A residual power series technique for solving systems of initial value problems

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Abstract. In this article, a residual power series technique for the power series solution of systems of initial value problems is introduced. The new approach provides the solution in the form of a rapidly convergent series with easily computable components using symbolic computation software. The proposed technique obtains Taylor expansion of the solution of a system and reproduces the exact solution when the solution is polynomial. Numerical examples are included to demonstrate the efficiency, accuracy, and applicability of the presented technique. The results reveal that the technique is very effective, straightforward, and simple.

Keywords: Systems of initial value problems; Residual power series; Taylor expansion

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1. Introduction

In real life situations quantities and their rate of changes depend on more than one variable. For example, the rabbit population, though it may be represented by a single number, depends on the size of predator populations and the availability of food. In order to represent and study such complicated problems we need to use more than one dependent variable and more than one equation. Systems of differential equations are the tools to use. These kinds of equations can be found in almost all branches of sciences, engineering, and technology, such as electromagnetic, solid state physics, plasma physics, elasticity, fluid dynamics, oscillation theory, mathematical biology, chemical kinetics, biomechanics, and control theory [1-6].

In the present paper, we invested the residual concept in the power series method to obtain a simple technique (we call it residual power series (RPS) [7-15]) to find out the coefficients of the series solutions. This technique helps us to construct a power series solution for strongly linear and nonlinear systems. The RPS technique is effective and easy to use for solving linear and nonlinear systems of initial value problems (IVPs) without linearization, perturbation, or discretization. Different from the classical power series method, the RPS technique does not need to compare the coefficients of the corresponding terms and recursion relations are not required. This technique computes the coefficient of the power series by a chain of linear equations of n-variable, where n is number of equations in the given system. The RPS technique is different from the traditional higher order Taylor series method. The Taylor series method is computationally expensive for large orders. The RPS technique is an alternative procedure for obtaining analytic Taylor series solution of systems of IVPs. By using residual error concept, we get a series solution, in practice a truncated series solution.

The RPS technique has the following characteristics [7-15]; first, the technique obtains Taylor expansion of the solution; as a result, the exact solution is available when the solution is polynomial. Moreover the solutions and all its derivatives are applicable for each arbitrary point in the given interval. Second, it does not require any modification while switching from the first order to the higher order; as a result the technique can be applied directly to the given problem by choosing an appropriate value for the initial guesses approximations. Third, the RPS technique needs small computational requirements with high precision and less time.

The purpose of this paper is to obtain symbolic approximate power series solutions for system of IVPs which is as follows:

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$$
\begin{aligned}
&x_1'(t) = f_1\big(t, x_1(t), x_2(t), \dots, x_n(t)\big),\\
&x_2'(t) = f_2\big(t, x_1(t), x_2(t), \dots, x_n(t)\big),\\
&\vdots\\
&x_n'(t) = f_n\big(t, x_1(t), x_2(t), \dots, x_n(t)\big),
\end{aligned} \tag{1}
$$

subject to the initial conditions

$$x_1(t_0) = x_1, x_2(t_0) = x_2, \dots, x_n(t_0) = x_n, \tag{2}$$

where $t \in [t_0, t_0 + a]$, $f_i: [t_0, t_0 + a] \times \mathbb{R}^n \to \mathbb{R}$ are nonlinear continuous functions of $t, x_1, x_2, \dots, x_n$, $x_i(t)$ are unknown functions of independent variable t to be determined, and t_0, a are real finite constants with $a > 0$. Throughout this paper, we assume that f_i, x_i are analytic functions on the given interval. Also, we assume that f_i satisfies all the necessary requirements for the existence of a unique solution.

In general, systems of IVPs do not always have solutions which we can obtain using analytical methods. In fact, many of real physical phenomena encountered, are almost impossible to solve by this technique. Due to this, some authors have proposed numerical methods to approximate the solutions of systems of IVPs. To mention a few, the homotopy analysis method has been applied to solve system (1) and (2) as described in [16]. In [17] the authors have developed the homotopy perturbation method. In [18] also, the author has provided the differential transformation technique to further investigation to the above system. Furthermore, the reproducing kernel Hilbert space method is carried out in [19]. Recently, a class of collocation methods for solving system (1) and (2) is proposed in [20].

However, none of previous studies propose a methodical way to solve systems of IVPs (1) and (2). Moreover, previous studies require more effort to achieve the results and usually they are suited for linear form of system (1) and (2). On the other hand, the applications of other versions of series solutions to linear and nonlinear problems can be found in [21-26] and references therein. Also, for numerical solvability of different categories of differential equations one can consult the references [27, 28].

The outline of the paper is as follows: in the next section, we present the basic idea of the RPS technique. In section 3, numerical examples are given to illustrate the capability of proposed approach. This article ends in section 4 with some concluding remarks.

2. Solution of systems of IVPs by RPS technique

In this section, we employ our technique of the RPS to find out series solution for systems of IVPs subject to given initial conditions. We first formulate and analyze the RPS technique for solving such systems of IVPs. After that, a convergence theorem is presented in order to capture the behavior of the solution.

The RPS technique consists in expressing the solutions of system of IVPs (1) and (2) as a power series expansion about the initial point $t = t_0$. To achieve our goal, we suppose that these solutions take the form

$$x_i(t) = \sum_{m=0}^{\infty} x_{i,m}(t), i = 1,2, \dots, n,$$

where $x_{i,m}$ are terms of approximations and are given as $x_{i,m}(t) = c_{i,m}(t - t_0)^m$.

Obviously, when $m = 0$, since $x_{i,0}(t)$ satisfy the initial conditions (2), where $x_{i,0}(t)$ are the initial guesses approximations of $x_i(t)$, we have $c_{i,0} = x_{i,0}(t_0) = x_i(t_0), i = 1,2, \dots, n$.

If we choose $x_{i,0}(t) = x_i(t_0)$ as initial guesses approximations of $x_i(t)$, then we can calculate $x_{i,m}(t)$ for $m = 1,2, \dots$ and approximate the solutions $x_i(t)$ of system of IVPs (1) and (2) by the kth-truncated series

$$x_i^k(t) = \sum_{m=0}^{k} c_{i,m}(t - t_0)^m, i = 1,2, \dots, n. \tag{3}$$

Prior to applying the RPS technique, we rewrite system of IVPs (1) and (2) in the form of the following:

$$x_i'(t) - f_i\big(t, x_1(t), x_2(t), \dots, x_n(t)\big) = 0, i = 1,2, \dots, n. \tag{4}$$

The subsisting of kth-truncated series $x_i^k(t)$ into Eq. (4) leads to the following definition for the kth residual functions:

$$\text{Res}_i^k(t) = \sum_{m=1}^{k} m c_{i,m}(t-t_0)^{m-1} - f_i\left(t, \sum_{m=0}^{k} c_{1,m}(t-t_0)^m, \sum_{m=0}^{k} c_{2,m}(t-t_0)^m, \dots, \sum_{m=0}^{k} c_{n,m}(t-t_0)^m\right), i = 1,2,\dots,n, \quad (5)$$

and the following ∞th residual functions:

$$\text{Res}_i^\infty(t) = \lim_{k\to\infty} \text{Res}_i^k(t), i = 1,2,\dots,n.$$

It easy to see that, $\text{Res}_i^\infty(t) = 0$ for each $t \in [t_0, t_0 + a]$. This show that $\text{Res}_i^\infty(t)$ are infinitely many times differentiable at $t = t_0$. On the other hand, $\frac{d^s}{dt^s}\text{Res}_i^\infty(t_0) = \frac{d^s}{dt^s}\text{Res}_i^k(t_0) = 0$, for each $s = 1,2,\dots,k$. In fact, this relation is a fundamental rule in RPS technique and its applications.

Now, in order to obtain the 1st-approximate solutions, we put $k = 1$ and substitute $t = t_0$ into Eq. (5) and using the fact that $\text{Res}_i^\infty(t_0) = \text{Res}_i^1(t_0) = 0$, to conclude

$$c_{i,1} = f_i(t_0, c_{1.0}, c_{2.0}, \dots, c_{n.0}) = f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big), i = 1,2,\dots,n.$$

Thus, using 1st-truncated series the first approximation for system of IVPs (1) and (2) can be written as

$$x_i^1(t) = x_i(t_0) + f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big)(t-t_0), i = 1,2,\dots,n.$$

Similarly, to find the 2nd approximation, we put $k = 2$ and $x_j^2(t) = \sum_{m=0}^{2} c_{j,m}(t-t_0)^m$, $i = 1,2,\dots,n$. On the other hand, we differentiate both sides of Eq. (5) with respect to t and substitute $t = t_0$, to get

$$\frac{d}{dt}\text{Res}_i^2(t_0) = 2c_{i,2} - \frac{\partial}{\partial t} f_i(t_0, c_{1.0}, c_{2.0}, \dots, c_{n.0}) - \sum_{j=1}^{n} c_{j,1}\frac{\partial}{\partial x_j^2} f_i\big(t_0, c_{1,0}, c_{2,0}, \dots, c_{n,0}\big), i = 1,2,\dots,n.$$

In fact $\frac{d}{dt}\text{Res}_i^2(t_0) = \frac{d}{dt}\text{Res}_i^\infty(t_0) = 0$. Thus, we can write

$$c_{i,2} = \frac{1}{2}\left[\frac{\partial}{\partial t} f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big) + \sum_{j=1}^{n} c_{j,1}\frac{\partial}{\partial x_j^2} f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big)\right], i = 1,2,\dots,n.$$

Hence, using 2nd-truncated series the second approximation for system of IVPs (1) and (2) can be written as

$$x_i^2(t) = x_i(t_0) + f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big)(t-t_0) + \frac{1}{2}\left[\frac{\partial}{\partial t} f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big) + \sum_{j=1}^{n} c_{j,1}\frac{\partial}{\partial x_j^2} f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big)\right](t-t_0)^2, i = 1,2,\dots,n.$$

This procedure can be repeated till the arbitrary order coefficients of RPS solutions for system of IVPs (1) and (2) are obtained. Moreover, higher accuracy can be achieved by evaluating more components of the solution. In other words, choose large k in the truncation series (3). The next theorem shows convergence of the RPS technique.

Theorem 2.1. Suppose that $x_i(t), i = 1,2,\dots,n$ are the exact solutions for system of IVPs (1) and (2). Then, the approximate solutions obtained by the RPS technique are just the Taylor expansion of $x_i(t), i = 1,2,\dots,n$.

Proof. Assume that the approximate solutions for system of IVPs (1) and (2) are as follows:

$$\tilde{x}_i(t) = c_{i.0} + c_{i.1}(t-t_0) + c_{i.2}(t-t_0)^2 + \cdots, i = 1,2,\dots,n. \quad (6)$$

In order to prove the theorem, it is enough to show that the coefficients $c_{i.m}$ in Eq. (6) take the form

$$c_{i.m} = \frac{1}{m!} x_i^{(m)}(t_0), i = 1,2,\dots,n, \quad (7)$$

for each $m = 0,1,\dots$, where $x_i(t)$ are the exact solutions for system of IVPs (1) and (2). Clear that for $m = 0$ the initial conditions (2) give

$$c_{i.0} = x_i(t_0), i = 1,2,\dots,n. \quad (8)$$

Moreover, for $m = 1$, substitute $t = t_0$ into Eq. (1), we obtain

$$x_i'(t_0) = f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big), i = 1,2,\dots,n.$$

On the other hand, from Eqs. (6) and (8), we can write

$$\tilde{x}_i(t) = x_i(t_0) + c_{i.1}(t - t_0) + c_{i.2}(t - t_0)^2 + \cdots, i = 1,2, \dots, n, \tag{9}$$

by substituting Eq. (9) into Eq. (1) and then setting $t = t_0$, we get

$$c_{i.1} = f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big) = x_i^{'}(t_0), i = 1,2, \dots, n. \tag{10}$$

Further, for $m = 2$, differentiating both sides of Eq. (1) with respect to t, we obtain

$$x_i^{''}(t) = \frac{\partial}{\partial t} f_i\big(t, x_1(t), x_2(t), \dots, x_n(t)\big) + \sum_{j=1}^{n} x_j^{'}(t) \frac{\partial}{\partial x_j} f_i\big(t, x_1(t), x_2(t), \dots, x_n(t)\big), i = 1,2, \dots, n, \tag{11}$$

by substituting $t = t_0$ in Eq. (11), we can conclude that

$$x_i''(t_0) = \frac{\partial}{\partial t} f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big) + \sum_{j=1}^{n} x_j'(t_0) \frac{\partial}{\partial x_j} f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big), i = 1,2, \dots, n. \tag{12}$$

According to Eqs. (9) and (10), we can write the approximation for system of IVPs (1) and (2) as follows:

$$\tilde{x}_i(t) = x_i(t_0) + x_i^{'}(t_0)(t - t_0) + c_{i.2}(t - t_0)^2 + \cdots, i = 1,2, \dots, n, \tag{13}$$

by substituting Eq. (13) into Eq. (11) and setting $t = t_0$, we obtain

$$2c_{i.2} = \frac{\partial}{\partial t} f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big) + \sum_{j=1}^{n} x_j'(t_0) \frac{\partial}{\partial x_j} f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big), i = 1,2, \dots, n. \tag{14}$$

Finally, by comparing Eqs. (12) and (14), we can conclude that $c_{i.2} = \frac{1}{2} x_i''(t_0), i = 1,2, \dots, n$. By continuing the above procedure, we can easily prove Eq. (7) for $m = 3,4, \dots$. So, the proof of the theorem is complete.

Corollary 2.1. If some of $x_i(t), i = 1,2, \dots, n$ is a polynomial, then the RPS technique will be obtained the exact solution.

It will be convenient to have a notation for the error in the approximation $x_i(t) \approx x_i^k(t)$. Accordingly, we will let $\mathrm{Rem}_i^k(t)$ denote the difference between $x_i(t)$ and its kth Taylor polynomial; that is,

$$\mathrm{Rem}_i^k(t) = x_i(t) - x_i^k(t) = \sum_{m=k+1}^{\infty} \frac{x_i^{(m)}(t_0)}{m!} (t - t_0)^m, i = 1,2, \dots, n.$$

The functions $\mathrm{Rem}_i^k(t)$ are called the kth remainder for the Taylor series of $x_i(t)$. In fact, it often happens that the remainders $\mathrm{Rem}_i^k(t)$ become smaller and smaller, approaching zero, as k gets large.

3. Numerical results and discussion

The proposed method provides an analytical approximate solution in terms of an infinite power series. However, there is a practical need to evaluate this solution, and to obtain numerical values from the infinite power series. The consequent series truncation and the practical procedure are conducted to accomplish this task, transforms the otherwise analytical results into an exact solution, which is evaluated to a finite degree of accuracy. In this section, we consider five examples to demonstrate the performance and efficiency of the present technique. Throughout this paper, all the symbolic and numerical computations performed by using Maple 13 software package.

To show the accuracy of the present method for our problems, we report four types of error. The first one is the residual error, $\mathrm{Res}_i^k(t)$, and defined as

$$\mathrm{Res}_i^k(t) \coloneqq \left| \frac{d}{dt} x_i^k(t) - f_i\left(t, x_1^k(t), x_2^k(t), \dots, x_n^k(t)\right) \right|,$$

while the exact, Ext, relative, Rel, and consecutive, Con, errors are defined, respectively, by

$$\mathrm{Ext}_i^k(t) := \left| x_{i,\mathrm{exact}}(t) - x_i^k(t) \right|,$$

$$\mathrm{Rel}_i^k(t) := \frac{\left| x_{i,\mathrm{exact}}(t) - x_i^k(t) \right|}{\left| x_{i,\mathrm{exact}}(t) \right|},$$

$$\mathrm{Con}_i^k(t) := \left| x_i^{k+1}(t) - x_i^k(t) \right|,$$

for $i = 1,2,\ldots,n$, where $t \in [t_0, t_0 + a]$, x_i^k are the kth-order approximation of $x_{i,\text{exact}}(t)$ obtained by the RPS technique, and $x_{i,\text{exact}}(t)$ are the exact solution.

In most real life situations, the differential equation that models the problem is too complicated to solve exactly, and there is a practical need to approximate the solution. In the next two examples, the exact solutions cannot be found analytically.

Example 3.1. Consider the nonlinear SIR model [29]:

$$
\begin{aligned}
S'(t) &= -\beta S(t)I(t), \\
I'(t) &= \beta S(t)I(t) - \gamma I(t), \\
R'(t) &= \gamma I(t),
\end{aligned} \tag{15}
$$

subject to the initial conditions

$$S(0) = N_S, I(0) = N_I, R(0) = N_R, \tag{16}$$

where β, γ and N_S, N_I, N_R are positive real numbers.

The SIR model is one common epidemiological model for the spread of disease, which consists of a system of three differential equations that describe the changes in the number of susceptible, infected, and recovered individuals in a given population. This was introduced as far back as 1927 by Kermack and McKendrick [30], and despite of its simplicity, it is a good model for many infectious diseases. The reader is asked to refer to [29-37] in order to know more details about mathematical epidemiology, including its history and kinds, basics of SIR epidemic models, method of solutions, and so forth.

As we mentioned earlier, if we select the initial guesses approximations as $S_0(t) = N_S$, $I_0(t) = N_I$, and $R_0(t) = N_R$ then the Taylor series expansions of solutions for Eqs. (15) and (16) are as follows:

$$
\begin{aligned}
S(t) &= \sum_{m=0}^{\infty} c_{1,m} t^m = N_S + c_{1,1}t + c_{1,2}t^2 + c_{1,3}t^3 + \cdots, \\
I(t) &= \sum_{m=0}^{\infty} c_{2,m} t^m = N_I + c_{2,1}t + c_{2,2}t^2 + c_{2,3}t^3 + \cdots, \\
R(t) &= \sum_{m=0}^{\infty} c_{3,m} t^m = N_R + c_{3,1}t + c_{3,2}t^2 + c_{3,3}t^3 + \cdots.
\end{aligned}
$$

According to kth residual functions in Eq. (5), we can write

$$
\begin{aligned}
\text{Res}_S^k(t) &= \sum_{m=1}^{k} m c_{1,m} t^{m-1} - \left[-\beta \left(\sum_{m=0}^{k} c_{1,m} t^m \right) \left(\sum_{m=0}^{k} c_{2,m} t^m \right) \right], \\
\text{Res}_I^k(t) &= \sum_{m=1}^{k} m c_{2,m} t^{m-1} - \left[\beta \left(\sum_{m=0}^{k} c_{1,m} t^m \right) \left(\sum_{m=0}^{k} c_{2,m} t^m \right) - \gamma \sum_{m=0}^{k} c_{2,m} t^m \right], \\
\text{Res}_R^k(t) &= \sum_{m=1}^{k} m c_{3,m} t^{m-1} - \gamma \left[\sum_{m=0}^{k} c_{2,m} t^m \right].
\end{aligned} \tag{17}
$$

In order to find the 1st-approximate solutions, we put $k = 1$ through Eq. (17) and using the fact that $\text{Res}_S^k(0) = \text{Res}_I^k(0) = \text{Res}_R^k(0) = 0$, to conclude

$$
\begin{aligned}
c_{1,1} - [-\beta N_S N_I] &= 0, \\
c_{2,1} - [\beta N_S N_I - \gamma N_I] &= 0, \\
c_{3,1} - [-\gamma N_I] &= 0.
\end{aligned}
$$

Based on the above equations, we can write the first approximations of the RPS solution for Eqs. (15) and (16) as

$$
\begin{aligned}
S^1(t) &= N_S - \beta N_S N_I t, \\
I^1(t) &= N_I + (\beta N_S N_I - \gamma N_I)t, \\
R^1(t) &= N_R + \gamma N_I t.
\end{aligned}
$$

By continuing with the similar fashion, the second approximations of the RPS solution for Eqs. (15) and (16) take the form

$$S^2(t) = N_S - \beta N_S N_I t + c_{1,2}t^2,$$
$$I^2(t) = N_I + (\beta N_S N_I - \gamma N_I)t + c_{2,2}t^2, \quad (18)$$
$$R^2(t) = N_R - \gamma N_I t + c_{3,2}t^2.$$

In order to find the values of the coefficients $c_{1,2}$, $c_{2,2}$, and $c_{3,2}$ in Eq. (18), we put $k = 2$ through Eq. (17) and using the fact that $\frac{d}{dt}\mathrm{Res}_S^2(0) = \frac{d}{dt}\mathrm{Res}_I^2(0) = \frac{d}{dt}\mathrm{Res}_R^2(0) = 0$, to obtain the following results:

$$2c_{1,2} - [-\beta(N_S)(\beta N_S N_I - \gamma N_I) - \beta(-\beta N_S N_I)(N_I)] = 0,$$
$$2c_{2,2} - [\beta(N_S)(\beta N_S N_I - \gamma N_I) + \beta(-\beta N_S N_I)(N_I) - \gamma(-\gamma N_I + \beta N_S N_I)] = 0,$$
$$2c_{3,1} - [\gamma(-\gamma N_I + \beta N_S N_I)] = 0.$$

Based on the above equations, we can write the second approximations of the RPS solution for Eqs. (15) and (16) as follows:

$$S^2(t) = N_S - \beta N_S N_I t + \frac{1}{2}(\beta(N_S)(\gamma N_I - \beta N_S N_I) + \beta^2 N_S N_I^2)t^2,$$
$$I^2(t) = N_I + (\beta N_S N_I - \gamma N_I)t + \frac{1}{2}\big(\beta N_S(\beta N_S N_I - \gamma N_I) - \beta^2 N_S N_I^2 + \gamma(\gamma N_I - \beta N_S N_I)\big)t^2,$$
$$R^2(t) = N_R + \gamma N_I t + \frac{1}{2}\gamma(-\gamma N_I + \beta N_S N_I)t^2.$$

For numerical results, the following values, for parameters, are considered [38]: $N_S = 499$, $N_I = 1$, $N_R = 1$, and $\beta = 0.001$, $\gamma = 0.1$. By continuing with the similar fashion, the 10th-order approximations of the RPS solution for $S(t)$, $I(t)$, and $R(t)$ lead to the following results:

$$S^{10}(t) = 499 - 0.499t - 0.099301t^2 - 0.013099249t^3 - 1.2810842802 \times 10^{-3}t^4 - 9.7848148692 \times 10^{-5}t^5 - 5.9089889702 \times 10^{-6}t^6 - 2.6871034875 \times 10^{-7}t^7 - 6.7536536974 \times 10^{-9}t^8 + 2.6455233662 \times 10^{-10}t^9 + 5.2266673677 \times 10^{-11}t^{10},$$
$$I^{10}(t) = 1 + 0.399t + 0.079351t^2 + 1.0454215667 \times 10^{-2}t^3 + 1.0197288885 \times 10^{-3}t^4 + 7.7453570922 \times 10^{-5}t^5 + 4.618096121476 \times 10^{-6}t^6 + 2.0273754701 \times 10^{-7}t^7 + 4.2194343597 \times 10^{-9}t^8 - 3.1143494062 \times 10^{-10}t^9 - 4.9152324271 \times 10^{-11}t^{10},$$
$$R^{10}(t) = 1 + 0.1t + 0.01995t^2 + 2.6450333333333333333 \times 10^{-3}t^3 + 2.6135539166666666667 \times 10^{-4}t^4 + 2.039457777 \times 10^{-5}t^5 + 1.2908928487055555556 \times 10^{-6}t^6 + 6.5972801735 \times 10^{-8}t^7 + 2.5342193377 \times 10^{-9}t^8 + 4.6882603997 \times 10^{-11}t^9 - 3.1143494062 \times 10^{-12}t^{10}.$$

These results are plotted in Figure 1 for the three components $S(t)$, $I(t)$, $R(t)$, and the summation $S(t) + I(t) + R(t)$, respectively. Figure 1.a illustrates the case when we introduce a small number of infectives $I(0) = 1$ into a susceptible population. An epidemic will occur and the number of infectives increases; the maximum infective population $I_{\max} = 242.11811$ will occur where S has decreased to the value 85.33824. As time goes on ∞ you travel along the curve to the right, eventually approaching $S = 0$ and the disease died out. The epidemic will end as S approaching to 0 with I and R approaching some positive value $I = 3.7283$ and $R = 497.27160$. Meanwhile, the number of immune population increases, but the size of the population over the period of the epidemic is constant and equal to 500 as shown in Figure 1.b.

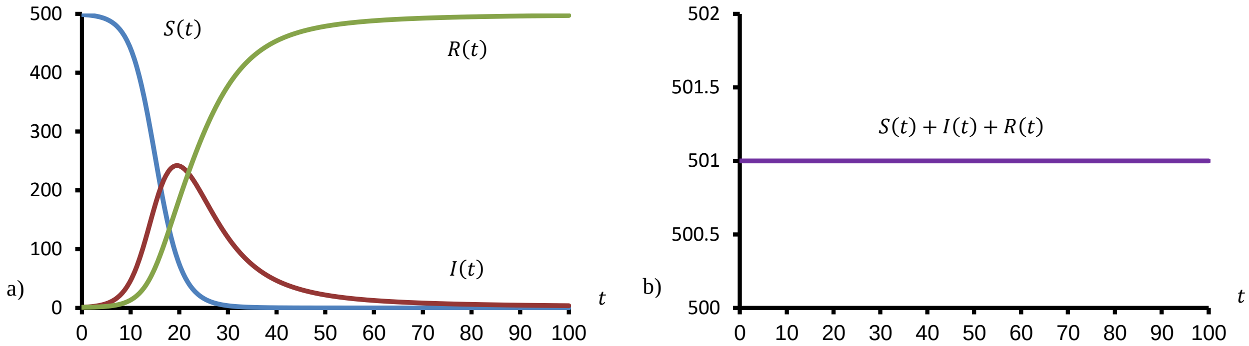

Figure 1. Plots of 50th terms RPS approximations for SIR model (15) and (16): a) $S(t)$, $I(t)$, and $R(t)$ versus time; b) $S(t)+I(t)+R(t)$ versus time.

We mention here that, the RPS solution is the same as the Adomian decomposition solution obtained in [34], the homotopy perturbation solution obtained in [35], variational iteration solution obtained in [36], and the homotopy analysis solution obtained in [37] when $\hbar_i = -1$ and $\mu_i = 1$, $i = 1,2,3$.

Example 3.2. Consider the nonlinear Genesio system [39]:

$$\begin{aligned} x^{'}(t) &= y(t), \\ y^{'}(t) &= z(t), \\ z^{'}(t) &= -cx(t) - by(t) - az(t) + x^2(t), \end{aligned} \tag{19}$$

subject to the initial conditions

$$x(0) = G_x, y(0) = G_y, x(0) = G_z, \tag{20}$$

where a, b, and c are positive real numbers, satisfying $ab < c$.

The Genesio system, proposed by Genesio and Tesi [32], is one of paradigms of chaos since it captures many features of chaotic systems. It includes a simple square part and three simple ordinary differential equations that depend on three positive real parameters. The reader is kindly requested to go through [39-44] in order to know more details about Genesio system, including its history and kinds, method of solutions, its applications, and so forth.

According to RPS technique, the initial guesses approximations of Eqs. (19) and (20) are $x_0(t) = G_x$, $y_0(t) = G_y$, and $z_0(t) = G_z$. Thus, the first few approximations of the RPS solution for Eqs. (19) and (20) are

$$\begin{aligned} x^1(t) &= G_x + G_y t, \\ y^1(t) &= G_y + G_z t, \\ z^1(t) &= G_z - \left(cG_x - (G_x)^2 + aG_z + bG_y\right)t, \\ x^2(t) &= G_x + G_y t + \frac{1}{2} G_z t^2, \\ y^2(t) &= G_y + G_z t - \frac{1}{2}\left(cG_x - (G_x)^2 + aG_z + bG_y\right)t^2, \\ z^2(t) &= G_z - \left(cG_x - (G_x)^2 + aG_z + bG_y\right)t - \frac{1}{2}\left(a\left(cG_x - (G_x)^2 + aG_z + bG_y\right) + 2G_xG_y + aG_z - bG_z - cG_y\right)t^2. \end{aligned}$$

For numerical results, the following values, for parameters, are considered [16]: $G_x = 0.2$, $G_y = -0.3$, $G_z = 0.1$, and $a = 1.2$, $b = 2.92$, $c = 6$. If we collect the above results, then the 10th-order approximations of the RPS solution for $x(t)$, $y(t)$, and $z(t)$ are as follows:

$$\begin{aligned} x^{10}(t) = {} & 0.2 - 0.3t + 0.05t^2 - 6.7333333333 \times 10^{-2} t^3 + 7.8033333333 \times 10^{-2} t^4 - 0.012064 t^5 \\ & - 2.2902222222 \times 10^{-3} t^6 - 6.4525841270 \times 10^{-4} t^7 + 25788923809523809524 \\ & \times 10^{-4} t^8 + 5.6070795062 \times 10^{-5} t^9 - 2.4439052416 \times 10^{-5} t^{10}, \\ y^{10}(t) = {} & -0.3 + 0.1t - 0.202t^2 + 3.1213333333 \times 10^{-1} t^3 - 6.032 \times 10^{-2} t^4 \\ & - 1.3741333333333333333 \times 10^{-2} t^5 - 4.5168088889 \times 10^{-3} t^6 + 2.0631139048 \\ & \times 10^{-3} t^7 + 5.0463715556 \times 10^{-4} t^8 - 2.4439052416 \times 10^{-4} t^9 + 8.4373889295 \times 10^{-5} t^{10}, \end{aligned}$$

$$z^{10}(t) = 0.1 - 0.404t + 0.9364t^2 - 0.24128t^3 - 6.8706666667 \times 10^{-2}t^4 - 2.7100853333 \times 10^{-2}t^5 + 1.4441797333 \times 10^{-2}t^6 + 4.0370972444 \times 10^{-3}t^7 - 2.1995147175 \times 10^{-3}t^8 + 8.4373889295 \times 10^{-4}t^9 - 2.4064938515 \times 10^{-4}t^{10}.$$

While one cannot know the error without knowing the solution, in most cases the consecutive error can be used as a reliable indicator in the iteration progresses. In Tables 1, the value of consecutive error functions $\mathrm{Con}_x^k(t)$, $\mathrm{Con}_y^k(t)$, and $\mathrm{Con}_z^k(t)$ for the two consecutive approximate consecutive solutions has been calculated for various t in $[0,1]$ with step size 0.1 to measure the difference between consecutive solutions obtained from the 10th-order RPS solutions for Eqs. (19) and (20). However, the computational results below provide a numerical estimate for the convergence of the RPS technique. Also, it is clear that the accuracy obtained using present method is in advanced by using only few terms approximations. In addition, we can conclude that higher accuracy can be achieved by evaluating more components of the solution. On the other hand, based on this heuristic, we terminate the iteration in our method.

Table 1: The values of consecutive error function $\mathrm{Con}^k(t)$ when $k = 10$ for different values of t.

t_i	$\mathrm{Con}_x^{10}(t)$	$\mathrm{Con}_y^{10}(t)$	$\mathrm{Con}_z^{10}(t)$
0	0	0	0
0.1	8.32667×10^{-17}	2.22045×10^{-16}	5.55112×10^{-17}
0.2	1.57097×10^{-13}	4.48031×10^{-13}	8.64239×10^{-14}
0.3	1.35878×10^{-11}	3.87549×10^{-11}	7.47563×10^{-12}
0.4	3.21718×10^{-10}	9.17597×10^{-10}	1.77000×10^{-10}
0.5	3.74529×10^{-9}	1.06822×10^{-8}	2.06056×10^{-9}
0.6	2.78278×10^{-8}	7.93699×10^{-8}	1.53101×10^{-8}
0.7	1.51668×10^{-7}	4.32584×10^{-7}	8.34435×10^{-8}
0.8	6.58878×10^{-7}	1.87924×10^{-6}	3.62497×10^{-7}
0.9	2.40704×10^{-6}	6.86530×10^{-6}	1.32429×10^{-6}
1	7.67035×10^{-6}	2.18772×10^{-5}	4.22002×10^{-6}

From the table, it can be seen that the RPS technique provides us with the accurate approximate solution for Eqs. (19) and (20). Also, we can note that the approximate solution more accurate at the beginning values of the independent interval $[0,1]$.

Numerical comparisons are studied next. Figure 2, shows a comparison between the numerical solution of Genesio system for 10th-order RPS approximation together with Runge-Kutta method (RKM) of order four and Predictor-Corrector method (PCM) of order four. Throughout this figure, the step size for the RKM and PCM is fixed at 0.01. The starting values of the PCM obtained from the classical fourth-order RKM. It is demonstrated that the RPS solutions agree very well with the solutions obtained by the RKM and PCM.

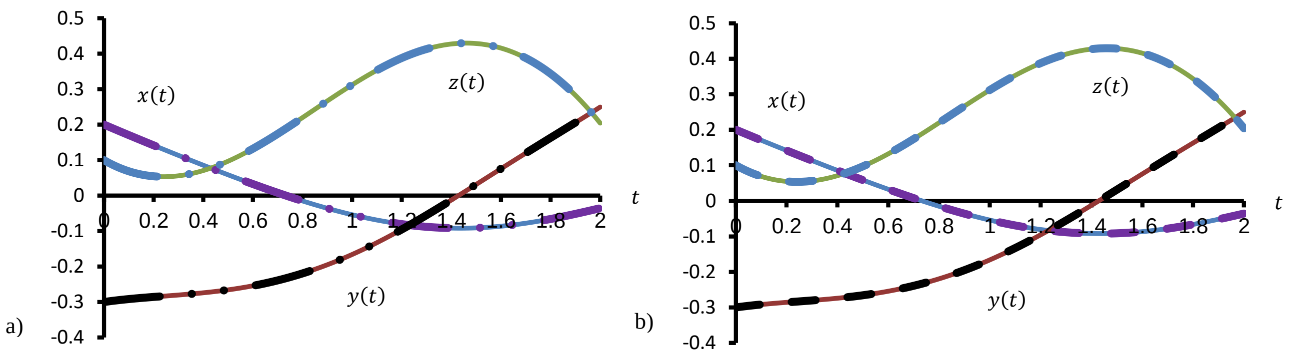

Figure 2. Plots of RPS solution vs. RKM and PCM solutions for Genesio system (19) and (20) versus time: a) solid line: 10th terms RPS approximations, dashed-dot-dotted line: RKM solution; b) solid line: 10th terms RPS approximations, dashed line: PCM solution.

Example 3.3. Consider the nonlinear system of second-order IVP [45]:

$$x_1''(t) = -4t^2 x_1(t) - \frac{2x_2(t)}{\sqrt{x_1^2(t)+x_2^2(t)}},$$
$$x_2''(t) = -4t^2 x_2(t) + \frac{2x_1(t)}{\sqrt{x_1^2(t)+x_2^2(t)}}, \quad (21)$$

subject to the initial conditions

$$x_1(0) = 1, x_1'(0) = 0, x_2(0) = 0, x_2'(0) = 0. \quad (22)$$

As we mentioned earlier, if we select the initial guesses approximations as $x_{1,0}(t) = 1$, $x_{1,1}(t) = 0$, $x_{2,0}(t) = 0$, and $x_{2,1}(t) = 0$, then the first few terms approximations of the RPS solution for Eqs. (21) and (22) are

$$x_{1,2}(t) = 0, x_{1,3}(t) = 0, x_{1,4}(t) = -\frac{1}{2}t^4, x_{1,5}(t) = 0, \dots,$$
$$x_{2,2}(t) = t^2, x_{2,3}(t) = 0, x_{2,4}(t) = 0, x_{2,5}(t) = 0, \dots.$$

If we collect the above results, then the 20th-truncated series of the RPS solution for $x_1(t)$ and $x_2(t)$ are as follows:

$$x_1^{20}(t) = 1 - \frac{1}{2}t^4 + \frac{1}{24}t^8 - \frac{1}{720}t^{12} + \frac{1}{40320}t^{16} - \frac{1}{3628800}t^{20} = \sum_{j=0}^{5}(-1)^j \frac{(t^2)^{2j}}{(2j)!},$$
$$x_2^{20}(t) = \frac{1}{2}t^2 - \frac{1}{6}t^6 + \frac{1}{120}t^{10} - \frac{1}{5040}t^{14} + \frac{1}{362880}t^{18} = \sum_{j=0}^{4}(-1)^j \frac{(t^2)^{1+2j}}{(1+2j)!}.$$

Thus, the exact solutions of Eqs. (21) and (22) have the general form which are coinciding with the exact solutions

$$x_1(t) = \sum_{j=0}^{\infty}(-1)^j \frac{(t^2)^{2j}}{(2j)!} = \cos t^2,$$
$$x_2(t) = \sum_{j=0}^{\infty}(-1)^j \frac{(t^2)^{1+2j}}{(1+2j)!} = \sin t^2.$$

Let us now carry out the error analysis of the RPS technique for this example. Figure 3 shows the exact solution $x_{1,\text{exact}}(t)$, $x_{2,\text{exact}}(t)$ and the four iterates approximations $x_1^k(t)$, $x_2^k(t)$ for $k = 5,10,15,20$. These graphs exhibit the convergence of the approximate solutions to the exact solutions with respect to the order of the solutions.

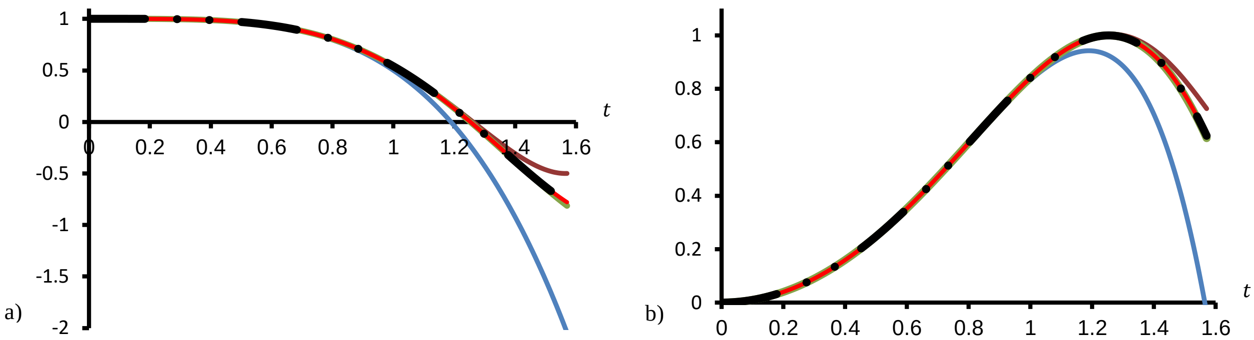

Figure 3. Plots of RPS solution for Eqs. (21) and (22) blue, brown, green, and red solid lines, denote four iterates approximations when $k = 5,10,15,20$, respectively, and black dashed-dot-dotted line, denote exact solution: a) $x_1^k(t)$ and $x_{1,\text{exact}}(t)$, b) $x_2^k(t)$ and $x_{2,\text{exact}}(t)$.

In Figure 4, we plot the error functions $\text{Ext}_1^k(t)$ and $\text{Ext}_2^k(t)$ for $k = 5,10,15,20$ which are approaching the axis $y = 0$ as the number of iterations increase. These graphs show that the exact errors are getting smaller as the order of the solutions is increasing, in other words, as we progress through more iterations. On the other hand,

Figure 5 shows the residual error functions $\mathrm{Res}_1^k(t)$ and $\mathrm{Res}_2^k(t)$ for $k = 5,10,15,20$ for the two consecutive solutions. These error indicators confirm the convergence of the method with respect to the order of the solutions.

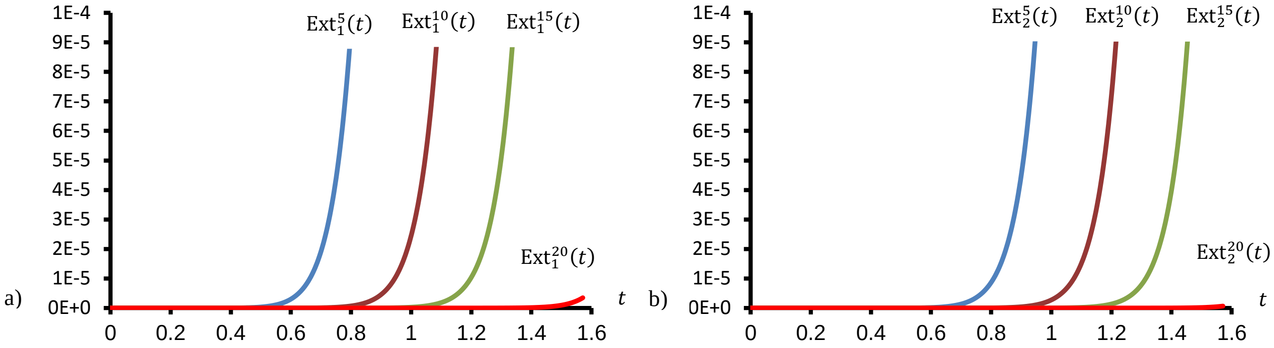

Figure 4. Plots of exact error functions for Eqs. (21) and (22) when $k = 5,10,15,20$: a) $\mathrm{Ext}_1^k(t)$, b) $\mathrm{Ext}_2^k(t)$.

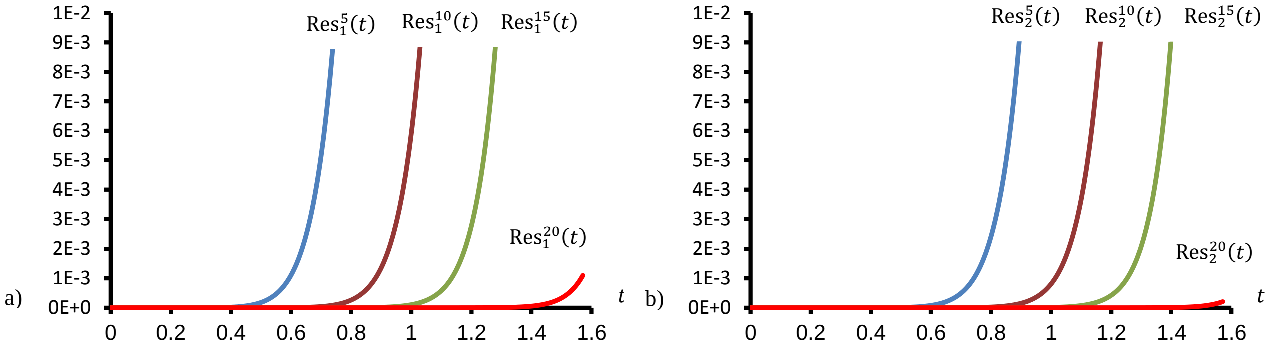

Figure 5. Plots of residual error functions for Eqs. (21) and (22) when $k = 5,10,15,20$: a) $\mathrm{Res}_1^k(t)$, b) $\mathrm{Res}_2^k(t)$.

Example 3.4. Consider the nonlinear system of second-order IVP [46]:

$$x_1''(t) = 1 - \cos t + \sin x_2'(t) + \cos x_2'(t),$$
$$x_2''(t) = \frac{1}{4 + x_1^2(t)} - \frac{5}{5 - \sin^2 t}, \tag{23}$$

subject to the initial conditions

$$x_1(0) = 1, x_1'(0) = 0, x_2(0) = 0, x_2'(0) = \pi. \tag{24}$$

Assuming that the initial guesses approximations have the form $x_{1,0}(t) = 1 + t$ and $x_{2,0}(t) = \pi t$. Then, the 10th-truncated series of the RPS solutions of $x_1(t)$ and $x_2(t)$ for Eqs. (23) and (24) are as follows:

$$x_1^{10}(t) = 1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{720} + \frac{t^8}{40320} - \frac{t^{10}}{3628800} = \sum_{j=0}^{5} (-1)^j \frac{(t)^{2j}}{(2j)!}$$

$$x_1^{10}(t) = \pi t.$$

It easy to see that, the 10th-truncated series of the RPS solutions for $x_1(t)$ and $x_2(t)$ above agree well with the general form

$$x_1(t) = \sum_{j=0}^{\infty} (-1)^j \frac{(t)^{2j}}{(2j)!} = \cos(t),$$

$$x_2(t) = \pi t.$$

So, the exact solutions of Eqs. (23) and (24) will be $x_1(t) = \cos(t)$ and $x_2(t) = \pi t$.

Our next goal is to show how the value of k in the truncation series (3) affects the RPS approximate solutions. To determine this effect an error analysis is performed. We calculate the approximations $x_1^k(t)$ and $x_2^k(t)$ for various k and obtain the exact error functions. The maximum and average errors when $k = 5,10,20$ for Eqs. (23) and (24) have been listed in Table 2 for $t_i = \frac{1}{10} i$, $i = 0,1,2,\dots,10$.

Table 2: The maximum error functions of $x_1(t)$ and $x_2(t)$ when $k = 5,10,15,20$.

Description	$k = 5$	$k = 10$	$k = 15$	$k = 20$
$\max\{\mathrm{Ext}_1^k(t_i)\}$	1.36436×10^{-3}	2.07625×10^{-9}	4.77396×10^{-14}	1.11022×10^{-16}
$\max\{\mathrm{Ext}_2^k(t_i)\}$	0	0	0	0
$\max\{\mathrm{Res}_1^k(t_i)\}$	4.03023×10^{-2}	2.73497×10^{-7}	1.12955×10^{-11}	7.99893×10^{-12}
$\max\{\mathrm{Res}_2^k(t_i)\}$	8.01106×10^{-5}	1.21799×10^{-10}	2.82828×10^{-12}	7.07071×10^{-13}
$\max\{\mathrm{Rel}_1^k(t_i)\}$	2.52518×10^{-3}	3.84276×10^{-9}	8.83572×10^{-14}	2.05483×10^{-16}
$\max\{\mathrm{Rel}_2^k(t_i)\}$	0	0	0	0
$\overline{\mathrm{Ext}_1^k(t_\iota)}$	4.51099×10^{-4}	2.58193×10^{-10}	3.63598×10^{-15}	2.11471×10^{-17}
$\overline{\mathrm{Ext}_2^k(t_\iota)}$	0	0	0	0
$\overline{\mathrm{Res}_1^k(t_\iota)}$	4.89750×10^{-3}	1.94374×10^{-8}	2.08501×10^{-12}	1.46313×10^{-12}
$\overline{\mathrm{Res}_2^k(t_\iota)}$	8.13844×10^{-6}	8.42147×10^{-12}	3.05697×10^{-13}	3.05595×10^{-13}
$\overline{\mathrm{Rel}_1^k(t_\iota)}$	2.15903×10^{-4}	2.39813×10^{-10}	4.99000×10^{-15}	3.09419×10^{-17}
$\overline{\mathrm{Rel}_2^k(t_\iota)}$	0	0	0	0

4. Conclusion

The main concern of this work has been to propose an efficient algorithm for the solutions of system of IVPs. The main goal has been achieved by introducing the RPS technique to solve this class of differential equations. We can conclude that the RPS technique is powerful and efficient technique in finding approximate solution for linear and nonlinear IVPs. The proposed algorithm produced a rapidly convergent series with easily computable components using symbolic computation software. There is an important point to make here, the results obtained by the RPS technique are very effective and convenient in linear and nonlinear cases with less computational work and time. This confirms our belief that the efficiency of our technique gives it much wider applicability for general classes of linear and nonlinear problems.

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