = G_{2,2}(v_1,v_2;v_1,v_3)1 - \alpha G_{0,0}(v_1v_2v_3) - \beta b_{13}(v_1v_2) + \beta b_{21}(v_1v_3) + \text{contributions over points}$$

where $G_{1,1}(v_i;v_j) = (v_j - v_i)^2 = a_{jj} + a_{ii} - 2a_{ij} := b_{ij}$ and β is dimensionless. One can argue that the contributions over the points should vanish as they are given by $(v_1)(G_{1,2}(v_2,v_1v_3) + G_{1,1}(v_1v_2,v_3)) + (v_2)G_{1,2}(v_1,v_1v_3) + G_{2,1}(v_1v_2,v_1)(v_3)$ where we have ignored the "Sign" factors and logically $G_{1,2}(v_1,v_1v_3) = G_{2,1}(v_1v_2,v_1) = 0$ and we assume $G_{1,2}(v_2,v_1v_3) + G_{1,1}(v_1v_2,v_3)$ to vanish. Finally, using symmetry assumptions, we can posit on dimensional grounds that

$$G_{2,2}(v_1,v_2;v_1,v_3) = \delta\alpha^2(b_{12} + b_{13} \pm b_{23}) + \kappa b_{12}b_{13}.$$

Therefore we have that

$$(v_1v_2).(v_1v_3) = (\delta\alpha^2(b_{12}+b_{13}\pm b_{23}) + \kappa b_{12}b_{13})1 - \beta b_{13}(v_1v_2) + \beta b_{21}(v_1v_3) - \alpha\zeta(v_1v_2v_3)$$

and we now make the following linear extension

$$\vec{v}.\vec{w} = \delta\alpha^2\vec{v}\cdot\vec{w}1 - \alpha\zeta\frac{1}{2}\epsilon(\cdot,\vec{v},\vec{w})$$

where we have only retained the terms linear in $\vec{v},\vec{w}$ as it should for a real product of free vectors gripping in the same point. Similarly, for faces

$$(v_1v_2v_3).(v_1v_2v_4) = -\gamma\alpha b_{14}(v_1v_2v_3) + \gamma\alpha b_{13}(v_1v_2v_4) + \gamma\alpha b_{24}(v_1v_2v_3) - \gamma\alpha b_{23}(v_1v_3v_4) +$$
$$\text{contributions over lines and points} + G_{3,3}(v_1v_2v_3;v_1v_2v_4)1.$$

Replacing planes $(v_1v_2v_3)$ by bivectors $(v_2 - v_1)\wedge(v_3 - v_1)$ and demanding linearity when regarding them as free vectors results in a formula which is Hodge dual to the one for the perpendicular vectors $\frac{1}{2}\epsilon(\cdot, v_2 - v_1, v_3 - v_1)$. Hence, we are left with *one* algebra for free edges, which can be mapped by means of Hodge duality to the one for the planes which is

$$\vec{v}.\vec{w} = \delta\alpha^2\vec{v}\cdot\vec{w}1 - \alpha\zeta\frac{1}{2}\epsilon(\cdot,\vec{v},\vec{w}).$$

All this can be thought of as classical considerations; the quantal aspect emerges through "loss of information", that is concentating upon the Lie brackets

$$[\vec{v},\vec{w}] = \vec{v}.\vec{w} - \vec{w}.\vec{v} = \alpha\zeta\frac{1}{2}(\epsilon(\cdot,\vec{w},\vec{v}) - \epsilon(\cdot,\vec{v},\vec{w}))$$

instead of on the products. We shall construct the canonical momentum in a while though. Therefore, denoting by $\vec{L} = (L_x, L_y, L_z)$ the normal vector associated to a plane, we have that

$$\vec{L}.\vec{L} = \delta\text{Area face}.1 + \frac{1}{2}\alpha^2\zeta\vec{L}\times\vec{L} = \delta\text{Area face}.1 + \frac{1}{2}l_p^2\zeta\vec{L}$$

where the last equality comes from the fact that the "vectorvalued product" generated by $\vec{L}$ with itself must be proportional to itself. Demanding, moreover, that all vectors $\vec{v}$ must be self adjoint $v_i^\dagger = v_i$ leads to $\vec{L} = \frac{l_p^2}{2}\vec{\sigma}$ and

$$\vec{L}.\vec{L} = \frac{3}{4}l_p^4 + \frac{1}{2}il_p^2\vec{L}.$$

Therefore, we have that

$$\left[\vec{L}_j, \vec{L}_k\right] = il_p^2 \epsilon_{jk}{}^{l} L_l$$

which is nothing but the $SU(2)$ Lie algebra and our quantal starting point.

The reader notices that the three dimensional case is rather exceptional given that a bivector can be seen as a vector by means of Hodge duality making use of the element $\sigma_1\sigma_2\sigma_3$. This is not the case in four dimensions and therefore, the entire real Clifford algebra $\mathrm{Cl}(1,3)$ needs to be taken into account, going beyond the Lie algebra of the Lorentz group $SO(1,3)$. This point seems to have been missed by many researchers.

3 The boundary operator and conjugate momentum.

As is usual, the boundary operator is defined by means of its action on a simplex $(v_0 v_1 \ldots v_n)$; it is given by

$$\partial(v_0 v_1 \ldots v_n) = \sum_{i=0}^{n}(-1)^i (v_0 \ldots v_{i-1} v_{i+1} \ldots v_n)$$

and the reader extends this definition by linearity to $\mathcal{V}^n$. Obviously $\partial^2 = 0$ which is the defining property of homology theory. Given that in the framework above, no coordinates exist, and given that the entire edifice of Heisenbergian quantum theory is grounded in the language of symplectic manifolds, there is no obvious substitute for the canonical commutation relations. Also, in the above $SU(2)$ non-abelian geometry, the "inertial" coordinates of the surface vector of the boundary triangle of a quantal tetrahedron did not commute. Hodge duality therefore subtely mixes "traditional" canonical and conjugate coordinates. There are a few other operators which are worthwhile studying and which reproduce a kind of Heisenberg commutation relations taken together with the configuration space variables. One of them is the boundary operator ∂, the others constitute refinements of the operator

$$\delta z = \{\text{sum of all simplices in z whose vertices are at least one edge removed from } \partial z\}$$

which effectively removes one "top" slice from the simplicial complex. For example, define the multiplication operator x_w on $\mathcal{V}^n$ as sending every simplex $(v_0 v_1 \ldots v_n)$ to $(w v_0 v_1 \ldots v_n)$ linearly extended to sums and ∂_w as the operator removing the w vertex. Given now any simplicial complex z which does not contain the w vertex, then

$$\partial_w x_w(z) - x_w \partial_w(z) = z$$

which is the necessary Heisenberg relation. One can extend this formalism to multiple vertices $w_1, \ldots, w_k$ given that the operator x_w is a projection $x_w^2 = x_w$ in contrast to ordinary real numbers.

This constitutes a rather important distinction with usual quantization procedures in the sense that there fundamental multiplication operators are $\mathbb{R}$ valued. Those operators can be retrieved from ours by means of identifying x_w for *different* w which is a method for ressurecting the continuum by means of considering infinite "time-dimensions". The reader may enjoy defining different "multiplication operators" canonically conjugate to δ_α. This is sufficient to construct a theory of quantum gravity which shall necessarily contain a homological flavour due to the presence of the boundary operator.