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The category of neutrosophic sets

Kul Hur [1], Pyung Ki Lim [2], Jeong Gon Lee [3], Junhui Kim [4,*]

[1]Division of Mathematics and Informational Statistics, Institute of Basic Natural Science, Wonkwang University 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea. E-mail: kulhur@wku.ac.kr

[2]Division of Mathematics and Informational Statistics, Institute of Basic Natural Science, Wonkwang University 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea. E-mail: pklim@wku.ac.kr

[3]Division of Mathematics and Informational Statistics, Institute of Basic Natural Science, Wonkwang University 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea. E-mail: jukolee@wku.ac.kr

[4]Department of Mathematics Education, Wonkwang University 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea. E-mail: junhikim@wku.ac.kr

* Corresponding author

Abstract: We introduce the category $\mathbf{NSet(H)}$ consisting of neutrosophic H-sets and morphisms between them. And we study $\mathbf{NSet(H)}$ in the sense of a topological universe and prove that it is Cartesian closed over $\mathbf{Set}$, where $\mathbf{Set}$ denotes the category consisting of ordinary sets and ordinary mappings between them. Furthermore, we investigate some relationships between two categories $\mathbf{ISet(H)}$ and $\mathbf{NSet(H)}$.

Keywords: Neutrosophic crisp set, Cartesian closed category, Topological universe.

1 Introduction

In 1965, Zadeh [20] had introduced a concept of a fuzzy set as the generalization of a crisp set. In 1986, Atanassov [1] proposed the notion of intuitionistic fuzzy set as the generalization of fuzzy sets considering the degree of membership and non-membership. Moreover, in 1998, Smarandache [19] introduced the concept of a neutrosophic set considering the degree of membership, the degree of indeterminacy and the degree of non-membership.

After that time, many researchers [3, 4, 5, 6, 8, 9, 13, 15, 16, 17] have investigated fuzzy sets in the sense of category theory, for instance, $\mathbf{Set(H)}$, $\mathbf{Set_f(H)}$, $\mathbf{Set_g(H)}$, $\mathbf{Fuz(H)}$. Among them, the category $\mathbf{Set(H)}$ is the most useful one as the "standard" category, because $\mathbf{Set(H)}$ is very suitable for describing fuzzy sets and mappings between them. In particular, Carrega [3], Dubuc [4], Eytan [5], Goguen [6], Pittes [15], Ponasse [16, 17] had studied $\mathbf{Set(H)}$ in topos view-point. However Hur et al. investigated $\mathbf{Set(H)}$ in topological view-point. Moreover, Hur et al. [9] introduced the category $\mathbf{ISet(H)}$ consisting of intuitionistic H-fuzzy sets and morphisms between them, and studied $\mathbf{ISet(H)}$ in the sense of topological universe. In particular, Lim et al. [13] introduced the new category $\mathbf{VSet(H)}$ and investigated it in the sense of topological universe. Recently, Lee et al. [10] define the category composed of neutrosophic crisp sets and morphisms between neutrosophic crisp sets and study its some properties.

The concept of a topological universe was introduced by Nel [14], which implies a Cartesian closed category and a concrete quasitopos. Furthermore the concept has already been up to effective use for several areas of mathematics.

In this paper, we introduce the category $\mathbf{NSet(H)}$ consisting of neutrosophic H-sets and morphisms between them. And we study $\mathbf{NSet(H)}$ in the sense of a topological universe and prove that it is Cartesian closed over $\mathbf{Set}$, where $\mathbf{Set}$ denotes the category consisting of ordinary sets and ordinary mappings between them. Furthermore, we investigate some relationships between two categories $\mathbf{ISet(H)}$ and $\mathbf{NSet(H)}$.

2 Preliminaries

In this section, we list some basic definitions and well-known results from [7, 12, 14] which are needed in the next sections.

Definition 2.1 [12] Let $\mathbf{A}$ be a concrete category and $((Y_j, \xi_j))_J$ a family of objects in A indexed by a class J. For any set X, let $(f_j : X \to Y_j)_J$ be a source of mappings indexed by J. Then an $\mathbf{A}$-structure ξ on X is said to be initial with respect to (in short, w.r.t.) $(X, (f_j), ((Y_j, \xi_j)))_J$, if it satisfies the following conditions:

(i) for each $j \in J$, $f_j : (X, \xi) \to (Y_j, \xi_j)$ is an $\mathbf{A}$-morphism,

(ii) if (Z, ρ) is an $\mathbf{A}$-object and $g : Z \to X$ is a mapping such that for each $j \in J$, the mapping $f_j \circ g : (Z, \rho) \to (Y_j, \xi_j)$ is an $\mathbf{A}$-morphism, then $g : (Z, \rho) \to (X, \xi)$ is an $\mathbf{A}$-morphism.

In this case, $(f_j : (X, \xi) \to (Y_j, \xi_j))_J$ is called an initial source in $\mathbf{A}$.

Dual notion: cotopological category.

Result 2.2 ([12], Theorem 1.5) A concrete category $\mathbf{A}$ is topological if and only if it is cotopological.

Result 2.3 ([12], Theorem 1.6) Let $\mathbf{A}$ be a topological category over $\mathbf{Set}$, then it is complete and cocomplete.

Definition 2.4 [12] Let $\mathbf{A}$ be a concrete category.

(i) The $\mathbf{A}$-fibre of a set X is the class of all $\mathbf{A}$-structures on X.

(ii) $\mathbf{A}$ is said to be properly fibred over $\mathbf{Set}$ if it satisfies the followings:

(a) (Fibre-smallness) for each set X, the $\mathbf{A}$-fibre of X is a set,

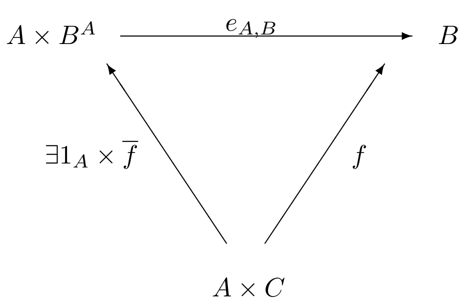
(b) (Terminal separator property) for each singleton set X, the $\mathbf{A}$-fibre of X has precisely one element,

(c) if ξ and η are $\mathbf{A}$-structures on a set X such that $id : (X, \xi) \to (X, \eta)$ and $id : (X, \eta) \to (X, \xi)$ are $\mathbf{A}$-morphisms, then $\xi = \eta$.

Definition 2.5 [7] A category $\mathbf{A}$ is said to be Cartesian closed if it satisfies the following conditions:

(i) for each $\mathbf{A}$-object A and B, there exists a product $A \times B$ in $\mathbf{A}$,

(ii) exponential objects exist in A, i.e., for each $\mathbf{A}$-object A, the functor $A \times - : \mathbf{A} \to \mathbf{A}$ has a right adjoint, i.e., for any $\mathbf{A}$-object B, there exist an $\mathbf{A}$-object B^A and an $\mathbf{A}$-morphism $e_{A,B} : A \times B^A \to B$ (called the evaluation) such that for any $\mathbf{A}$-object C and any $\mathbf{A}$-morphism $f : A \times C \to B$, there exists a unique $\mathbf{A}$-morphism $\bar{f} : C \to B^A$ such that $e_{A,B} \circ (id_A \times \bar{f}) = f$, i.e., the diagram commutes:



Definition 2.6 [7] A category $\mathbf{A}$ is called a topological universe over $\mathbf{Set}$ if it satisfies the following conditions:

(i) $\mathbf{A}$ is well-structured, i.e., (a) $\mathbf{A}$ is a concrete category; (b) $\mathbf{A}$ satisfies the fibre-smallness condition; (c) $\mathbf{A}$ has the terminal separator property,

(ii) $\mathbf{A}$ is cotopological over $\mathbf{Set}$,

(iii) final episinks in $\mathbf{A}$ are preserved by pullbacks, i.e., for any episink $(g_j : X_j \to Y)_J$ and any $\mathbf{A}$-morphism $f : W \to Y$, the family $(e_j : U_j \to W)_J$, obtained by taking the pullback f and g_j, for each $j \in J$, is again a final episink.

Definition 2.7 [2, 11] A lattice H is called a complete Heyting algebra if it satisfies the following conditions:

(i) it is a complete lattice,

(ii) for any $a, b \in H$, the set $\{x \in H : x \wedge a \leq b\}$ has the greatest element denoted by $a \to b$ (called the relative pseudo-complement of a and b), i.e., $x \wedge a \leq b$ if and only if $x \leq (a \to b)$.

In particular, if H is a complete Heyting algebra with the least element 0 then for each $a \in H$, $N(a) = a \to 0$ is called negation or the paudo-complement of a.

Result 2.8 ([2], Ex. 6 in p. 46) Let H be a complete Heyting algebra and $a, b \in H$.

(1) If $a \leq b$, then $N(b) \leq N(a)$, where $N : H \to H$ is an involutive order reversing operation in $(H, \leq)$.

(2) $a \leq NN(a)$.

(3) $N(a) = NNN(a)$.

(4) $N(a \vee b) = N(a) \wedge N(b)$ and $N(a \wedge b) = N(a) \vee N(b)$.

Throughout this paper, we will use H as a complete Heyting algebra with the least element 0 and the greatest element 1.

Definition 2.9 [9] Let X be a set. Then A is called an intuitionistic H-fuzzy set (in short, IHFS) in X if it satisfies the following conditions:

(i) A is of the form $A = (\mu, \nu)$, where $\mu, \nu : X \to H$ are mappings,

(ii) $\mu \leq N(\nu)$, i.e., $\mu(x) \leq N(\nu)(x)$ for each $x \in X$.

In this case, the pair (X, A) is called an intuitionistic H-fuzzy space (in short, IHFSp). We will denote the set of all IHFSs as $IHFS(X)$.

Definition 2.10 [9] The concrete category $\mathbf{ISet}(\mathbf{H})$ is defined as follows:

(i) each object is an IHFSp (X, A_X), where $A_X = (\mu_{A_X}, \nu_{A_X}) \in IHFS(X)$,

(ii) each morphism is a mapping $f : (X, A_X) \to (Y, A_Y)$ such that $\mu_{A_X} \leq \mu_{A_Y} \circ f$ and $\nu_{A_X} \geq \nu_{A_Y} \circ f$, i.e., $\mu_{A_X}(x) \leq \mu_{A_Y} \circ f(x)$ and $\nu_{A_X}(x) \geq \nu_{A_Y} \circ f(x)$, for each $x \in X$. In this case, the morphism $f : (X, A_X) \to (Y, A_Y)$ is called an $\mathbf{ISet}(\mathbf{H})$-mapping.

3 Neutrosophic sets

In [18], Salama and Smarandache introduced the concept of a neutrosophic crisp set in a set X and defined the inclusion between two neutrosophic crisp sets, the intersection [union] of two neutrosophic crisp sets, the complement of a neutrosophic crisp set, neutrosophic empty [resp., whole] set as more than two types. And they studied some properties related to neutrosophic set operations. However, by selecting only one type, we define the inclusion, the intersection [union] and the neutrosophic empty [resp., whole] set again and obtain some properties.

Definition 3.1 Let X be a non-empty set. Then A is called a neutrosophic set (in short, NS) in X, if A has the form $A = (T_A, I_A, F_A)$, where
$T_A : X \to]^-0, 1^+[$, $I_A : X \to]^-0, 1^+[$, $F_A : X \to]^-0, 1^+[$.
Since there is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, for each $x \in X$,

$$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+.$$

Moreover, for each $x \in X$, $T_A(x)$ [resp., $I_A(x)$ and $F_A(x)$] represent the degree of membership [resp., indeterminacy and nonmembership] of x to A.

The neutrosophic empty [resp., whole] set, denoted by 0_N [resp., 1_N] is an NS in X defined by $0_N = (0, 0, 1)$ [resp., $1_N = (1, 1, 0)$], where $0, 1 : X \to]^-0, 1^+[$ are defined by $0(x) = 0$ and $1(x) = 1$ respectively. We will denote the set of all NSs in X as $NS(X)$.

From Example 2.1.1 in [18], we can see that every IFS (intutionistic fuzzy set) A in a non-empty set X is an NS in X having the form

$$A = (T_A, 1 - (T_A + F_A), F_A),$$

where $(1 - (T_A + F_A))(x) = 1 - (T_A(x) + F_A(x))$.

Definition 3.2 Let $A = (T_A, I_A, F_A)$, $B = (T_B, I_B, F_B) \in NS(X)$. Then

(i) A is said to be contained in B, denoted by $A \subset B$, if
$T_A(x) \leq T_B(x)$, $I_A(x) \leq I_B(x)$ and $F_A(x) \geq F_B(x)$ for each $x \in X$,

(ii) A is said to equal to B, denoted by $A = B$, if

$$A \subset B \text{ and } B \subset A,$$

(iii) the complement of A, denoted by A^c, is an NCS in X defined as:

$$A^c = (F_A, 1 - I_A, T_A),$$

(iv) the intersection of A and B, denoted by $A \cap B$, is an NCS in X defined as:

$$A \cap B = (T_A \wedge T_B, I_A \wedge I_B, F_A \vee F_B),$$

where $(T_A \wedge T_B)(x) = T_A(x) \wedge T_B(x)$, $(F_A \vee F_B) = F_A(x) \vee F_B(x)$ for each $x \in X$,

(v) the union of A and B, denoted by $A \cup B$, is an NCS in X defined as:

$$A \cup B = (T_A \vee T_B, I_A \vee I_B, F_A \wedge F_B).$$

Let $(A_j)_{j \in J} \subset NS(X)$, where $A_j = (T_{A_j}, I_{A_j}, F_{A_j})$. Then

(vi) the intersection of $(A_j)_{j \in J}$, denoted by $\bigcap_{j \in J} A_j$ (simply, $\bigcap A_j$), is an NS in X defined as:

$$\bigcap A_j = (\bigwedge T_{A_j}, \bigwedge I_{A_j}, \bigvee F_{A_j}),$$

(vii) the union of $(A_j)_{j \in J}$, denoted by $\bigcup_{j \in J} A_j$ (simply, $\bigcup A_j$), is an NCS in X defined as:

$$\bigcup A_j = (\bigvee T_{A_j}, \bigvee I_{A_j}, \bigwedge F_{A_j}).$$

The followings are the immediate results of Definition 3.2.

Proposition 3.3 *Let $A, B, C \in NS(X)$. Then*
(1) $0_N \subset A \subset 1_N$,
(2) *if* $A \subset B$ *and* $B \subset C$, *then* $A \subset C$,
(3) $A \cap B \subset A$ *and* $A \cap B \subset B$,
(4) $A \subset A \cup B$ *and* $B \subset A \cup B$,
(5) $A \subset B$ *if and only if* $A \cap B = A$,
(6) $A \subset B$ *if and only if* $A \cup B = B$.

Also the followings are the immediate results of Definition 3.2.

Proposition 3.4 *Let $A, B, C \in NS(X)$. Then*
(1) (Idempotent laws): $A \cup A = A$, $A \cap A = A$,
(2) (Commutative laws): $A \cup B = B \cup A$, $A \cap B = B \cap A$,
(3) (Associative laws): $A \cup (B \cup C) = (A \cup B) \cup C$,
$A \cap (B \cap C) = (A \cap B) \cap C$,
(4) (Distributive laws): $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
(5) (Absorption laws): $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$,
(6) (De Morgan's laws): $(A \cup B)^c = A^c \cap B^c$,
$(A \cap B)^c = A^c \cup B^c$,
(7) $(A^c)^c = A$,
(8) (8a) $A \cup 0_N = A$, $A \cap 0_N = 0_N$,
(8b) $A \cup 1_N = 1_N$, $A \cap 1_N = A$,
(8c) $1_N^c = 0_N$, $0_N^c = 1_N$,
(8d) *in general*, $A \cup A^c \neq 1_N$, $A \cap A^c \neq 0_N$.

Proposition 3.5 *Let $A \in NS(X)$ and let $(A_j)_{j \in J} \subset NS(X)$. Then*
(1) $(\bigcap A_j)^c = \bigcup A_j^c$, $(\bigcup A_j)^c = \bigcap A_j^c$,
(2) $A \cap (\bigcup A_j) = \bigcup (A \cap A_j)$, $A \cup (\bigcap A_j) = \bigcap (A \cup A_j)$.

Proof. (1) Let $A_j = (T_{A_j}, I_{A_j}, F_{A_j})$.
Then $\bigcap A_j = (\bigwedge T_{A_j}, \bigwedge I_{A_j}, \bigvee F_{A_j})$.

Thus

$$(\bigcap A_j)^c = (\bigvee F_{A_j}, 1 - \bigwedge I_{A_j}, \bigwedge T_{A_j}) = (\bigvee F_{A_j}, \bigvee(1 - I_{A_j}), \bigwedge T_{A_j}) = \bigcup A_j^c$$

Similarly, the second part is proved.
(2) Let $A = (T_A, I_A, F_A)$ and $A_j = (T_{A_j}, I_{A_j}, F_{A_j})$.
Then

$$A \cup (\bigcap A_j) = (T_A \vee (\bigwedge T_{A_j}, I_A \vee (\bigwedge I_{A_j}), F_A \wedge (\bigvee F_{A_j})) = (\bigwedge(T_A \vee T_{A_j}), \bigwedge(I_A \vee I_{A_j}), \bigvee(F_A \wedge F_{A_j}) = \bigcap(A \cup A_j).$$

Similarly, the first part is proved. □

Definition 3.6 Let $f : X \to Y$ be a mapping and let $A \subset X$, $B \subset Y$. Then

(i) the image of A under f, denoted by $f(A)$, is an NS in Y defined as:

$$f(A) = (f(T_A), f(I_A), f(F_A)),$$

where for each $y \in Y$,

$$[f(T_A)](y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} T_A(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi, \end{cases}$$

(ii) the preimage of B, denoted by $f^{-1}(B)$, is an NCS in X defined as:

$$f^{-1}(B) = (f^{-1}(T_B), f^{-1}(I_B), f^{-1}(F_B)),$$

where $f^{-1}(T_B)(x) = T_B(f(x))$ for each $x \in X$,
in fact, $f^{-1}(B) = (T_B \circ f, I_B \circ f, F_B \circ f)$.

Proposition 3.7 *Let $f : X \to Y$ be a mapping and let $A, B, C \in NCS(X)$, $(A_j)_{j \in J} \subset NCS(X)$ and $D, E, F \in NCS(Y)$, $(D_k)_{k \in K} \subset NCS(Y)$. Then the followings hold:*
(1) *if $B \subset C$, then $f(B) \subset f(C)$ and if $E \subset F$, then $f^{-1}(E) \subset f^{-1}(F)$.*
(2) *$A \subset f^{-1}f(A))$ and if f is injective, then $A = f^{-1}f(A))$,*
(3) *$f(f^{-1}(D)) \subset D$ and if f is surjective, then $f(f^{-1}(D)) = D$,*
(4) *$f^{-1}(\bigcup D_k) = \bigcup f^{-1}(D_k)$, $f^{-1}(\bigcap D_k) = \bigcap f^{-1}(D_k)$,*
(5) *$f(\bigcup D_k) = \bigcup f(D_k)$, $f(\bigcap D_k) \subset \bigcap f(D_k)$,*
(6) *$f(A) = 0_N$ if and only if $A = 0_N$ and hence $f(0_N) = 0_N$, in particular if f is surjective, then $f(1_{X,N}) = 1_{Y,N}$,*
(7) *$f^{-1}(1_{Y,N}) = 1_{X,N}$, $f^{-1}(0_{Y,N}) = 0_{X,N}$.*

4 Properties of $\mathbf{NSet(H)}$

Definition 4.1 A is called a neutrosophic H-set (in short, NHS) in a non-empty set X if it satisfies the following conditions:

(i) A has the form $A = (T_A, I_A, F_A)$, where $T_A, I_A, F_A) : X \to H$ are mappings,

(ii) $T_A \leq N(F_A)$ and $I_A \geq N(F_A)$.

In this case, the pair (X, A) is called a neutrosophic H-space (in short, NHSp). We will denote the set of all the NHSs as $NHS(X)$.

Definition 4.2 Let (X, A_X), (Y, A_Y) be two NHSps and let $f : X \to Y$ be a mapping. Then $f : (X, A_X) \to (Y, A_Y)$ is called a morphism if $A_X \subset f^{-1}(A_Y)$, i.e.,
$T_{A_X} \leq T_{A_Y} \circ f$, $I_{A_X} \leq I_{A_Y} \circ f$ and $F_{A_X} \geq F_{A_Y} \circ f$.
In particular, $f : (X, A_X) \to (Y, A_Y)$ is called an epimorphism [resp., a monomorphism and an isomorphism], if it is surjective [resp., injective and bijective].

The following is the immediate result of Definition 4.2.

Proposition 4.3 *For each NHSp (X, A_X), the identity mapping $id : (X, A_X) \to (X, A_X)$ is a morphism.*

Proposition 4.4 *Let (X, A_X), (Y, A_Y), (Z, A_Z) be NHSps and let $f : X \to Y$, $g : Y \to Z$ be mappings. If $f : (X, A_X) \to (Y, A_Y)$ and $f : (Y, A_Y) \to (Z, A_Z)$ are morphisms, then $g \circ f : (X, A_X) \to (Z, A_Z)$ is a morphism.*

Proof. Let $A_X = (T_{A_X}, I_{A_X}, F_{A_X})$, $A_Y = (T_{A_Y}, I_{A_Y}, F_{A_Y})$ and $A_Z = (T_{A_Z}, I_{A_Z}, F_{A_Z})$. Then by the hypotheses and Definition 4.2, $A_X \subset f^{-1}(A_Y)$ and $A_Y \subset g^{-1}(A_Z)$, i.e.,
$T_{A_X} \leq T_{A_Y} \circ f$, $I_{A_X} \leq I_{A_Y} \circ f$, $F_{A_X} \geq F_{A_Y} \circ f$
and
$T_{A_Y} \leq T_{A_Z} \circ g$, $I_{A_Y} \leq I_{A_Z} \circ g$, $F_{A_Z} \geq F_{A_Z} \circ g$.
Thus $T_{A_X} \leq (T_{A_Z} \circ g) \circ f$, $I_{A_X} \leq (I_{A_Z} \circ g) \circ f$, $F_{A_X} \geq (F_{A_Z} \circ g) \circ f$.
So $T_{A_X} \leq T_{A_Z} \circ (g \circ f)$, $I_{A_X} \leq I_{A_Z} \circ (g \circ f)$, $F_{A_X} \geq F_{A_Z} \circ (g \circ f)$.
Hence $g \circ f$ is a morphism. □

From Propositions 4.3 and 4.4, we can form the concrete category $\mathbf{NSet(H)}$ consisting of NHSs and morphisms between them. Every $\mathbf{NSet(H)}$-morphism will be called an $\mathbf{NSet(H)}$-mapping.

Lemma 4.5 *The category* $\mathbf{NSet}$ *is topological over* $\mathbf{Set}$.

Proof. Let X be any set and let $((X_j, A_j))_{j \in J}$ be any family of NHSps indexed by a class J, where $A_j = (T_{A_j}, I_{A_j}, F_{A_j})$. Suppose $(f_j : X \to (X_j, A_j)_J$ is a source of ordinary mappings. We define mappings $T_{A_X}, I_{A_X}, F_{A_X} : X \to H$ as follows: for each $x \in X$,
$T_{A_X}(x) = \bigwedge(T_{A_j} \circ f_j)(x)$, $I_{A_X}(x) = \bigwedge(I_{A_j} \circ f_j)(x)$, $F_{A_X}(x) = \bigvee(F_{A_j} \circ f_j)(x)$.

Let $j \in J$ and $x \in X$.
Since $A_j = (T_{A_j}, I_{A_j}, F_{A_j}) \in NHS(X)$, $T_{A_j} \leq N(F_{A_X})$ and $I_{A_j} \geq N(F_{A_X})$. Then

$$\begin{aligned} N(F_{A_X}(x)) &= N(\bigvee(F_{A_j} \circ f_j)(x)) \\ &= \bigwedge N(F_{A_j}(f_j(x))) \\ &\geq \bigwedge T_{A_j}(f_j(x)) \\ &= \bigwedge T_{A_j} \circ f_j(x) \\ &= T_{A_X}(x) \end{aligned}$$

and

$$\begin{aligned} N(F_{A_X}(x)) &= \bigwedge N(F_{A_j}(f_j(x))) \\ &\leq \bigwedge I_{A_j}(f_j(x)) \\ &= \bigwedge I_{A_j} \circ f_j(x) \\ &= I_{A_X}(x) \end{aligned}$$

Thus $T_{A_X} \leq N(F_{A_X})$ and $I_{A_X} \geq N(F_{A_X})$.
So $A_X = \bigcap f_j^{-1}(A_j) \in NHS(X)$ and thus (X, A_X) is an NHSp. Moreover, by the definition of A_X,

$$T_{A_X} \leq T_{A_j} \circ f_j, I_{A_X} \leq I_{A_j} \circ f_j, F_{A_X} \geq F_{A_j} \circ f_j.$$

Hence $A_X \subset f_j^{-1}(A_j)$.
Therefore each $f_j : (X, A_X) \to (X_j, A_j)$ is an $\mathbf{NSet(H)}$-mapping.
Now let (Y, A_Y) be any NHSp and suppose $g : Y \to X$ is an ordinary mapping for which $f_j \circ g : (Y, A_Y) \to (X_j, A_j)$ is an $\mathbf{NSet(H)}$-mapping for each $j \in J$. Then

$A_Y \subset (f_j \circ g)^{-1}(A_j) = g^{-1}(f_j^{-1}(A_j))$ for each $j \in J$.

Thus

$$A_Y \subset g^{-1}(\bigcap f_j^{-1}(A_j)) = g^{-1}(A_X).$$

So $g : (Y, A_Y) \to (X, A_X)$ is an $\mathbf{NSet(H)}$-mapping. Hence $(f_j : (X, A_X) \to (X_j, A_j))_J$ is an initial source in $\mathbf{NSet(H)}$. This completes the proof. □

Example 4.6 (1) Let X be a set, let (Y, A_Y) be an NHSp and let $f : X \to Y$ be an ordinary mapping. Then clearly, there exists a unique NHS $A_X \in NHS(X)$ for which $f : (X, A_X) \to (Y, A_Y)$ is an $\mathbf{NSet(H)}$-mapping. In fact, $A_X = f^{-1}(A_Y)$.

In this case, A_X is called the inverse image under f of the NHS structure A_Y.

(2) Let $((X_j, A_j))_{j\in J}$ be any family of NHSps and let $X = \Pi_{j\in J} X_j$. For each $j \in J$, let $pr_j : X \to X_j$ be the ordinary projection. Then there exists a unique NHS $A_X \in NHS(X)$ for which $pr_j : (X, A_X) \to (X_j, A_j)$ is an $\mathbf{NSet(H)}$-mapping for each $j \in J$.

In this case, A_X is called the product of $(A_j)_J$, denoted by

$$A_X = \Pi_{j\in J} A_j = (\Pi_{j\in J} T_{A_j}, \Pi_{j\in J} I_{A_j}, \Pi_{j\in J} F_{A_j})$$

and (X, A_X) is called the product NHSp of $((X_j, A_j))_J$.
In fact, $A_X = \bigcap_{j\in J} pr^{-1}(A_j)$
and

$$\Pi_{j\in J} T_{A_j} = \bigwedge T_{A_j} \circ pr_j, \quad \Pi_{j\in J} I_{A_j} = \bigwedge I_{A_j} \circ pr_j,$$
$$\Pi_{j\in J} F_{A_j} = \bigvee F_{A_j} \circ pr_j.$$

In particular, if $J = \{1, 2\}$, then

$$\Pi_{j\in J} T_{A_j} = T_{A_1} \times T_{A_2} = (T_{A_1} \circ pr_1) \wedge (T_{A_2} \circ pr_2),$$
$$\Pi_{j\in J} I_{A_j} = I_{A_1} \times I_{A_2} = (I_{A_1} \circ pr_1) \wedge (I_{A_2} \circ pr_2),$$
$$\Pi_{j\in J} F_{A_j} = F_{A_1} \times F_{A_2} = (F_{A_1} \circ pr_1) \vee (F_{A_2} \circ pr_2).$$

The following is the immediate result of Lemma 4.5 and Result 2.3.

Corollary 4.7 *The category* $\mathbf{NSet(H)}$ *is complete and cocomplete.*

The following is obvious from Result 2.2. But we show directly it.

Corollary 4.8 *The category* $\mathbf{NCSet}$ *is cotopological over* $\mathbf{Set}$.

Proof. Let X be any set and let $((X_j, A_j))_J$ be any family of NHSps indexed by a class J. Suppose $(f_j : X_j \to X)_J$ is a sink of ordinary mappings. We define mappings $T_{A_X}, I_{A_X}, F_{A_X} : X \to H$ as follows: for each $x \in X$,

$$T_{A_X}(x) = \begin{cases} \bigvee_J \bigvee_{x_j \in f_j^{-1}(x)} T_{A_j}(x_j) & \text{if } f_j^{-1}(x) \neq \phi \text{ for all } j \\ 0 & \text{if } f_j^{-1}(x) = \phi \text{ for some } j, \end{cases}$$

$$I_{A_X}(x) = \begin{cases} \bigvee_J \bigvee_{x_j \in f_j^{-1}(x)} I_{A_j}(x_j) & \text{if } f_j^{-1}(x) \neq \phi \text{ for all } j \\ 0 & \text{if } f_j^{-1} = \phi \text{ for some } j, \end{cases}$$

$$F_{A_X}(x) = \begin{cases} \bigwedge_J \bigwedge_{x_j \in f_j^{-1}(x)} F_{A_j}(x_j) & \text{if } f_j^{-1} \neq \phi \text{ for all } j \\ 1 & \text{if } f_j^{-1} = \phi \text{ for some } j. \end{cases}$$

Since $((X_j, A_j))_J$ is a family of NHSps, $T_{A_j} \leq N(F_{A_j})$ and $I_{A_j} \geq N(F_{A_j})$ for each $j \in J$. We may assume that $f_j^{-1} \neq \phi$ without loss of generality. Let $x \in X$. Then

$$\begin{aligned} N(F_{A_X}(x)) &= N(\bigwedge_J \bigwedge_{x_j \in f_j^{-1}(x)} F_{A_j}(x_j)) \\ &= \bigvee_J \bigvee_{x_j \in f_j^{-1}(x)} N(F_{A_j}(x_j)) \\ &\geq \bigvee_J \bigvee_{x_j \in f_j^{-1}(x)} T_{A_j}(x_j). \\ &= T_{A_X}(x). \end{aligned}$$

and

$$\begin{aligned} N(F_{A_X}(x)) &= \bigvee_J \bigvee_{x_j \in f_j^{-1}(x)} N(F_{A_j}(x_j)) \\ &\leq \bigvee_J \bigvee_{x_j \in f_j^{-1}(x)} I_{A_j}(x_j). \\ &= I_{A_X}(x). \end{aligned}$$

Thus $T_{A_X} \leq N(F_{A_X})$ and $I_{A_X} \geq N(F_{A_X})$.
So (X, A_X) is an NHSp. Moreover, for each $j \in J$,

$$f_j^{-1}(A_X) = f_j^{-1}(\bigcup f_j(A_j)) = \bigcup f_j^{-1}(f_j(A_j)) \supset A_j.$$

Hence each $f_j : (X_j, A_j) \to (X, A_X)$ is an $\mathbf{NSet(H)}$-mapping.
Now for each NHSp (Y, A_Y), let $g : X \to Y$ be an ordinary mapping for which each $g \circ f_j : (X_j, A_j) \to (Y, A_Y)$ is an $\mathbf{NSet(H)}$-mapping. Then clearly for each $j \in J$,

$A_j \subset (g \circ f_j)^{-1}(A_Y)$, i.e., $A_j \subset f_j^{-1}(g^{-1}(A_Y))$.

Thus $\bigcup A_j \subset \bigcup f_j^{-1}(g^{-1}(A_Y))$.
So $f_j(\bigcup A_j) \subset f_j(\bigcup f_j^{-1}(g^{-1}(A_Y)))$. By Proposition 3.7 and the definition of A_X,

$$f_j(\bigcup A_j) = \bigcup f_j(A_j) = A_X$$

and

$$f_j(\bigcup f_j^{-1}(g^{-1}(A_Y))) = \bigcup(f_j \circ f_j^{-1})(g^{-1}(A_Y)) = g^{-1}(A_Y).$$

Hence $A_X \subset g^{-1}(A_Y)$. Therefore $g : (X, A_X) \to (Y, A_Y)$ is an $\mathbf{NSet(H)}$-mapping. This completes the proof. □

Example 4.9 (1) Let $(X, A_X) \in \mathbf{NSet(H)}$, let R be an ordinary equivalence relation on X and let $\varphi : X \to X/R$ be the canonical mapping. Then there exists the final NHS structure $A_{X/R}$ in X/R for which $\varphi : (X, A_X) \to (X/R, A_{X/R})$ is an $\mathbf{NSet(H)}$-mapping, where $A_{X/R} = (T_{A_{X/R}}, I_{A_{X/R}}, F_{A_{X/R}}) = (\varphi(T_{A_X}), \varphi(I_{A_X}), \varphi(F_{A_X}))$.

In this case, $A_{X/R}$ is called the neutrosophic H-quotient set structure of X by R.

(2) Let $((X_\alpha, A_\alpha))_{\alpha\in\Gamma}$ be a family of NHSs, let X be the sum of $(X_\alpha)_{\alpha\in\Gamma}$, i.e., $X = \bigcup(X_\alpha \times \{\alpha\})$ and let $j_\alpha : X_\alpha \to X$ the canonical (injective) mapping for each $\alpha \in \Gamma$. Then there exists the final NHS A_X in X. In fact, $A_X = (T_{A_X}, I_{A_X}, F_{A_X})$, where for each $(x, \alpha) \in X$,

$$T_{A_X}(x, \alpha) = \bigvee_\Gamma T_{A_\alpha}(x), \quad I_{A_X}(x, \alpha) = \bigvee_\Gamma I_{A_\alpha}(x),$$
$$F_{A_X}(x, \alpha) = \bigwedge_\Gamma F_{A_\alpha}(x).$$

In this case, A_X is called the sum of $((X_\alpha, A_\alpha))_{\alpha\in\Gamma}$.

Lemma 4.10 *Final episinks in* $\mathbf{NSet(H)}$ *are prserved by pullbacks.*

Proof. Let $(g_j : (X_j, A_j) \to (Y, A_Y))_J$ be any final episink in $\mathbf{NSet(H)}$ and let $f : (W, A_W) \to (Y, A_Y)$ be any $\mathbf{NSet(H)}$-mapping. For each $j \in J$, let

$$U_j = \{(w, x_j) \in W \times X_j : f(w) = g_j(x_j)\}.$$

For each $j \in J$, we define mappings $T_{A_{U_j}}, I_{A_{U_j}}, F_{A_{U_j}} : U_j \to H$ as follows: for each $(w, x_j) \in U_j$,

$$T_{A_{U_j}}(w, x_j) = T_{A_W}(w) \wedge T_{A_j}(x_j),$$

$$I_{A_{U_j}}(w, x_j) = I_{A_W}(w) \wedge I_{A_j}(x_j),$$

$$F_{A_{U_j}}(w, x_j) = F_{A_W}(w) \vee F_{A_j}(x_j).$$

Then clearly, $A_{U_j} = (T_{A_{U_j}}, I_{A_{U_j}}, F_{A_{U_j}}) = (A_W \times A_j)_* \in NHS(U_j)$. Thus (U_j, A_{U_j}) is an NHSp, where $(A_W \times A_j)_*$ denotes the restriction of $A_W \times A_j$ under U_j.

Let e_j and p_j be ordinary projections of U_j. Let $j \in J$. Then clearly,

$$A_{U_j} \subset e_j^{-1}(A_Y) \text{ and } A_{U_j} \subset p_j^{-1}(A_j).$$

Thus $e_j : (U_j, A_{U_j}) \to (W, A_W)$ and $p_j : (U_j, A_{U_j}) \to (X_j, A_j)$ are $\mathbf{NSet(H)}$-mappings. Moreover, $g_h \circ p_h = f \circ e_j$ for each $j \in J$, i.e., the diagram is a pullback square in $\mathbf{NCSet}$:

$$\begin{array}{ccc}
(U_j, A_{U_j}) & \xrightarrow{\quad p_j \quad} & (X_j, A_j) \\
\Big\downarrow e_j & & \Big\downarrow g_j \\
(W, A_W) & \xrightarrow[\quad f \quad]{} & (Y, A_Y).
\end{array}$$

Now in order to prove that $(e_j)_J$ is an episink in $\mathbf{NSet(H)}$, i.e., each e_j is surjective, let $w \in W$. Since $(g_j)_J$ is an episink, there exists $j \in J$ such that $g_j(x_j) = f(w)$ for some $x_j \in X_j$. Thus $(w, x_j) \in U_j$ and $w = e_j(w, x_j)$. So $(e_j)_J$ is an episink in $\mathbf{NSet(H)}$.

Finally, let us show that $(e_j)_J$ is final in $\mathbf{NSet(H)}$. Let A_W^* be the final structure in W w.r.t. $(e_j)_J$ and let $w \in W$. Then

$$\begin{aligned}
T_{A_W}(w) &= T_{A_W}(w) \wedge T_{A_W}(w) \\
&\leq T_{A_W}(w) \wedge f^{-1}(T_{A_Y}(w)) \\
&\qquad \text{[since } f : (W, A_W) \to (Y, A_Y))_J) \text{ is an } \mathbf{NSet(H)}\text{-mapping]} \\
&= T_{A_W}(w) \wedge T_{A_Y}(f(w)) \\
&= T_{A_W}(w) \wedge (\textstyle\bigvee_J \bigvee_{x_j \in g_j^{-1}(f(w))} T_{A_j}(x_j)) \\
&\qquad \text{[since } (g_j)_J \text{ is final in } \mathbf{NSet(H)}] \\
&= \textstyle\bigvee_J \bigvee_{x_j \in g_j^{-1}(f(w))}(T_{A_W}(w) \wedge T_{A_j}(x_j)) \\
&= \textstyle\bigvee_J \bigvee_{(w,x_j) \in e_j^{-1}(w)}(T_{U_j}(w, x_j)) \\
&= T_{A_W^*}(w).
\end{aligned}$$

Thus $T_{A_W} \leq T_{A_W^*}$. Similarly, we can see that $I_{A_W} \leq I_{A_W^*}$ and $F_{A_W} \geq F_{A_W^*}$. So $A_W \subset A_W^*$. On the other hand, since $e_j : (U_j, A_{U_j}) \to (W, A_W^*)$ is final, $id_W : (W, A_W^*) \to (W, A_W)$ is an $\mathbf{NSet(H)}$-mapping. So $A_W^* \subset A_W$. Hence $A_W = A_W^*$. This completes the proof. □

For any singleton set $\{a\}$, since the NHS structure $A_{\{a\}}$ on $\{a\}$ is not unique, the category $\mathbf{NSet(H)}$ is not properly fibred over $\mathbf{Set}$. Then by Lemmas 4.5,4.9 and Definition 2.6, we obtain the following result.

Theorem 4.11 *The category* $\mathbf{NSet(H)}$ *satisfies all the conditions of a topological universe over* $\mathbf{Set}$ *except the terminal separator property.*

Theorem 4.12 *The category* $\mathbf{NSet(H)}$ *is Cartesian closed over* $\mathbf{Set}$.

Proof. From Lemma 4.5, it is clear that $\mathbf{NSet(H)}$ has products. So it is sufficient to prove that $\mathbf{NSet(H)}$ has exponential objects.

For any NHSs $\mathbf{X} = (X, A_X)$ and $\mathbf{Y} = (Y, A_Y)$, let Y^X be the set of all ordinary mappings from X to Y. We define mappings $T_{A_{Y^X}}, I_{A_{Y^X}}, F_{A_{Y^X}} : Y^X \to H$ as follows: for each $f \in Y^X$,

$$T_{A_{Y^X}}(f) = \bigvee\{h \in H : T_{A_X}(x) \wedge h \leq T_{A_Y}(f(x)),$$
$$\text{for each } x \in X\},$$

$$I_{A_{Y^X}}(f) = \bigvee\{h \in H : I_{A_X}(x) \wedge h \leq I_{A_Y}(f(x)),$$

$$\text{for each } x \in X\},$$

$$F_{A_{Y^X}}(f) = \bigwedge\{h \in H : F_{A_X}(x) \vee h \geq F_{A_Y}(f(x)),$$

$$\text{for each } x \in X\}.$$

Then clearly, $A_{Y^X} = (T_{A_{Y^X}}, I_{A_{Y^X}}, F_{A_{Y^X}}) \in NHS(Y^X)$ and thus (Y^X, A_{Y^X}) is an NHSp. Let $\mathbf{Y^X} = (Y^X, A_{Y^X})$ and let $f \in Y^X$, $x \in X$. Then by the definition of A_{Y^X},

$$T_{A_X}(x) \wedge T_{A_{Y^X}}(f) \leq T_{A_Y}(f(x)),$$

$$I_{A_X}(x) \wedge I_{A_{Y^X}}(f) \leq I_{A_Y}(f(x)),$$

$$F_{A_X}(x) \vee F_{A_{Y^X}}(f) \geq F_{A_Y}(f(x)).$$

We define a mapping $e_{X,Y} : X \times Y^X \to Y$ as follows: for each $(x, f) \in X \times Y^X$,

$$e_{X,Y}(x, f) = f(x).$$

Then clearly, $A_X \times A_{Y^X} \in NHS(X \times Y^X)$, where $A_X = (T_{A_X}, I_{A_X}, F_{A_X})$
and for each $(x, f) \in X \times Y^X$,

$$T_{A_X \times A_{Y^X}}(x, f) = T_{A_X}(x) \wedge T_{A_{Y^X}}(f),$$
$$I_{A_X \times A_{Y^X}}(x, f) = I_{A_X}(x) \wedge I_{A_{Y^X}}(f),$$
$$F_{A_X \times A_{Y^X}}(x, f) = F_{A_X}(x) \vee F_{A_{Y^X}}(f).$$

Let us show that $A_X \times A_{Y^X} \subset e^{-1}_{X,Y}(A_Y)$. Let $(x, f) \in X \times Y^X$. Then

$$e^{-1}_{X,Y}(A_Y)(x, f) = A_Y(e_{X,Y}(x, f)) = A_Y(f(x)).$$

Thus

$$\begin{aligned} T_{e^{-1}_{X,Y}(A_Y)}(x, f) &= T_{A_Y}(f(x)) \\ &\geq T_{A_X}(x) \wedge T_{A_{Y^X}}(f) \\ &= T_{A_X \times A_{Y^X}}(x, f), \end{aligned}$$

$$\begin{aligned} I_{e^{-1}_{X,Y}(A_Y)}(x, f) &= I_{A_Y}(f(x)) \\ &\geq I_{A_X}(x) \wedge I_{A_{Y^X}}(f) \\ &= I_{A_X \times A_{Y^X}}(x, f), \end{aligned}$$

$$\begin{aligned} F_{e^{-1}_{X,Y}(A_Y)}(x, f) &= F_{A_Y}(f(x)) \\ &\leq F_{A_X}(x) \vee F_{A_{Y^X}}(f) \\ &= F_{A_X \times A_{Y^X}}(x, f). \end{aligned}$$

So $A_X \times A_{Y^X} \subset e^{-1}_{X,Y}(A_Y)$. Hence $e_{X,Y} : \mathbf{X} \times \mathbf{Y^X} \to \mathbf{Y}$ is an $\mathbf{NSet(H)}$-mapping, where
$\mathbf{X} \times \mathbf{Y^X} = (X \times Y^X, A_X \times A_{Y^X})$ and $\mathbf{Y} = (Y, A_Y)$.

For any $\mathbf{Z} = (Z, A_Z) \in \mathbf{NSet(H)}$, let $h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ be an $\mathbf{NSet(H)}$-mapping where $\mathbf{X} \times \mathbf{Z} = (X \times Z, A_X \times A_Z)$. We define a mapping $\bar{h} : Z \to Y^X$ as follows:

$$(\bar{h}(z))(x) = h(x, z),$$

for each $z \in Z$ and each $x \in X$. Let $(x, z) \in X \times Z$. Then

$$\begin{aligned} T_{A_X \times A_Z}(x, z) &= T_{A_X}(x) \wedge T_{A_Z}(z) \\ &\leq T_{A_Y}(h(x, z)) \text{ [since } h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y} \\ &\qquad \text{is an } \mathbf{NSet(H)}\text{-mapping]} \\ &= T_{A_Y}(\bar{h}(z))(x). \end{aligned}$$

Thus by the definition of A_{Y^X},

$$T_{A_Z}(z) \leq T_{A_{Y^X}}(\bar{h}(z)) = \bar{h}^{-1}(T_{A_{Y^X}})(z).$$

So $T_{A_Z} \leq \bar{h}^{-1}(T_{A_{Y^X}})$. Similarly, we can see that $I_{A_Z} \leq \bar{h}^{-1}(I_{A_{Y^X}})$ and $F_{A_Z} \geq \bar{h}^{-1}(F_{A_{Y^X}})$. Hence $\bar{h} : \mathbf{Z} \to \mathbf{Y^X}$ is an $\mathbf{NSet(H)}$-mapping, where $\mathbf{Y^X} = (Y^X, A_{Y^X})$. Furthermore, we can prove that $\bar{h}$ is a unique $\mathbf{NSet(H)}$-mapping such that $e_{X,Y} \circ (id_X \times \bar{h}) = h$. □

5 The relation between NSet(H) and ISet(H)

Lemma 5.1 *Define $G_1, G_2 : \mathbf{NSet(H)} \to \mathbf{ISet(H)}$ by:*

$$G_1(X, (T, I, F)) = (X, (T, F)),$$

$$G_2(X, (T, I, F)) = (X, (T, N(T)))$$

and

$$G_1(f) = G_2(f) = f.$$

Then G_1 and G_2 are functors.

Proof. It is clear that $G_1(X, (T, I, F)) = (X, (T, F)) \in \mathbf{ISet(H)}$ for each $(X, (T, I, F) \in \mathbf{NSet(H)}$.

Let $(X, (T_X, I_X, F_X)), (Y, (T_Y, I_Y, F_Y)) \in \mathbf{NSet(H)}$ and let $f : (X, (T_X, I_X, F_X)) \to (Y, (T_Y, I_Y, F_Y))$ be an $\mathbf{NSet(H)}$-mapping. Then

$$T_X \leq T_Y \circ f \text{ and } F_X \geq F_Y \circ f.$$

Thus $G_1(f) = f$ is an $\mathbf{ISet(H)}$-mapping. So $G_1 : \mathbf{NSet(H)} \to \mathbf{ISet(H)}$ is a functor.

Now let $(X, (T, I, F)) \in \mathbf{NSet(H)}$ and consider $(X, (T, N(T)))$. Then by Result 2.8, $T \leq NN(T)$. Thus $G_2(X, (T, I, F)) = (X, (T, N(T))) \in \mathbf{NSet(H)}$.

Let $(X, (T_X, I_X, F_X)), (Y, (T_Y, I_Y, F_Y)) \in \mathbf{NSet(H)}$ and let $f : (X, (T_X, I_X, F_X)) \to (Y, (T_Y, I_Y, F_Y))$ be an $\mathbf{NSet(H)}$-mapping. Then $T_X \leq T_Y \circ f$. Thus $N(T_X) \geq N(T_Y) \circ f$.
So $G_2(f) = f : (X, (T_X, N(T_X)) \to (Y, (T_Y, N(T_Y))$ is an $\mathbf{ISet(H)}$-mapping. Hence $G_2 : \mathbf{NSet(H)} \to \mathbf{ISet(H)}$ is a functor. □

Lemma 5.2 *Define* $F_1 : \mathbf{ISet}(\mathbf{H}) \to \mathbf{NSet}(\mathbf{H})$ *by:*
$F_1(X, (\mu, \nu)) = (X, (\mu, N(\nu), \nu))$ *and* $F_1(f) = f$.
Then F_1 *is a functor.*

Proof. Let $(X, (\mu, \nu)) \in \mathbf{ISet}(\mathbf{H})$. Then
$\mu \leq N(\nu)$ and $N(\nu) \leq N(\nu)$.
Thus $F_1(X, (\mu, \nu)) = (X, (\mu, N(\nu), \nu)) \in \mathbf{NSet}(\mathbf{H})$.
Let $(X, (\mu_X, \nu_X)), (Y, (\mu_Y, \nu_Y)) \in \mathbf{ISet}(\mathbf{H})$ and let $f : (X, (\mu_X, \nu_X)) \to (Y, (\mu_Y, \nu_Y))$ be an $\mathbf{ISet}(\mathbf{H})$-mapping.
Consider the mapping

$$F_1(f) = f : F_1(X, (\mu_X, \nu_X)) \to F_1(Y, (\mu_Y, \nu_Y)),$$

where

$$F_1(X, (\mu_X, \nu_X)) = (X, (\mu_X, N(\nu_X), \nu_X))$$

and

$$F_1(Y, (\mu_Y, \nu_Y)) = (Y, (\mu_Y, N(\nu_Y), \nu_Y)).$$

Since $f : (X, (\mu_X, \nu_X)) \to (Y, (\mu_Y, \nu_Y))$ is an $\mathbf{ISet}(\mathbf{H})$-mapping, $\mu_X \leq \mu_Y \circ f$ and $\nu_X \geq \nu_Y \circ f$. Thus $N(\nu_X) \leq N(\nu_Y) \circ f$. So $F_1(f) = f : (X, (\mu_X, N(\nu_X), \nu_X)) \to (Y, (\mu_Y, N(\nu_Y), \nu_Y))$ is an $\mathbf{NSet}(\mathbf{H})$-mapping. Hence F_1 is a functor. □

Lemma 5.3 *Define* $F_2 : \mathbf{ISet}(\mathbf{H}) \to \mathbf{NSet}(\mathbf{H})$ *by:*

$$F_2(X, (\mu, \nu)) = (X, (\mu, N(\nu), N(\mu)) \text{ and } F_2(f) = f.$$

Then F_2 *is a functor.*

Proof. Let $(X, (\mu, \nu)) \in \mathbf{ISet}(\mathbf{H})$. Then $\mu \leq N(\nu)$ and $\mu \leq NN(\mu)$, by Result 2.8. Also by Result 2.8, $NN(\mu) \leq NNN(\nu) = N(\nu)$. Thus $\mu \leq NN(\mu) \leq N(\nu)$. So $F_2(X, (\mu, \nu)) = (X, (\mu, N(\nu), N(\mu))) \in \mathbf{NSet}(H)$.

Let $(X, (\mu_X, \nu_X)), (Y, (\mu_Y, \nu_Y)) \in \mathbf{ISet}(H)$ and $f : (X, (\mu_X, \nu_X)) \to (Y, (\mu_Y, \nu_Y))$ be an $\mathbf{ISet}(H)$-mapping. Then
$\mu_X \leq \mu_Y \circ f^2$ and $\nu_X \geq \nu_Y \circ f^2$.
Thus $N(\nu_X) \leq N(\nu_Y) \circ f^2$. So $L(f) = f : (X, (\mu_X, N(\nu_X), N(\mu_X))) \to (Y, (\mu_Y, N(\nu_Y), N(\mu_Y)))$ is an $\mathbf{NSet}(H)$-mapping. Hence F_2 is a functor. □

Theorem 5.4 *The functor* $F_1 : \mathbf{ISet}(\mathbf{H}) \to \mathbf{NSet}(\mathbf{H})$ *is a left adjoint of the functor* $G_1 : \mathbf{NSet}(\mathbf{H}) \to \mathbf{ISet}(\mathbf{H})$.

Proof. For each $(X, (\mu, \nu)) \in \mathbf{ISet}(\mathbf{H})$, $1_X : (X, (\mu, \nu)) \to G_1F_1(X, (\mu, \nu)) = (X, (\mu, \nu))$ is an $\mathbf{ISet}(\mathbf{H})$-mapping. Let $(Y, (T_Y, I_Y, F_Y)) \in \mathbf{NSet}(\mathbf{H})$ and let $f : (X, (\mu, \nu)) \to G_1(Y, (T_Y, I_Y, F_Y)) = (Y, (T_Y, F_Y))$ be an $\mathbf{ISet}(\mathbf{H})$-mapping.

We will show that $f : F_1(X, (\mu, \nu)) = (X, (\mu, N(\nu), \nu)) \to (Y, (T_Y, I_Y, F_Y))$ is an $\mathbf{NSet}(\mathbf{H})$-mapping. Since $f : (X, (\mu, \nu)) \to (Y, (T_Y, F_Y))$ is an $\mathbf{ISet}(\mathbf{H})$-mapping,
$\mu \leq T_Y \circ f$ and $\nu \geq F_Y \circ f$.
Then $N(\nu) \leq N(F_Y) \circ f$. Since $(Y, (T_Y, I_Y, F_Y)) \in \mathbf{NSet}(\mathbf{H})$, $I_Y \geq N(F_Y))$. Thus $N(\nu) \leq I_Y \circ f$. So $f : F_1(X, (\mu, \nu)) = (X, (\mu, N(\nu), \nu)) \to (Y, (T_Y, I_Y, F_Y))$ is an $\mathbf{NSet}(\mathbf{H})$-mapping. Hence 1_X is a G_1-universal mapping for $(X, (\mu, \nu)) \in \mathbf{ISet}(\mathbf{H})$. This completes the proof. □

For each $(X, (\mu, \nu)) \in \mathbf{ISet}(\mathbf{H})$, $F_1(X, (\mu, \nu)) = (X, (\mu, N(\nu), \nu))$ is called a neutrosophic H-space induced by $(X, (\mu, \nu))$. Let us denote the category of all induced neutrosophic H-spaces and $\mathbf{NSet}(\mathbf{H})$-mappings as $\mathbf{NSet}^*(\mathbf{H})$. Then $\mathbf{NSet}^*(\mathbf{H})$ is a full subcategory of $\mathbf{NSet}(\mathbf{H})$.

Theorem 5.5 *Two categories* $\mathbf{ISet}(\mathbf{H})$ *and* $\mathbf{NSet}^*(\mathbf{H})$ *are isomorphic.*

Proof. From Lemma 5.2, it is clear that $F_1 : \mathbf{ISet}(\mathbf{H}) \to \mathbf{NSet}^*(\mathbf{H})$ is a functor. Consider the restriction $G_1 : \mathbf{NSet}^*(\mathbf{H}) \to \mathbf{ISet}(\mathbf{H})$ of the functor G_1 in Lemma 5.1. Let $(X, (\mu, \nu)) \in \mathbf{ISet}(\mathbf{H})$. Then by Lemma 5.2, $F_1(X, (\mu, \nu)) = (X, (\mu, N(\nu), \nu))$. Thus $G_1F_1(X, (\mu, \nu)) = G_1(X, (\mu, N(\nu), \nu)) = (X, (\mu, \nu))$. So $G_1 \circ F_1 = 1_{\mathbf{ISet}(\mathbf{H})}$.

Now let $(X, (T_X, I_X, F_X)) \in \mathbf{NSet}^*(\mathbf{H})$. Then by definition of $\mathbf{NSet}^*(\mathbf{H})$, there exists $(X, (\mu, N(\nu), \nu))$ such that

$$F_1(X, (\mu, \nu)) = (X, (\mu, N(\nu), \nu)) = (X, (T_X, I_X, F_X)).$$

Thus by Lemma 5.1,

$$\begin{aligned} G_1(X, (T_X, I_X, F_X)) &= G_1(X, (\mu, N(\nu), \nu)) \\ &= (X, (\mu, \nu)). \end{aligned}$$

So

$$\begin{aligned} F_1G_1(X, (T_X, I_X, F_X)) &= F_1(X, (\mu, \nu)) \\ &= (X, (T_X, I_X, F_X)). \end{aligned}$$

Hence $F_1 \circ G_1 = 1_{\mathbf{NSet}^*(\mathbf{H})}$. Therefore $F_1 : \mathbf{ISet}(\mathbf{H}) \to \mathbf{NSet}^*(\mathbf{H})$ is an isomorphism. This completes the proof. □

6 Conclusions

In the future, we will form a category $\mathbf{NCRel}$ composed of neutrosophic crisp relations and morphisms between them [resp., $\mathbf{NRel}(\mathbf{H})$ composed of neutrosophic relations and morphisms between them, $\mathbf{NCTop}$ composed of neutrosophic crisp topological spaces and morphisms between them and $\mathbf{NTop}$ composed of neutrosophic topological spaces and morphisms between them] and investigate each category in view points of topological universe. Moreover, we will form some subcategories of each category and study their properties.

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