Preprint with edited references section, fully unabbreviated and corrected:
Robert Benjamin Easter and Eckhard Hitzer. *Double Conformal Geometric Algebra.* Advances in Applied Clifford Algebras, first online, 20 April 2017.
DOI: 10.1007/s00006-017-0784-0.

Double Conformal Geometric Algebra

Robert Benjamin Easter and Eckhard Hitzer

Abstract. This paper introduces the Double Conformal / Darboux Cyclide Geometric Algebra (DCGA), based in the $\mathcal{G}_{8,2}$ Clifford geometric algebra. DCGA is an extension of CGA and has entities representing points and general (quartic) Darboux cyclide surfaces in Euclidean 3D space, including circular tori and all quadrics, and all surfaces formed by their inversions in spheres. Dupin cyclides are quartic surfaces formed by inversions in spheres of torus, cylinder, and cone surfaces. Parabolic cyclides are cubic surfaces formed by inversions in spheres that are centered on points of other surfaces. All DCGA entities can be transformed by versors, and reflected in spheres and planes.

Mathematics Subject Classification (2010). 15A66, 53A30, 14J26, 53A05, 51N20, 51K05.

Keywords. Conformal geometric algebra, Darboux Dupin cyclide, Quadric surface.

1. Introduction

This paper introduces an application of the Geometric Algebra $\mathcal{G}_{8,2}$ [12, 13], named in this paper the *Double Conformal / Darboux Cyclide Geometric Algebra* (DCGA) $\mathcal{G}_{8,2}$. The DCGA $\mathcal{G}_{8,2}$ contains two orthogonal subalgebras of the *Conformal Geometric Algebra* (CGA) $\mathcal{G}_{4,1}$. Readers not familiar with CGA should consult some of the following standard references: the first three chapters of [26] by Hestenes, Rockwood and Li, give the first comprehensive introduction to CGA. The book [20], describes the wider context of CGA related to Grassmann algebra, Clifford algebra, and Cayley algebra. The PhD thesis [23] describes applications to pose estimation, see also [16]. And, there are a number of self contained textbooks on the subject [4, 14, 22]. A tutorial introduction to Clifford algebra and CGA is given in [18]. A survey of applications is presented in [19]. We thank the reviewers for pointing out reference [11], which has a related approach[1] limited to the representation of quadrics in geometric algebra based on bivectors in $\mathcal{G}_{4,4}$. By using $\mathcal{G}_{8,2}$, our

[1] We expect that ongoing research on reformulating DCGA will potentially clarify the relationship with the bivector representation in [11] fully.

treatment goes further by including Darboux cyclides, studying intersections and specifying a commutator method for differentiation.

The first $\mathcal{G}_{4,1}$ CGA subalgebra, called CGA1 $\mathcal{C}^1$, is the geometric algebra of the five unit vector elements $\mathbf{e}_i : 1 \leq i \leq 5$, with $\mathbf{e}_i \cdot \mathbf{e}_j = +1$ for $i = j, 1 \leq i \leq 4$, $\mathbf{e}_i \cdot \mathbf{e}_j = -1$ for $i = j = 5$, and zero otherwise. The second $\mathcal{G}_{4,1}$ CGA subalgebra, called CGA2 $\mathcal{C}^2$, is defined similarly with the unit vector elements $\mathbf{e}_i : 6 \leq i \leq 10$. The CGA1 $\mathcal{C}^1$ and CGA2 $\mathcal{C}^2$ unit pseudoscalars are

$$\mathbf{I}_{\mathcal{C}^1} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\mathbf{e}_5, \qquad \mathbf{I}_{\mathcal{C}^2} = \mathbf{e}_6\mathbf{e}_7\mathbf{e}_8\mathbf{e}_9\mathbf{e}_{10}. \tag{1}$$

The $\mathcal{G}_{8,2}$ DCGA metric is therefore

$$m = \operatorname{diag}(1, 1, 1, 1, -1, 1, 1, 1, 1, -1) = [m_{ij}] = [\mathbf{e}_i \cdot \mathbf{e}_j]. \tag{2}$$

The DCGA $\mathcal{D}$ unit pseudoscalar is $\mathbf{I}_{\mathcal{D}} = \mathbf{I}_{\mathcal{C}^1}\mathbf{I}_{\mathcal{C}^2} = \mathbf{I}_{\mathcal{C}^1} \wedge \mathbf{I}_{\mathcal{C}^2}$. For the indexing scheme $\mathbf{X}_{\mathcal{C}^k_i}$, $k \in \{1, 2\}$ indicates an element $\mathbf{X}$ of CGA1 or CGA2, and i is an iterator. A Euclidean 3D vector $\mathbf{p}_{\mathcal{E}^1}$ in the $\mathcal{G}_3$ *Algebra of Physical Space*[2] *1* (APS1) $\mathcal{E}^1$ [12], subalgebra of $\mathcal{G}_{4,1}$ CGA1 $\mathcal{C}^1$, is formed with conventional $(x, y, z) \cong (p_x, p_y, p_z)$ scalar components as

$$\mathbf{p}_{\mathcal{E}^1} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3, \quad \mathbf{p}_{\mathcal{E}^2} = p_x\mathbf{e}_6 + p_y\mathbf{e}_7 + p_z\mathbf{e}_8 \tag{3}$$

and its paired copy or double $\mathbf{p}_{\mathcal{E}^2}$ in the $\mathcal{G}_3$ APS2 $\mathcal{E}^2$ subalgebra of $\mathcal{G}_{4,1}$ CGA2 $\mathcal{C}^2$. The $\mathcal{G}_3$ APS1 $\mathcal{E}^1$ and $\mathcal{G}_3$ APS2 $\mathcal{E}^2$ unit pseudoscalars are $\mathbf{I}_{\mathcal{E}^1} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, and $\mathbf{I}_{\mathcal{E}^2} = \mathbf{e}_6\mathbf{e}_7\mathbf{e}_8$, respectively. CGA1 $\mathcal{C}^1$ and CGA2 $\mathcal{C}^2$ are paired copies or doubles. Any CGA entity, versor, or pseudoscalar $A_{\mathcal{C}^k}$ in CGA1 as $A_{\mathcal{C}^1}$ is paired with identical scalar components on corresponding elements in CGA2 as $A_{\mathcal{C}^2}$, and their geometric or outer product $A_{\mathcal{D}} = A_{\mathcal{C}^1}A_{\mathcal{C}^2} = A_{\mathcal{C}^1} \wedge A_{\mathcal{C}^2}$, is the corresponding doubled DCGA $\mathcal{D}$ entity, versor, or pseudoscalar $A_{\mathcal{D}}$. The doubled DCGA $\mathcal{D}$ entities $\mathbf{X}_{\mathcal{D}} = A_{\mathcal{D}}$ are called the DCGA *standard entities* or *bi-CGA entities* $\mathbf{X}_{\mathcal{D}}$.

The paper is organized as follows. Section 2 reviews the concepts of conformal geometric algebra (CGA), in order to introduce the necessary notation for constructing double CGA (DCGA). Section 3 introduces DCGA, shows how to describe points, form geometric entities of CGA in DCGA by outer products of points, formulates Darboux cyclides as bivectors in DCGA, and describes versor operations, intersections, differentiation operators, and conic sections. We conclude in Section 4. After the references, an appendix is added with more technical details on certain types of Darboux cyclides as bivectors in DCGA.

2. Conformal Geometric Algebra (CGA)

We briefly review the concepts of conformal geometric algebra (CGA), in order to introduce the necessary notation for constructing double CGA (DCGA).

---

[2] The algebra of physical space can be interpreted as the algebra of directions in space. It uses measured directions from the origin to denote locations, as opposed to a separate algebraic concept of points. We do thank the anonymous reviewer for this helpful clarification.

The CGA1 $\mathcal{C}^1$ and CGA2 $\mathcal{C}^2$ entities follow the ordinary $\mathcal{G}_{4,1}$ *Conformal Geometric Algebra*.

2.1. CGA point entity

The $\mathcal{G}_{4,1}$ CGA null 1-vector *point*

$$\mathbf{P}_{\mathcal{C}} = \mathcal{C}(\mathbf{p}_{\mathcal{E}}) = \mathcal{C}_{4,1}(\mathbf{p}_{\mathcal{E}}) = \mathbf{p}_{\mathcal{E}} + \tfrac{1}{2}\mathbf{p}_{\mathcal{E}}^2\mathbf{e}_\infty + \mathbf{e}_o \tag{4}$$

is derived by stereographic embedding and homogenization of the point $\mathbf{p}_{\mathcal{E}}$ [22]. The Euclidean 3D point $\mathbf{p}_{\mathcal{E}}$ in $\mathcal{G}_3$ APS $\mathcal{E}$ is $\mathbf{p}_{\mathcal{E}} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$. In $\mathcal{G}_{4,1}$ CGA, the point at the origin $\mathbf{e}_o$ and at infinity $\mathbf{e}_\infty$ are (without index k for CGA1 or CGA2)

$$\mathbf{e}_o = \tfrac{1}{2}(-\mathbf{e}_4 + \mathbf{e}_5), \qquad \mathbf{e}_\infty = \mathbf{e}_4 + \mathbf{e}_5. \tag{5}$$

In this section, we omit the index k on $\mathbf{e}_o$ and $\mathbf{e}_\infty$ and assume these are CGA points, not DCGA points. The elements $\mathbf{e}_4$ and $\mathbf{e}_5$ are sometimes denoted $\mathbf{e}_+ \cong \mathbf{e}_4$ and $\mathbf{e}_- \cong \mathbf{e}_5$ since $\mathbf{e}_4^2 = +1$ and $\mathbf{e}_5^2 = -1$.

A normalized (*unit scale*) point, with unit scale on the component $\mathbf{e}_o$, is

$$\hat{\mathbf{P}}_{\mathcal{C}} = \mathbf{P}_{\mathcal{C}}/(-\mathbf{P}_{\mathcal{C}} \cdot \mathbf{e}_\infty) = \mathcal{C}(\mathbf{p}_{\mathcal{E}}). \tag{6}$$

The *projection* (inverse embedding) of a point $\mathbf{P}_{\mathcal{C}} = \mathcal{C}(\mathbf{p}_{\mathcal{E}})$ is

$$\mathbf{p}_{\mathcal{E}} = (\hat{\mathbf{P}}_{\mathcal{C}} \cdot \mathbf{I}_{\mathcal{E}})\mathbf{I}_{\mathcal{E}}^{-1}, \tag{7}$$

where the $\mathcal{G}_3$ APS $\mathcal{E}$ unit pseudoscalar is $\mathbf{I}_{\mathcal{E}} = \mathbf{I}_{\mathcal{E}^1}$.

2.2. CGA IPNS surface entities

A CGA point $\mathbf{T}_{\mathcal{C}} = \mathcal{C}(\mathbf{t}_{\mathcal{E}}) = \mathcal{C}_{4,1}(\mathbf{t}_{\mathcal{E}})$ is on a CGA *geometric inner product null space* (IPNS) surface $\mathbf{S}_{\mathcal{C}}$ iff $\mathbf{T}_{\mathcal{C}} \cdot \mathbf{S}_{\mathcal{C}} = 0$ [22]. The CGA IPNS *sphere* vector $\mathbf{S}_{\mathcal{C}}$, centered at CGA point $\hat{\mathbf{P}}_{\mathcal{C}} = \mathcal{C}(\mathbf{p}_{\mathcal{E}})$, with radius r or through CGA point $\hat{\mathbf{Q}}_{\mathcal{C}} = \mathcal{C}(\mathbf{q}_{\mathcal{E}})$, is defined as

$$\mathbf{S}_{\mathcal{C}} = \hat{\mathbf{S}}_{\mathcal{C}} = \hat{\mathbf{P}}_{\mathcal{C}} - \tfrac{1}{2}r^2\mathbf{e}_\infty = \hat{\mathbf{P}}_{\mathcal{C}} + (\hat{\mathbf{Q}}_{\mathcal{C}} \cdot \hat{\mathbf{P}}_{\mathcal{C}})\mathbf{e}_\infty. \tag{8}$$

Formed with normalized points $\hat{\mathbf{P}}_{\mathcal{C}}$ and $\hat{\mathbf{Q}}_{\mathcal{C}}$ (6), the sphere is *unit scale* $\mathbf{S}_{\mathcal{C}} = \hat{\mathbf{S}}_{\mathcal{C}}$, where $\hat{\mathbf{S}}_{\mathcal{C}}^2 = r^2$. The CGA IPNS *plane* vector $\mathbf{\Pi}_{\mathcal{C}}$, normal to unit vector $\hat{\mathbf{n}}_{\mathcal{E}}$, at distance d from the origin or through 3D point $\mathbf{p}_{\mathcal{E}}$, is defined as

$$\mathbf{\Pi}_{\mathcal{C}} = \hat{\mathbf{\Pi}}_{\mathcal{C}} = \hat{\mathbf{n}}_{\mathcal{E}} + d\mathbf{e}_\infty = \hat{\mathbf{n}}_{\mathcal{E}} + (\mathbf{p}_{\mathcal{E}} \cdot \hat{\mathbf{n}}_{\mathcal{E}})\mathbf{e}_\infty. \tag{9}$$

Formed with unit normal vector $\hat{\mathbf{n}}_{\mathcal{E}}$, the plane is *unit scale* $\mathbf{\Pi}_{\mathcal{C}} = \hat{\mathbf{\Pi}}_{\mathcal{C}}$, where $\hat{\mathbf{\Pi}}_{\mathcal{C}}^2 = 1$.

The CGA IPNS 2-blade *line* $\mathbf{L}_{\mathcal{C}}$, in the direction of the unit vector $\hat{\mathbf{d}}_{\mathcal{E}}$, perpendicular to unit bivector $\hat{\mathbf{d}}_{\mathcal{E}}^* = \hat{\mathbf{d}}^{*\mathcal{E}} = \hat{\mathbf{d}}_{\mathcal{E}}/\mathbf{I}_{\mathcal{E}}$, and through Euclidean 3D point $\mathbf{p}_{\mathcal{E}}$, is defined as

$$\mathbf{L}_{\mathcal{C}} = \hat{\mathbf{L}}_{\mathcal{C}} = \hat{\mathbf{d}}_{\mathcal{E}}^* - (\mathbf{p}_{\mathcal{E}} \cdot \hat{\mathbf{d}}_{\mathcal{E}}^*)\mathbf{e}_\infty. \tag{10}$$

Formed with unit direction vector $\hat{\mathbf{d}}_{\mathcal{E}}$, the line is *unit scale* $\mathbf{L}_{\mathcal{C}} = \hat{\mathbf{L}}_{\mathcal{C}}$, where $\hat{\mathbf{L}}_{\mathcal{C}}^2 = -1$. The line $\mathbf{L}_{\mathcal{C}}$ can be factored into the intersection of two planes as $\mathbf{L}_\mathcal{C} = \mathbf{\Pi}_{\mathcal{C}_1} \wedge \mathbf{\Pi}_{\mathcal{C}_2}$. The CGA IPNS 2-blade *circle* $\mathbf{C}_\mathcal{C} = \mathbf{S}_\mathcal{C} \wedge \mathbf{\Pi}_\mathcal{C}$, is the intersection of a sphere $\mathbf{S}_\mathcal{C}$ and plane $\mathbf{\Pi}_\mathcal{C}$.

2.3. CGA OPNS surface entities

A CGA point $\mathbf{T}_\mathcal{C} = \mathcal{C}(\mathbf{t}_\mathcal{E}) = \mathcal{C}_{4,1}(\mathbf{t}_\mathcal{E})$ is on a CGA *geometric outer product null space* (OPNS) surface $\mathbf{S}^*_\mathcal{C} = \mathbf{S}^{*\mathcal{C}}_\mathcal{C} = \mathbf{S}_\mathcal{C}\mathbf{I}^{-1}_\mathcal{C}$ iff $\mathbf{T}_\mathcal{C} \wedge \mathbf{S}^*_\mathcal{C} = 0$ [22], where the $\mathcal{G}_{4,1}$ CGA $\mathcal{C}$ unit pseudoscalar is $\mathbf{I}_\mathcal{C} = \mathbf{I}_{\mathcal{C}^1}$ (1). The CGA *point* null-vector $\mathbf{P}_\mathcal{C} = \mathcal{C}(\mathbf{p}_\mathcal{E})$ (4) has the square $\mathbf{P}^2_\mathcal{C} = 0 = \mathbf{P}_\mathcal{C} \cdot \mathbf{P}_\mathcal{C} + \mathbf{P}_\mathcal{C} \wedge \mathbf{P}_\mathcal{C}$ and is both a IPNS and OPNS point vector entity. The wedge of up to five points forms the various CGA OPNS entities.

The CGA OPNS 2-blade *flat point* $\mathbb{P}^*_\mathcal{C}$ is the wedge of one finite CGA point and point $\mathbf{e}_\infty$: $\mathbb{P}^*_\mathcal{C} = \mathbf{P}_\mathcal{C} \wedge \mathbf{e}_\infty$. The CGA OPNS 2-blade *point pair* $\boldsymbol{PP}^*_\mathcal{C}$ is the wedge of two finite CGA points $\boldsymbol{PP}^*_\mathcal{C} = \mathbf{P}_{\mathcal{C}_-} \wedge \mathbf{P}_{\mathcal{C}_+}$. The *point pair decomposition* [4] gives the two normalized points (6) as

$$\hat{\mathbf{P}}_{\mathcal{C}_\pm} = \left(\boldsymbol{PP}^*_\mathcal{C} \pm \sqrt{(\boldsymbol{PP}^*_\mathcal{C})^2}\right)(-\mathbf{e}_\infty \cdot \boldsymbol{PP}^*_\mathcal{C})^{-1}. \tag{11}$$

The CGA OPNS 3-blade *line* $\mathbf{L}^*_\mathcal{C}$ is the wedge of two CGA points $\mathbf{P}_{\mathcal{C}_i}$ on the line and the point $\mathbf{e}_\infty$

$$\mathbf{L}^*_\mathcal{C} = \mathbf{P}_{\mathcal{C}_1} \wedge \mathbf{P}_{\mathcal{C}_2} \wedge \mathbf{e}_\infty = \mathbf{L}_\mathcal{C}/\mathbf{I}_\mathcal{C} \tag{12}$$

and is the CGA dual of the CGA IPNS 2-blade line $\mathbf{L}_\mathcal{C}$. The CGA OPNS 3-blade *circle* $\mathbf{C}^*_\mathcal{C}$ is the wedge of three CGA points $\mathbf{P}_{\mathcal{C}_i}$ on the circle

$$\mathbf{C}^*_\mathcal{C} = \mathbf{P}_{\mathcal{C}_1} \wedge \mathbf{P}_{\mathcal{C}_2} \wedge \mathbf{P}_{\mathcal{C}_3} = \mathbf{C}_\mathcal{C}/\mathbf{I}_\mathcal{C} \tag{13}$$

and is the CGA dual of the CGA IPNS 2-blade circle $\mathbf{C}_\mathcal{C}$.

The CGA OPNS 4-blade *plane* $\mathbf{\Pi}^*_\mathcal{C}$ is the wedge of three non-collinear CGA points $\mathbf{P}_{\mathcal{C}_i}$ on the plane and the point $\mathbf{e}_\infty$

$$\mathbf{\Pi}^*_\mathcal{C} = \mathbf{P}_{\mathcal{C}_1} \wedge \mathbf{P}_{\mathcal{C}_2} \wedge \mathbf{P}_{\mathcal{C}_3} \wedge \mathbf{e}_\infty = \mathbf{\Pi}_\mathcal{C}/\mathbf{I}_\mathcal{C} \tag{14}$$

and is the CGA dual of CGA IPNS plane vector $\mathbf{\Pi}_\mathcal{C}$. The CGA OPNS 4-blade *sphere* $\mathbf{S}^*_\mathcal{C}$ is the wedge of four non-coplanar CGA points $\mathbf{P}_{\mathcal{C}_i}$ on the sphere

$$\mathbf{S}^*_\mathcal{C} = \mathbf{P}_{\mathcal{C}_1} \wedge \mathbf{P}_{\mathcal{C}_2} \wedge \mathbf{P}_{\mathcal{C}_3} \wedge \mathbf{P}_{\mathcal{C}_4} = \mathbf{S}_\mathcal{C}/\mathbf{I}_\mathcal{C} \tag{15}$$

and is the CGA dual of the CGA IPNS sphere vector $\mathbf{S}_\mathcal{C}$.

2.4. CGA versor operations

The translator $T_\mathcal{C}$, rotor $R_\mathcal{C}$, and dilator $D_\mathcal{C}$ are called even *versors* $V_\mathcal{C}$. Their operation on a CGA entity $\mathbf{X}_\mathcal{C}$ has the form $\mathbf{X}'_\mathcal{C} = V_\mathcal{C}\mathbf{X}_\mathcal{C}V^{-1}_\mathcal{C}$, called a *versor "sandwich" operation*. This section gives the translated forms of the rotor $T_\mathcal{C}R_\mathcal{C}T^{-1}_\mathcal{C}$ and dilator $T_\mathcal{C}D_\mathcal{C}T^{-1}_\mathcal{C}$.

The CGA 2-versor *translator* $T_\mathcal{C}$, for translation by a three-dimensional Euclidean vector $\mathbf{d}_\mathcal{E} = d_x\mathbf{e}_1 + d_y\mathbf{e}_2 + d_z\mathbf{e}_3$, is defined as

$$T_\mathcal{C} = e^{\frac{1}{2}\mathbf{e}_\infty\mathbf{d}_\mathcal{E}} = 1 + \tfrac{1}{2}\mathbf{e}_\infty\mathbf{d}_\mathcal{E}. \tag{16}$$

The translator can also be defined by reflections in two parallel planes as

$$T^2_\mathcal{C} = \hat{\mathbf{\Pi}}_{\mathcal{C}_2}\hat{\mathbf{\Pi}}_{\mathcal{C}_1} = \hat{\mathbf{\Pi}}_{\mathcal{C}_2} \cdot \hat{\mathbf{\Pi}}_{\mathcal{C}_1} + \hat{\mathbf{\Pi}}_{\mathcal{C}_2} \wedge \hat{\mathbf{\Pi}}_{\mathcal{C}_1}, \tag{17}$$

which translates by *twice* the vector displacement $\mathbf{d}_{\mathcal{E}} = (d_2 - d_1)\hat{\mathbf{n}}_{\mathcal{E}}$ between the planes.

The CGA 2-versor *rotor* $R_{\mathcal{C}^k}$ for rotation by right-hand rule by θ radians around the unit vector axis $\hat{\mathbf{n}}_{\mathcal{E}^k}$ through the origin is

$$R_{\mathcal{C}^k} = e^{\frac{1}{2}\theta\hat{\mathbf{n}}^*_{\mathcal{E}^k}} = \cos(\theta/2) + \sin(\theta/2)\hat{\mathbf{n}}^*_{\mathcal{E}^k}, \tag{18}$$

where the $\mathcal{G}_3$ APS $\mathcal{E}^k$ dual of the unit vector $\hat{\mathbf{n}}_{\mathcal{E}^k}$ is

$$\hat{\mathbf{n}}^*_{\mathcal{E}^k} = \hat{\mathbf{n}}_{\mathcal{E}^k}\mathbf{I}^{-1}_{\mathcal{E}^k}, \tag{19}$$

which is the unit bivector that generates rotation around axis $\hat{\mathbf{n}}_{\mathcal{E}^k}$.

The general CGA 2-versor *translated-rotor* $R_{\mathcal{C}}$, for rotation around the normed line $\hat{\mathbf{L}}_{\mathcal{C}}$ (10), which does not need to pass through the origin, by θ radians, is similarly defined as

$$R_{\mathcal{C}} = e^{\frac{1}{2}\theta\hat{\mathbf{L}}_{\mathcal{C}}} = \cos(\theta/2) + \sin(\theta/2)\hat{\mathbf{L}}_{\mathcal{C}}. \tag{20}$$

The rotation is by right-hand rule around the normed line direction $\hat{\mathbf{d}}_{\mathcal{E}}$ by θ radians. The rotor can also be defined by reflections in two general non-parallel planes $\hat{\mathbf{\Pi}}_{\mathcal{C}_i}$ as

$$R^2_{\mathcal{C}} = \hat{\mathbf{\Pi}}_{\mathcal{C}_2}\hat{\mathbf{\Pi}}_{\mathcal{C}_1} = \hat{\mathbf{\Pi}}_{\mathcal{C}_2} \cdot \hat{\mathbf{\Pi}}_{\mathcal{C}_1} + \hat{\mathbf{\Pi}}_{\mathcal{C}_2} \wedge \hat{\mathbf{\Pi}}_{\mathcal{C}_1}, \tag{21}$$

where $\hat{\mathbf{L}}_{\mathcal{C}}$ appears as the intersection line of the planes and θ is the angle between the planes. The successive reflections in two non-parallel planes rotate by *twice* the angle θ subtended between the planes.

The CGA 2-versor *translated-dilator* $D_{\mathcal{C}}$, by factor $d > 0$ centered on $\hat{\mathbf{P}}_{\mathcal{C}} = \mathcal{C}(\mathbf{p}_{\mathcal{E}})$, is defined as

$$D_{\mathcal{C}} = e^{\frac{\ln d}{2}\hat{\mathbf{P}}_{\mathcal{C}}\wedge\mathbf{e}_\infty} = \cosh\tfrac{\ln d}{2} + \sinh(\tfrac{\ln d}{2})\hat{\mathbb{P}}^*_{\mathcal{C}}, \tag{22}$$

where $\hat{\mathbb{P}}^*_{\mathcal{C}} = \hat{\mathbf{P}}_{\mathcal{C}} \wedge \mathbf{e}_\infty$ is a unit scale CGA OPNS 2-blade *flat point*. The dilator $D_{\mathcal{C}}$ can be factored into the product of two *concentric* CGA IPNS sphere vector entities $\mathbf{S}_{\mathcal{C}_i}$ *centered on* $\hat{\mathbf{P}}_{\mathcal{C}}$ as $D_{\mathcal{C}} = \mathbf{S}_{\mathcal{C}_2}\mathbf{S}_{\mathcal{C}_1}$, which dilates by factor $d = r_2^2/r_1^2$, with radius r_1 of $\mathbf{S}_{\mathcal{C}_1}$ and radius r_2 of $\mathbf{S}_{\mathcal{C}_2}$. An alternative form of the dilator, derived from inversions in two concentric spheres (60) as in [22], is

$$D_{\mathcal{C}} = \tfrac{1}{2}(d+1) + \tfrac{1}{2}(d-1)\hat{\mathbb{P}}^*_{\mathcal{C}}, \tag{23}$$

which also allows $d = 0$ on certain finite closed-surface entities that dilate by $d = 0$ into the point $\mathbf{P}_{\mathcal{C}}$.

Any *versor* V operates on any even grade[3] *entity* $\mathbf{X}$ using the *versor "sandwich" operation*

$$\mathbf{X}' = V\mathbf{X}V^{-1}. \tag{24}$$

The even parity CGA 2-versors $V_{\mathcal{C}^k}$ have unimodular exponential forms e^A such that their *reverse* $V^{\sim}_{\mathcal{C}^k}$ equals their inverse $V^{-1}_{\mathcal{C}^k}$. Note that for even parity CGA versors, equation (24) also holds for odd grade multivectors $\mathbf{X}$.

[3]Note that for the case of the doubling (1:1 pairing) procedure of DCGA, and for the exclusive use of bivectors as explained in Section 3.3 for Darboux cyclides, the entity $\mathbf{X}$ in DCGA is of even grade. An exception is explained in Section 3.8.

In general, when an odd parity versor is applied to an odd grade multivector $\mathbf{X}$, a sign change has to be taken into account, like in the simple case of reflecting a vector at a plane or sphere.

A k-versor V [13] can be factored into a product of k vectors as $V = \Pi_{i=1}^{k}\mathbf{a}_i$. The versor operation may be most computationally efficient as the succession of vector reflections $\mathbf{X}' = \mathbf{a}_1(\cdots(\mathbf{a}_k\mathbf{X}\mathbf{a}_k)\cdots)\mathbf{a}_1$, which is an application of the Cartan-Dieudonné theorem on orthogonal transformations. The vectors $\mathbf{a}_i$ are typically the CGA IPNS plane vector $\mathbf{\Pi}_\mathcal{C}$ or sphere $\mathbf{S}_\mathcal{C}$. In general, the associativity of the geometric products in versor transformations can greatly affect the computational efficiency of the operation.

3. Double Conformal Geometric Algebra (DCGA)

$\mathcal{G}_{8,2}$ DCGA extends $\mathcal{G}_{4,1}$ CGA with new DCGA IPNS *bivector entities* for quartic Darboux cyclides $\mathbf{\Omega}$ (including quartic Dupin cyclides $\mathbf{\Phi}$ and tori $\mathbf{O}$, cubic parabolic cyclides $\mathbf{\Psi}$, and general quadric surfaces $\mathbf{Q}$) as linear combinations $\mathbf{\Omega} = \sum \alpha_s T_s$ of the DCGA extraction elements T_s (Table 1). *Doubled* (paired) CGA versors (DCGA versors) can be applied to all DCGA entities, for details see Section 3.4. The new DCGA IPNS bivector entities can be intersected (by wedge product) with doubled CGA IPNS entities (DCGA IPNS entities), but the new DCGA IPNS bivector entities cannot be intersected with each other by means of a meet operator as in CGA [15], since they are generally not represented by blades, compare (39) and its explanation in Section 3.3.

3.1. DCGA point entity

The DCGA *null* 2-blade standard *point* entity $\mathbf{P}_\mathcal{D} = \mathcal{D}(\mathbf{p}_\mathcal{E})$ is the embedding of a vector $\mathbf{p}_{\mathcal{E}^1} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$, and its paired copy $\mathbf{p}_{\mathcal{E}^2} = p_x\mathbf{e}_6 + p_y\mathbf{e}_7 + p_z\mathbf{e}_8$, as

$$\mathbf{P}_\mathcal{D} = \mathcal{C}^1(\mathbf{p}_{\mathcal{E}^1})\mathcal{C}^2(\mathbf{p}_{\mathcal{E}^2}) = \mathcal{C}^1(\mathbf{p}_{\mathcal{E}^1}) \wedge \mathcal{C}^2(\mathbf{p}_{\mathcal{E}^2}) = \mathbf{P}_{\mathcal{C}^1}\mathbf{P}_{\mathcal{C}^2} = \mathbf{P}_{\mathcal{C}^1} \wedge \mathbf{P}_{\mathcal{C}^2}. \tag{25}$$

This is a DCGA standard entity and is a (doubled) paired form of the CGA point

$$\mathbf{P}_{\mathcal{C}^k} = \mathcal{C}^k(\mathbf{p}_{\mathcal{E}^k}) = \mathbf{p}_{\mathcal{E}^k} + (1/2)\mathbf{p}_{\mathcal{E}^k}^2\mathbf{e}_{\infty k} + \mathbf{e}_{ok}, \tag{26}$$

where the CGA1 $\mathcal{C}^1$ and CGA2 $\mathcal{C}^2$ points are given by setting $k = 1$ and $k = 2$, respectively. The CGA1 and CGA2 points at the origin $\mathbf{e}_{ok}$ and at infinity[4] $\mathbf{e}_{\infty k}$, with corresponding index $k = 1, 2$, are

$$\begin{aligned} \mathbf{e}_{o1} &= \tfrac{1}{2}(\mathbf{e}_5 - \mathbf{e}_4), \qquad \mathbf{e}_{\infty 1} = \mathbf{e}_4 + \mathbf{e}_5, \\ \mathbf{e}_{o2} &= \tfrac{1}{2}(\mathbf{e}_{10} - \mathbf{e}_9), \qquad \mathbf{e}_{\infty 2} = \mathbf{e}_9 + \mathbf{e}_{10}. \end{aligned} \tag{27}$$

The DCGA points at the origin and infinity (both without index) are defined as

$$\mathbf{e}_o = \mathbf{e}_{o1}\mathbf{e}_{o2} = \mathbf{e}_{o1} \wedge \mathbf{e}_{o2}, \qquad \mathbf{e}_\infty = \mathbf{e}_{\infty 1}\mathbf{e}_{\infty 2} = \mathbf{e}_{\infty 1} \wedge \mathbf{e}_{\infty 2}. \tag{28}$$

[4]We basically follow the standard definitions in CGA, see (5) and e.g. the Wikipedia entry on CGA, opened 25th Feb. 2017. Our intention is to make the introduction to DCGA as easy as possible, based on standard CGA.

As in CGA, these DCGA points also have the inner product $\mathbf{e}_\infty \cdot \mathbf{e}_o = -1$. All DCGA points are null 2-blades, $\mathbf{P}_\mathcal{D}^2 = 0$. The squared-squared distance d^4 between two DCGA points $\mathbf{P}_{\mathcal{D}_1}$ and $\mathbf{P}_{\mathcal{D}_2}$ is

$$d^4 \quad = \quad -4\mathbf{P}_{\mathcal{D}_1} \cdot \mathbf{P}_{\mathcal{D}_2}. \tag{29}$$

The projection of a DCGA point $\mathbf{P}_\mathcal{D} = \mathcal{D}(\mathbf{p}_\mathcal{E})$ to the embedded $\mathcal{G}_3$ APS1 vector $\mathbf{p}_{\mathcal{E}^1}$ is

$$\mathbf{p}_{\mathcal{E}^1} = \mathcal{D}^{-1}(\mathbf{p}_\mathcal{E}) = \frac{((\mathbf{P}_\mathcal{D} \cdot \mathbf{e}_{\infty 2}) \cdot \mathbf{I}_{\mathcal{E}^1})\mathbf{I}_{\mathcal{E}^1}^{-1}}{-(\mathbf{P}_\mathcal{D} \cdot \mathbf{e}_{\infty 2}) \cdot \mathbf{e}_{\infty 1}} = (\hat{\mathbf{P}}_{\mathcal{C}^1} \cdot \mathbf{I}_{\mathcal{E}^1})\mathbf{I}_{\mathcal{E}^1}^{-1}. \tag{30}$$

The DCGA null 2-blade *test point* $\mathbf{T}_\mathcal{D} = \mathcal{D}(\mathbf{t})$, testing for position on surfaces, is according to (25) the embedding of the symbolic Euclidean 3D *test point vector* $\mathbf{t}_{\mathcal{E}^1} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$, as

$$\mathbf{T}_\mathcal{D} = \mathcal{D}(\mathbf{t}_\mathcal{E}) = \mathcal{C}^1(\mathbf{t}_{\mathcal{E}^1})\mathcal{C}^2(\mathbf{t}_{\mathcal{E}^2}) = \mathcal{C}^1(\mathbf{t}_{\mathcal{E}^1}) \wedge \mathcal{C}^2(\mathbf{t}_{\mathcal{E}^2}). \tag{31}$$

Expanding $\mathbf{T}_\mathcal{D} = \mathcal{D}(\mathbf{t}) = \mathcal{C}^1(\mathbf{t}_{\mathcal{E}^1}) \wedge \mathcal{C}^2(\mathbf{t}_{\mathcal{E}^2})$ shows the 25 scalar components, of which 15 are unique,

$$\begin{aligned}
\mathbf{T}_\mathcal{D} &= (\mathbf{t}_{\mathcal{E}^1} + \tfrac{1}{2}\mathbf{t}^2\mathbf{e}_{\infty 1} + \mathbf{e}_{o1})(\mathbf{t}_{\mathcal{E}^2} + \tfrac{1}{2}\mathbf{t}^2\mathbf{e}_{\infty 2} + \mathbf{e}_{o2}) \qquad (32)\\
&= \mathbf{t}_{\mathcal{E}^1}\mathbf{t}_{\mathcal{E}^2} + \tfrac{1}{2}\mathbf{t}_{\mathcal{E}^1}\mathbf{t}^2\mathbf{e}_{\infty 2} - \tfrac{1}{2}\mathbf{t}_{\mathcal{E}^2}\mathbf{t}^2\mathbf{e}_{\infty 1} + \mathbf{t}_{\mathcal{E}^1}\mathbf{e}_{o2} - \mathbf{t}_{\mathcal{E}^2}\mathbf{e}_{o1}\\
&+ \tfrac{1}{2}\mathbf{t}^2(\mathbf{e}_{\infty 1}\mathbf{e}_{o2} - \mathbf{e}_{\infty 2}\mathbf{e}_{o1}) + \tfrac{1}{4}\mathbf{t}^4\mathbf{e}_{\infty 1}\mathbf{e}_{\infty 2} + \mathbf{e}_{o1}\mathbf{e}_{o2}\\
&= x(\mathbf{e}_1\mathbf{e}_{o2} - \mathbf{e}_6\mathbf{e}_{o1}) + y(\mathbf{e}_2\mathbf{e}_{o2} - \mathbf{e}_7\mathbf{e}_{o1}) + z(\mathbf{e}_3\mathbf{e}_{o2} - \mathbf{e}_8\mathbf{e}_{o1})\\
&+ x^2\mathbf{e}_{16} + y^2\mathbf{e}_{27} + z^2\mathbf{e}_{38}\\
&+ xy(\mathbf{e}_{17} + \mathbf{e}_{26}) + xz(\mathbf{e}_{18} + \mathbf{e}_{36}) + yz(\mathbf{e}_{28} + \mathbf{e}_{37})\\
&+ \tfrac{1}{2}x\mathbf{t}^2(\mathbf{e}_1\mathbf{e}_{\infty 2} - \mathbf{e}_6\mathbf{e}_{\infty 1}) + \tfrac{1}{2}y\mathbf{t}^2(\mathbf{e}_2\mathbf{e}_{\infty 2} - \mathbf{e}_7\mathbf{e}_{\infty 1})\\
&+ \tfrac{1}{2}z\mathbf{t}^2(\mathbf{e}_3\mathbf{e}_{\infty 2} - \mathbf{e}_8\mathbf{e}_{\infty 1})\\
&+ \tfrac{1}{2}\mathbf{t}^2(\mathbf{e}_{\infty 1}\mathbf{e}_{o2} - \mathbf{e}_{\infty 2}\mathbf{e}_{o1}) + \tfrac{1}{4}\mathbf{t}^4\mathbf{e}_{\infty 1}\mathbf{e}_{\infty 2} + \mathbf{e}_{o1}\mathbf{e}_{o2}\\
&= \tfrac{x}{2}(\mathbf{t}^2 - 1)\mathbf{e}_{19} + \tfrac{x}{2}(\mathbf{t}^2 + 1)\mathbf{e}_{1,10} + \tfrac{x}{2}(\mathbf{t}^2 - 1)\mathbf{e}_{46}\\
&+ \tfrac{x}{2}(\mathbf{t}^2 + 1)\mathbf{e}_{56} + \tfrac{y}{2}(\mathbf{t}^2 - 1)\mathbf{e}_{29} + \tfrac{y}{2}(\mathbf{t}^2 + 1)\mathbf{e}_{2,10}\\
&+ \tfrac{y}{2}(\mathbf{t}^2 - 1)\mathbf{e}_{47} + \tfrac{y}{2}(\mathbf{t}^2 + 1)\mathbf{e}_{57} + \tfrac{z}{2}(\mathbf{t}^2 - 1)\mathbf{e}_{39}\\
&+ \tfrac{z}{2}(\mathbf{t}^2 + 1)\mathbf{e}_{3,10} + \tfrac{z}{2}(\mathbf{t}^2 - 1)\mathbf{e}_{48} + \tfrac{z}{2}(\mathbf{t}^2 + 1)\mathbf{e}_{58}\\
&+ xy\mathbf{e}_{17} + xy\mathbf{e}_{26} + yz\mathbf{e}_{28} + yz\mathbf{e}_{37} + xz\mathbf{e}_{18}\\
&+ xz\mathbf{e}_{36} + x^2\mathbf{e}_{16} + y^2\mathbf{e}_{27} + z^2\mathbf{e}_{38} + \tfrac{1}{4}(\mathbf{t}^4 - 1)\mathbf{e}_{4,10}\\
&+ \tfrac{1}{4}(\mathbf{t}^4 - 1)\mathbf{e}_{59} + \tfrac{1}{4}(\mathbf{t}^4 - 2\mathbf{t}^2 + 1)\mathbf{e}_{49} + \tfrac{1}{4}(\mathbf{t}^4 + 2\mathbf{t}^2 + 1)\mathbf{e}_{5,10},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{t} = \mathbf{t}_{\mathcal{E}^1}, &= x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3, \quad \mathbf{t}_{\mathcal{E}^2} = x\mathbf{e}_6 + y\mathbf{e}_7 + z\mathbf{e}_8, \qquad (33)\\
\mathbf{t}^2 = x^2 + y^2 + z^2, &\quad \mathbf{t}^4 = x^4 + y^4 + z^4 + 2x^2y^2 + 2y^2z^2 + 2z^2x^2.
\end{aligned}$$

The vector $\mathbf{t}$ and its DCGA point embedding $\mathbf{T}_\mathcal{D} = \mathcal{D}(\mathbf{t})$ is a *test point* for position on surfaces. The 15 DCGA *extraction operators* (or elements) T_s (Table 1) are defined as inner product operators to extract the 15 unique

$T_x = \frac{1}{2}(\mathbf{e}_1\mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1}\mathbf{e}_6)$	$T_y = \frac{1}{2}(\mathbf{e}_2\mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1}\mathbf{e}_7)$	$T_z = \frac{1}{2}(\mathbf{e}_3\mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1}\mathbf{e}_8)$
$T_{x^2} = \mathbf{e}_6\mathbf{e}_1$	$T_{y^2} = \mathbf{e}_7\mathbf{e}_2$	$T_{z^2} = \mathbf{e}_8\mathbf{e}_3$
$T_{xy} = \frac{1}{2}(\mathbf{e}_7\mathbf{e}_1 + \mathbf{e}_6\mathbf{e}_2)$	$T_{yz} = \frac{1}{2}(\mathbf{e}_7\mathbf{e}_3 + \mathbf{e}_8\mathbf{e}_2)$	$T_{zx} = \frac{1}{2}(\mathbf{e}_8\mathbf{e}_1 + \mathbf{e}_6\mathbf{e}_3)$
$T_{x\mathbf{t}_{\boldsymbol{\mathcal{E}}}^{\mathbf{2}}} = \mathbf{e}_1\mathbf{e}_{o2} + \mathbf{e}_{o1}\mathbf{e}_6$	$T_{y\mathbf{t}_{\boldsymbol{\mathcal{E}}}^{\mathbf{2}}} = \mathbf{e}_2\mathbf{e}_{o2} + \mathbf{e}_{o1}\mathbf{e}_7$	$T_{z\mathbf{t}_{\boldsymbol{\mathcal{E}}}^{\mathbf{2}}} = \mathbf{e}_3\mathbf{e}_{o2} + \mathbf{e}_{o1}\mathbf{e}_8$
$T_1 = -\mathbf{e}_\infty$	$T_{\mathbf{t}_{\boldsymbol{\mathcal{E}}}^{\mathbf{2}}} = \mathbf{e}_{o2}\mathbf{e}_{\infty 1} + \mathbf{e}_{\infty 2}\mathbf{e}_{o1}$	$T_{\mathbf{t}_{\boldsymbol{\mathcal{E}}}^{\mathbf{4}}} = -4\mathbf{e}_o$

TABLE 1. The DCGA bivector *extraction operators* T_s.

$T^x = T_{x\mathbf{t}_{\boldsymbol{\mathcal{E}}}^{\mathbf{2}}}$	$T^y = T_{y\mathbf{t}_{\boldsymbol{\mathcal{E}}}^{\mathbf{2}}}$	$T^z = T_{z\mathbf{t}_{\boldsymbol{\mathcal{E}}}^{\mathbf{2}}}$
$T^{x^2} = -T_{x^2}$	$T^{y^2} = -T_{y^2}$	$T^{z^2} = -T_{z^2}$
$T^{xy} = -2T_{xy}$	$T^{yz} = -2T_{yz}$	$T^{zx} = -2T_{zx}$
$T^{x\mathbf{t}_{\boldsymbol{\mathcal{E}}}^{\mathbf{2}}} = T_x$	$T^{y\mathbf{t}_{\boldsymbol{\mathcal{E}}}^{\mathbf{2}}} = T_y$	$T^{z\mathbf{t}_{\boldsymbol{\mathcal{E}}}^{\mathbf{2}}} = T_z$
$T^1 = \frac{-1}{4}T_{\mathbf{t}_{\boldsymbol{\mathcal{E}}}^{\mathbf{4}}}$	$T^{\mathbf{t}_{\boldsymbol{\mathcal{E}}}^{\mathbf{2}}} = \frac{-1}{2}T_{\mathbf{t}_{\boldsymbol{\mathcal{E}}}^{\mathbf{2}}}$	$T^{\mathbf{t}_{\boldsymbol{\mathcal{E}}}^{\mathbf{4}}} = \frac{-1}{4}T_1$

TABLE 2. Reciprocals of DCGA bivector *extraction operators* T_s.

scalar components s (which also act as indices of T_s) from any DCGA point $\mathbf{T}_{\mathcal{D}}$ as $s = \mathbf{T}_{\mathcal{D}} \cdot T_s = T_s \cdot \mathbf{T}_{\mathcal{D}}$.

Two properties of the extraction operators T_s are

$$\mathbf{e}_\infty \cdot T_s = \begin{cases} 0: & T_s \neq T_{\mathbf{t}_{\mathcal{E}}^4} \\ 4: & T_s = T_{\mathbf{t}_{\mathcal{E}}^4} \end{cases}, \quad \mathbf{e}_o \cdot T_s = \begin{cases} 0: & T_s \neq T_1 \\ 1: & T_s = T_1 \end{cases}. \tag{34}$$

3.2. DCGA OPNS surface entities formed by outer products of DCGA points

Up to five paired DCGA points $\mathbf{P}_{\mathcal{D}}$ (25) can be wedged to form DCGA *geometric outer product null space* (OPNS) surface entities that have the same forms as the CGA OPNS surface entities. The DCGA OPNS 8-blade *sphere* $\mathbf{S}^{*\mathcal{D}}$ is defined as the wedge of four non-coplanar paired DCGA points $\mathbf{P}_{\mathcal{D}_i}$ on the sphere as

$$\mathbf{S}^{*\mathcal{D}} = \mathbf{P}_{\mathcal{D}_1} \wedge \mathbf{P}_{\mathcal{D}_2} \wedge \mathbf{P}_{\mathcal{D}_3} \wedge \mathbf{P}_{\mathcal{D}_4} = \mathbf{S}/\mathbf{I}_{\mathcal{D}}. \tag{35}$$

The DCGA OPNS 8-blade *plane* $\mathbf{\Pi}^{*\mathcal{D}}$ is defined as the wedge of three non-collinear DCGA points $\mathbf{P}_{\mathcal{D}_i}$ on the plane and the DCGA point at infinity $\mathbf{e}_\infty$ as

$$\mathbf{\Pi}^{*\mathcal{D}} = \mathbf{P}_{\mathcal{D}_1} \wedge \mathbf{P}_{\mathcal{D}_2} \wedge \mathbf{P}_{\mathcal{D}_3} \wedge \mathbf{e}_\infty = \mathbf{\Pi}/\mathbf{I}_{\mathcal{D}}. \tag{36}$$

The DCGA OPNS 6-blade *line* $\mathbf{L}^{*\mathcal{D}}$ is defined as the wedge of two DCGA points $\mathbf{P}_{\mathcal{D}_i}$ on the line and the DCGA point at infinity $\mathbf{e}_\infty$ as

$$\mathbf{L}^{*\mathcal{D}} = \mathbf{P}_{\mathcal{D}_1} \wedge \mathbf{P}_{\mathcal{D}_2} \wedge \mathbf{e}_\infty = \mathbf{L}/\mathbf{I}_{\mathcal{D}}. \tag{37}$$

The DCGA OPNS 6-blade *circle* $\mathbf{C}^{*\mathcal{D}}$ is defined as the wedge of three DCGA points $\mathbf{P}_{\mathcal{D}_i}$ on the circle as

$$\mathbf{C}^{*\mathcal{D}} = \mathbf{P}_{\mathcal{D}_1} \wedge \mathbf{P}_{\mathcal{D}_2} \wedge \mathbf{P}_{\mathcal{D}_3} = \mathbf{C}/\mathbf{I}_{\mathcal{D}}. \tag{38}$$

The DCGA OPNS 8-vector Darboux cyclide $\mathbf{\Omega}^{*\mathcal{D}}$ is the DCGA dual of the DCGA IPNS bivector Darboux cyclide $\mathbf{\Omega}$ of (39), i.e. $\mathbf{\Omega}^{*\mathcal{D}} = \mathbf{\Omega}\mathbf{I}_{\mathcal{D}}^{-1}$.

3.3. DCGA IPNS surface entities

A Darboux cyclide (52) is the most general implicit surface $F(x, y, z) = 0$, that can be formed as a general linear combination of the 15 extracted scalars s, constituting a quartic polynomial, see [21]. A general Darboux cyclide can be identified with the *inner product null space* (IPNS) [22] of the bivector Darboux cyclide entity $\mathbf{\Omega}$ (52)(39) in DCGA, which is a linear combination of the 15 bivector extraction elements T_s

$$\mathbf{\Omega} = \sum_s \alpha_s T_s, \tag{39}$$

where the $\alpha_s = T^s \cdot \mathbf{\Omega}$ are 15 real scalar coefficients or components, with the reciprocal bivector operators[5] T^s listed in Table 2, $T^s \cdot T_r = \delta_{s,r}$, $\delta_{r,s}$ the usual Kronecker symbol with $\delta_{s,s} = 1$, $\delta_{s,r} = 0$ for $s \neq r$.

Equation (39) represents an arbitrary linear combination of the 25 mixed bivectors possible in DCGA, where bivectors are constructed with one vector factor from $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{o1}, \mathbf{e}_{\infty 1}\}$ and one vector factor from $\{\mathbf{e}_6, \mathbf{e}_7, \mathbf{e}_8, \mathbf{e}_{o2}, \mathbf{e}_{\infty 2}\}$. Ten bivector pair combinations in Table 1 seem to reduce this arbitrary freedom of linear combination, but one can define more freely e.g. for the x extraction operator $T'_x = \alpha \mathbf{e}_1 \mathbf{e}_{\infty 2} + \beta \mathbf{e}_{\infty 1} \mathbf{e}_6$, $\alpha, \beta \in \mathbb{R}$, which gives $T'_x \cdot \mathbf{T}_{\mathcal{D}} = (\alpha + \beta)x$. Thus no new term in the variables x, y, z is generated, and the factor $(\alpha + \beta)$ is just part of the arbitrary constant $\alpha_x \in \mathbb{R}$ in (39).

Note furthermore, that only five of the 15 extraction operators in Table 1 are blades, the rest are non-simple bivectors, i.e. not blades. Therefore, their linear combinations (39), (40), etc., are not blades. In order to describe quadrics in DCGA, we fundamentally need to go beyond the limitations of representing objects only by blades. While thus, the expression of Darboux cyclides in DCGA begins with a coordinate based transcription, the resulting representation of surfaces by means of bivectors is computationally very versatile. It allows the free intersection of Darboux cyclides (including quadrics) with points, point pairs, lines, planes and spheres. It allows the use of versors for geometric operations of reflections, inversions, motor operators (translations and rotations), and scaling. Though it may be said, that the approach does not extend the elegance of representing objects merely as outer products of points of CGA to Darboux cyclides (though all entities and operations of CGA are fully embedded and operational in DCGA, thus nothing has to be unlearned), in its power of multivector representation of higher order geometric surfaces, DCGA goes far beyond CGA.

Darboux cyclides $\mathbf{\Omega}$ include quartic *Dupin cyclides* $\mathbf{\Phi} \subset \mathbf{\Omega}$, cubic *parabolic cyclides* $\mathbf{\Psi} \subset \mathbf{\Omega}$, and *general quadrics* $\mathbf{Q} \subset \mathbf{\Psi}$ in Euclidean 3D space [24][5].

All DCGA IPNS bivector surface entities that include a term $T_{\mathbf{t}^4_{\mathcal{E}}} = -4\mathbf{e}_o$ do not have $\mathbf{e}_\infty$ as a surface point and are finite closed-surface entities.

[5] Note, that instead of the conventional term reciprocal [3, 17], also the term pseudoinverse is used [6, 22]. Furthermore the notation T_s^+ has also been used [6] for our T^s.

However, all cubic parabolic cyclides $\mathbf{\Psi}$ (57) and quadrics $\mathbf{Q}$ are formed without $T_{\mathbf{t}^4_{\mathcal{E}}}$ and include the surface point $\mathbf{e}_\infty$.

As an *example*, centered at the origin and aligned with the coordinate axis, a DCGA IPNS bivector *ellipsoid* entity $\mathbf{E} \in \mathbf{Q} \subset \mathbf{\Omega}$ (see also (70)) can be formed as

$$\mathbf{E} = T_{x^2}/a^2 + T_{y^2}/b^2 + T_{z^2}/c^2 - T_1. \tag{40}$$

A DCGA point $\mathbf{P}_\mathcal{D}$ is on the ellipsoid $\mathbf{E}$ iff $\mathbf{P}_\mathcal{D} \cdot \mathbf{E} = 0$, or iff $\mathbf{P}_\mathcal{D} \wedge \mathbf{E}^{*\mathcal{D}} = \mathbf{P}_\mathcal{D} \wedge (\mathbf{E}\mathbf{I}_\mathcal{D}^{-1}) = 0$. This DCGA IPNS bivector *ellipsoid* entity includes the surface point $\mathbf{e}_\infty$ as a singular outlier surface point, while otherwise $\mathbf{E}$ is a finite closed-surface entity. The inversion $\mathbf{\Omega} = \mathbf{S}\mathbf{E}\mathbf{S}^\sim$ of the ellipsoid entity $\mathbf{E}$ in a DCGA standard sphere $\mathbf{S}$ (48) creates a Darboux cyclide surface entity $\mathbf{\Omega}$ (52)(39) that is finite and closed, but that has the sphere center point $\mathbf{P}_\mathcal{C} = \mathbf{S}\mathbf{e}_\infty\mathbf{S}^\sim$ as a singular outlier surface point, or *handle point*.

Note, that (40) represents a bivector, that cannot be rewritten as a blade[6]:

$$\mathbf{E} = \tfrac{1}{a^2}\mathbf{e}_{61} + \tfrac{1}{b^2}\mathbf{e}_{72} + \tfrac{1}{c^2}\mathbf{e}_{83} + \mathbf{e}_{\infty 1}\mathbf{e}_{\infty 2}. \tag{41}$$

This clearly shows, in view of the diagonal DCGA metric (2), that a mere multiplicative construction of $\mathbf{E}$ (and other Darboux cyclides represented in DCGA) by means of outer products of points is currently not in view. This counter example further shows, that in general the Darboux cyclides (39) are not simple blades (outer products of vectors). Yet, for some Darboux cyclides there are special parameter choices in (39), which produce blades, but in our experience special parameter choices always seem to only produce entities already described in (single) CGA. For an example see (46).

The DCGA IPNS bivector surface entities are constructed as various linear combinations of the inner product DCGA extraction elements T_s (Table 1) and are various forms of the DCGA IPNS bivector Darboux cyclide surface entity $\mathbf{\Omega}$ (52)(39). The DCGA entities are generally not blades and generally cannot be formed as the wedge of DCGA points.

A DCGA entity has two forms, where the first is its *dual* IPNS form $\mathbf{X}_\mathcal{D} = \mathbf{X}^{*\mathcal{D}}\mathbf{I}_\mathcal{D}$, and the second is its *undual* outer product null space (OPNS) [22] form $\mathbf{X}^{*\mathcal{D}} = \mathbf{X}\mathbf{I}_\mathcal{D}^{-1} = \mathbf{X}/\mathbf{I}_\mathcal{D}$. Division is right-side multiplication of an inverse element. The pseudoscalars are the dualization/undualization operators such that for CGA1, CGA2, and DCGA IPNS entities $\mathbf{X}_{\mathcal{C}^1}$, $\mathbf{X}_{\mathcal{C}^2}$, and $\mathbf{X}_\mathcal{D}$, respectively, their OPNS entities are

$$\mathbf{X}^*_{\mathcal{C}^1} = \mathbf{X}^{*\mathcal{C}^1} = \mathbf{X}_{\mathcal{C}^1}\mathbf{I}^{-1}_{\mathcal{C}^1}, \mathbf{X}^*_{\mathcal{C}^2} = \mathbf{X}^{*\mathcal{C}^2} = \mathbf{X}_{\mathcal{C}^2}\mathbf{I}^{-1}_{\mathcal{C}^2}, \mathbf{X}^*_\mathcal{D} = \mathbf{X}^{*\mathcal{D}} = \mathbf{X}_\mathcal{D}\mathbf{I}^{-1}_\mathcal{D}. \tag{42}$$

Any DCGA surface entity can be rotated, dilated, and translated using the DCGA versors. The reflection and inversion of any DCGA surface entity in any standard DCGA IPNS plane $\mathbf{\Pi}_\mathcal{D} = \mathbf{\Pi}_{\mathcal{C}^1}\mathbf{\Pi}_{\mathcal{C}^2}$ or sphere $\mathbf{S}_\mathcal{D} = \mathbf{S}_{\mathcal{C}^1}\mathbf{S}_{\mathcal{C}^2}$ can equally be used. All DCGA IPNS entities can be intersected (61) with the DCGA IPNS standard entities $\mathbf{X}_\mathcal{D} = A_\mathcal{D}$ (1).

[6]For a general reference to the question of determination of blade character, see [10]. We also note the opinion of Hestenes and Sobczyk in [13], page 30, that geometric algebra enables a direct characterization of blades, without the need for Plücker coordinates.

In the following Subsections, in order to give representative examples for the IPNS bivector representation of surfaces in DCGA, we successively discuss the bivector representations of toroids, ellipsoids, spheres, elliptic paraboloids, ellipses, Darboux cyclides, Dupin cyclides, and parabolic cyclides. This will sufficiently clarify the pattern of bivector surface representations, for the interested reader we therefore give details of the remaining following bivector surface representations in Appendix A: ellipsoid in general location, line, plane, circle, elliptic cylinder, elliptic cone, hyperbolic paraboloid, hyperboloid of one sheet, hyperboloid of two sheets, parabolic cylinder, hyperbolic cylinder, parallel planes pair, non-parallel planes pair, parabola, hyperbola, and horned Dupin cyclide.

3.3.1. DCGA IPNS toroid. The implicit quartic equation for a circular toroid (torus) at the origin and around the z-axis is

$$\mathbf{t}^4 + 2\mathbf{t}^2(R^2 - r^2) + (R^2 - r^2)^2 - 4R^2(x^2 + y^2) \quad = \quad 0, \tag{43}$$

where $\mathbf{t} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ is a test point, R is the major radius of a circle around the origin in the xy-plane, and r is the minor radius. The equation is true if the test point $\mathbf{t}$ is on the torus.

The DCGA IPNS bivector *toroid* surface entity $\mathbf{O}$ is defined in terms of T_s (Table 1) as

$$\mathbf{O} = T_{\mathbf{t}^4} + 2(R^2 - r^2)T_{\mathbf{t}^2} + (R^2 - r^2)^2 T_1 - 4R^2(T_{x^2} + T_{y^2}). \tag{44}$$

A test DCGA point $\mathbf{T}_{\mathcal{D}} = \mathcal{D}(\mathbf{t})$ is on the toroid surface represented by $\mathbf{O}$ iff $\mathbf{T}_{\mathcal{D}} \cdot \mathbf{O} = 0$. Using symbolic mathematics software, such as the *Geometric Algebra Module* [2] for *Sympy* [25] by Alan Bromborsky, the inner product $\mathbf{T}_{\mathcal{D}} \cdot \mathbf{O}$ generates the *scalar* implicit surface function $F(x, y, z)$ of the toroid when $\mathbf{t}$ is a symbolic (x, y, z) vector. The DCGA dual of $\mathbf{O}$ is

$$\mathbf{O}^{*\mathcal{D}} = \mathbf{O}/\mathbf{I}_{\mathcal{D}} = \mathbf{O}\mathbf{I}_{\mathcal{D}}^{-1}. \tag{45}$$

The DCGA OPNS 8-vector *toroid* surface entity is $\mathbf{O}^{*\mathcal{D}}$, where point $\mathbf{T}_{\mathcal{D}}$ is a surface point iff $\mathbf{T}_{\mathcal{D}} \wedge \mathbf{O}^{*\mathcal{D}} = 0$. All DCGA versors are applicable to $\mathbf{O}$ and it can be intersected with DCGA standard entities. With the extraction term $T_{\mathbf{t}^4} = -4\mathbf{e}_o$, the toroid $\mathbf{O}$ is manifestly a closed-surface that does not include $\mathbf{e}_\infty$.

We note, that the bivector $T_{x^2} + T_{y^2}$ in (44) is due to Table 1 manifestly not a blade, and further comparison of (44) and Table 1, clearly shows that in general for major radius $R \neq 0$ the toroid of (44) is not a blade. Only by setting the major radius in (44) to $R = 0$, do we obtain a bi-CGA IPNS 2-blade sphere (48) centered at the origin with radius r,

$$\begin{aligned}\mathbf{O}_{R=0} &= T_{\mathbf{t}^4} - 2r^2 T_{\mathbf{t}^2} + r^4 T_1 \\ &= -4\mathbf{e}_{o1}\mathbf{e}_{o2} - 2r^2(\mathbf{e}_{o2}\mathbf{e}_{\infty1} + \mathbf{e}_{\infty2}\mathbf{e}_{o1}) - r^4\mathbf{e}_{\infty1}\mathbf{e}_{\infty2} \\ &= -4(\mathbf{e}_{o1} - \tfrac{1}{2}r^2\mathbf{e}_{\infty1}) \wedge (\mathbf{e}_{o2} - \tfrac{1}{2}r^2\mathbf{e}_{\infty2}) = -4S.\end{aligned} \tag{46}$$

3.3.2. DCGA IPNS ellipsoid. The ellipsoid at the origin, with principal axis aligned with the coordinate directions $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is given by its DCGA bivector expression in (40). For details of the ellipsoid in general location, see Appendix A.1.

A DCGA 2-blade point $\mathbf{T}_{\mathcal{D}} = \mathcal{D}(\mathbf{t})$ is tested against the DCGA bivector ellipsoid $\mathbf{E}$ by $\mathbf{T}_{\mathcal{D}} \cdot \mathbf{E} < 0$ (inside), $\mathbf{T}_{\mathcal{D}} \cdot \mathbf{E} = 0$ (on ellipsoid), $\mathbf{T}_{\mathcal{D}} \cdot \mathbf{E} > 0$ (outside). These results indicate (similarly Section 3.3.4), that this may lead in DCGA to the definition of an oriented distance of points from an ellipsoid, similar to how the inner product between point and sphere in CGA can be interpreted as tangential distance between point and sphere [4]. But in the framework of DCGA, this currently remains a matter of further investigation, which may be of certain interest in fields like distance geometry [1].

3.3.3. DCGA IPNS sphere. The standard DCGA IPNS bivector *sphere* $\mathbf{S}$ will be defined as a bi-CGA sphere (48), *not* the DCGA IPNS ellipsoid $\mathbf{E}$ with equal semi-diameters $r = r_x = r_y = r_z$. The DCGA IPNS ellipsoid $\mathbf{E}$ with $r = r_x = r_y = r_z$ can be *reformulated* into the DCGA IPNS bivector *ellipsoid-based sphere* entity $\boldsymbol{\Theta}$ as

$$\boldsymbol{\Theta} = -2(p_x T_x + p_y T_y + p_z T_z) + T_{x^2} + T_{y^2} + T_{z^2} + (p_x^2 + p_y^2 + p_z^2 - r^2)T_1. \quad (47)$$

For $r = 0$, the sphere $\boldsymbol{\Theta}$ degenerates into a DCGA *non-null* bivector *point* entity. In the limit $r \to 0$ does $\mathbf{E}$ approach a point $\boldsymbol{\Theta}$ with $r = 0$. The presence of $T_{x^2}+T_{y^2}+T_{z^2}$, shows again (compare (41)), that independent of the value of r, the ellipsoid-based sphere representation (47) is manifestly never a blade.

The DCGA IPNS 2-blade standard *sphere* surface entity $\mathbf{S}$, also being called a bi-CGA IPNS 2-blade sphere, is defined as

$$\mathbf{S} = \mathbf{S}_{\mathcal{C}^1} \wedge \mathbf{S}_{\mathcal{C}^2} = \mathbf{S}_{\mathcal{C}^1}\mathbf{S}_{\mathcal{C}^2} = (\mathbf{P}_{\mathcal{C}^1} - \tfrac{1}{2}r^2\mathbf{e}_{\infty 1})(\mathbf{P}_{\mathcal{C}^2} - \tfrac{1}{2}r^2\mathbf{e}_{\infty 2}). \quad (48)$$

The CGA1 IPNS sphere $\mathbf{S}_{\mathcal{C}^1}$ and the CGA2 IPNS sphere $\mathbf{S}_{\mathcal{C}^2}$, both representing the same sphere with radius r at center position $\mathbf{P}_{\mathcal{D}} = \mathcal{D}(\mathbf{p})$, are wedged to form the DCGA IPNS sphere $\mathbf{S}$. The sphere bivector of equation (48) represents the DCGA dual of the OPNS 8-blade sphere of (35). If $r = 0$, then the sphere is degenerated into a DCGA *null* 2-blade point $\mathbf{P}_{\mathcal{D}} = \mathbf{P}_{\mathcal{C}^1}\mathbf{P}_{\mathcal{C}^2}$ (25) that would represent the center position of the sphere. This form of sphere can be intersected with any DCGA IPNS entity. A sphere that is formed using the DCGA IPNS ellipsoid can only be intersected with bi-CGA IPNS standard entities.

A DCGA 2-blade point $\mathbf{T}_{\mathcal{D}} = \mathcal{D}(\mathbf{t})$ is tested against a DCGA IPNS 2-blade standard sphere $\mathbf{S}$ by projecting both back to CGA1, followed by computing their inner product, which gives the squared tangent distance, and its sign tells whether the point is inside the sphere ($-$ sign), or outside ($+$ sign), respectively.

3.3.4. DCGA IPNS elliptic paraboloid. The elliptic paraboloid has a cone-like shape that opens up or down. The implicit quadric equation of a z-axis aligned elliptic paraboloid is

$$(x - p_x)^2/r_x^2 + (y - p_y)^2/r_y^2 - (z - p_z)/r_z = 0. \quad (49)$$

The DCGA IPNS bivector *z-axis aligned elliptic paraboloid* surface entity $\mathbf{V}^{||z}$ is defined as

$$\mathbf{V}^{||z} = \frac{-2p_x T_x}{r_x^2} + \frac{-2p_y T_y}{r_y^2} + \frac{-T_z}{r_z} + \frac{T_{x^2}}{r_x^2} + \frac{T_{y^2}}{r_y^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_y^2}{r_y^2} + \frac{p_z}{r_z}\right) T_1. \qquad (50)$$

A DCGA 2-blade point $\mathbf{T}_{\mathcal{D}} = \mathcal{D}(\mathbf{t})$ is tested against the DCGA bivector paraboloid $\mathbf{V}$ by $\mathbf{T}_{\mathcal{D}} \cdot \mathbf{V} < 0$ (inside), $\mathbf{T}_{\mathcal{D}} \cdot \mathbf{V} = 0$ (on paraboloid), $\mathbf{T}_{\mathcal{D}} \cdot \mathbf{V} > 0$ (outside). The presence of the term $T_{x^2}/r_x^2 + T_{y^2}/r_y^2$ in (50) shows similar to (41), that the elliptic paraboloid (50) is manifestly not a blade.

3.3.5. DCGA IPNS ellipse. The DCGA IPNS quadvector *xy-plane ellipse* 1D surface entity $\boldsymbol{\epsilon}^{||xy}$ is defined as

$$\boldsymbol{\epsilon}^{||xy} \quad = \quad \mathbf{\Pi}^{z=0} \wedge \mathbf{H}^{||z} \qquad (51)$$

where the DCGA IPNS 2-blade standard *plane* $\mathbf{\Pi}^{z=0}$ is for plane $z = 0$, and the DCGA IPNS bivector *elliptic cylinder* $\mathbf{H}^{||z}$ is as previously defined and directly represents an ellipse in the xy-plane. The invariant test $\mathbf{e}_\infty \cdot \boldsymbol{\epsilon}^{||xy} = 0$ seems to indicate that the ellipse reaches to infinity, but it actually indicates the handle point, already explained in the ellipsoid example of Section 3.3.

Note, that because the factor $\mathbf{H}^{||z}$ also contains the linear combination $T_{x^2}/r_x^2 + T_{y^2}/r_y^2$, and because the xy-plane bivector $\mathbf{\Pi}^{z=0} = \mathbf{e}_3\mathbf{e}_8$, the ellipse $\boldsymbol{\epsilon}^{||xy}$ is again manifestly not a blade.

3.3.6. DCGA IPNS Darboux cyclide. The implicit quartic equation for a *Darboux cyclide* [21] surface is

$$\begin{aligned} &A\mathbf{t}^4 + B\mathbf{t}^2 + Cx\mathbf{t}^2 + Dy\mathbf{t}^2 + Ez\mathbf{t}^2 + Fx^2 + Gy^2 + Hz^2 \\ &\quad + Ixy + Jyz + Kzx + Lx + My + Nz + O = 0 \end{aligned} \qquad (52)$$

where $\mathbf{t} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ is a test point and the $A \ldots O$ are 15 real scalar coefficients. The point $\mathbf{t}$ is on the Darboux cyclide surface if the equation is fulfilled.

The DCGA IPNS bivector *Darboux cyclide* surface entity $\mathbf{\Omega}$ is defined as

$$\begin{aligned} \mathbf{\Omega} =& AT_{\mathbf{t}^4} + BT_{\mathbf{t}^2} + CT_{x\mathbf{t}^2} + DT_{y\mathbf{t}^2} + ET_{z\mathbf{t}^2} + FT_{x^2} + GT_{y^2} + HT_{z^2} \\ &+ IT_{xy} + JT_{yz} + KT_{zx} + LT_x + MT_y + NT_z + OT_1. \end{aligned} \qquad (53)$$

Based on the several counter examples, which we have seen so far in this paper, we note that in general Darboux cyclides $\mathbf{\Omega}$ are not blades.

3.3.7. DCGA IPNS Dupin cyclide. The implicit quartic equation for a Dupin cyclide surface is

$$(\mathbf{t}^2 + (b^2 - \mu^2))^2 - 4(ax - c\mu)^2 - 4b^2y^2 \quad = \quad 0 \qquad (54)$$

where $\mathbf{t} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ is a surface test point. This gives

$$\mathbf{t}^4 + 2\mathbf{t}^2(b^2 - \mu^2) + (b^2 - \mu^2)^2 - 4(a^2x^2 - 2ac\mu x + c^2\mu^2) - 4b^2y^2 = 0. \qquad (55)$$

The DCGA IPNS bivector *Dupin cyclide* surface $\mathbf{\Phi}$ is defined as

$$\mathbf{\Phi} = T_{\mathbf{t}^4} + 2T_{\mathbf{t}^2}(b^2 - \mu^2) - 4a^2T_{x^2} - 4b^2T_{y^2} + 8ac\mu T_x + ((b^2 - \mu^2)^2 - 4c^2\mu^2)T_1.$$

The scalar parameters of the surface are a, b, c, μ, with b always squared. The Dupin cyclide can be described as a surface that envelops a family of spheres defined by two initial spheres $\mathbf{S}_1$ and $\mathbf{S}_2$ of *minor radii* r_1 and r_2, respectively, centered on a circle of *major radius* R around the origin in the xy-plane. To gain a more intuitive expression of the Dupin cyclide equation, we can define the parameters as

$$a = R, \;\; \mu = (1/2)(r_1 + r_2), \;\; c = (1/2)(r_1 - r_2), \;\; b^2 = a^2 - c^2. \tag{56}$$

When $r = r_1 = r_2$, the Dupin cyclide $\mathbf{\Phi}$ is exactly the same entity as the toroid $\mathbf{O}(R, r)$. The center point of $\mathbf{S}_1$ is $\mathcal{D}(-R\mathbf{e}_1)$. The center of $\mathbf{S}_2$ is $\mathcal{D}(R\mathbf{e}_1)$. The center of the ring or spindle hole in $\mathbf{\Phi}$ is $\mathcal{D}(c\mathbf{e}_1)$. The bounding sphere of radius $\mu + R$, that encloses $\mathbf{\Phi}$, is centered at $\mathcal{D}(-c\mathbf{e}_1)$. The entity $\mathbf{\Phi}$ can be a *ring cyclide* $(r_1 + r_2) < 2R$, *spindle cyclide* $(r_1 + r_2) > 2R$, *horn cyclide* $(r_1 + r_2) = 2R$, *ring torus* $(r_1 = r_2) < R$, *spindle torus* $(r_1 = r_2) > R$, or *horn torus* $(r_1 = r_2) = R$. All DCGA versors can be applied to $\mathbf{\Phi}$.

Comparing Table 1 with the above expression $\mathbf{\Phi}$ for the Dupin cyclide, we see that the presence of $-4a^2 T_{x^2} - 4b^2 T_{y^2}$ with general coefficients means, that $\mathbf{\Phi}$ is not a blade. The same conclusion follows from the presence of $8ac\mu T_x$.

3.3.8. DCGA IPNS parabolic cyclide. The DCGA IPNS bivector *parabolic cyclide* cubic surface entity $\mathbf{\Psi}$ can be defined as

$$\begin{aligned} \mathbf{\Psi} \;\; = \;\; & BT_{\mathbf{t}^2} + CT_{x\mathbf{t}^2} + DT_{y\mathbf{t}^2} + ET_{z\mathbf{t}^2} + FT_{x^2} + GT_{y^2} + HT_{z^2} + \\ & IT_{xy} + JT_{yz} + KT_{zx} + LT_x + MT_y + NT_z + OT_1. \end{aligned} \tag{57}$$

All DCGA versor operations can be applied to $\mathbf{\Psi}$, and it can be intersected with the DCGA IPNS standard entities. Without a term in $T_{\mathbf{t}^4} = -4\mathbf{e}_o$, the parabolic cyclide $\mathbf{\Psi}$ has surface point $\mathbf{e}_\infty$ and is generally an open-surface entity. The DCGA IPNS bivector ellipsoid $\mathbf{E}$ is a degenerate parabolic cyclide entity $\mathbf{\Psi}$ that becomes a closed-surface entity with a singular outlier surface point at $\mathbf{e}_\infty$, which actually indicates the handle point, already explained in the ellipsoid example of Section 3.3. An instance of the parabolic cyclide $\mathbf{\Psi}$ can be produced as the inversion $\mathbf{\Psi} = \mathbf{S}\mathbf{\Omega}\mathbf{S}^\sim$ of a DCGA IPNS bivector Darboux cyclide $\mathbf{\Omega}$ in a DCGA IPNS bivector standard sphere $\mathbf{S}$ that is *centered on a surface point* of $\mathbf{\Omega}$.

Since the ellipsoid $\mathbf{E}$ is a special case of a parabolic cyclide $\mathbf{\Psi}$, it follows that parabolic cyclides are in general not blades.

3.4. DCGA versor operations

The DCGA 4-versors $V_{\mathcal{D}} = V_{\mathcal{C}^1} V_{\mathcal{C}^2}$ inherit the property $V_{\mathcal{D}}^{-1} = V_{\mathcal{D}}^\sim = V_{\mathcal{C}^2}^\sim V_{\mathcal{C}^1}^\sim$.

The CGA translator $T_{\mathcal{C}}$, rotor $R_{\mathcal{C}}$, and isotropic dilator $D_{\mathcal{C}}$ have corresponding paired DCGA 4-versors

$$T_{\mathcal{D}} = T_{\mathcal{C}^1} T_{\mathcal{C}^2}, \quad R_{\mathcal{D}} = R_{\mathcal{C}^1} R_{\mathcal{C}^2}, \quad D_{\mathcal{D}} = D_{\mathcal{C}^1} D_{\mathcal{C}^2}, \tag{58}$$

where the translation vector components of $T_{\mathcal{C}^1}$ are the same as for $T_{\mathcal{C}^2}$, the rotation bivector components are the same in $R_{\mathcal{C}^1}$ and $R_{\mathcal{C}^2}$, and the dilation center and scaling factor are the same in $D_{\mathcal{C}^1}$ and $D_{\mathcal{C}^2}$. The geometric products of the CGA1 and CGA2 versors in (58), can be replaced in all three cases by the outer product, due to the complete orthogonality of multivectors in CGA1 and CGA2, e.g. $T_{\mathcal{C}^1}T_{\mathcal{C}^2} = T_{\mathcal{C}^1} \wedge T_{\mathcal{C}^2}$, etc.

The DCGA 2-versor *reflector* $\mathbf{\Pi}$ is the DCGA 2-blade standard plane $\mathbf{\Pi}$ (72). That means for $k = 1, 2$, the CGA IPNS plane vectors $\mathbf{\Pi}_{\mathcal{C}^k}$, normal to $\hat{\mathbf{n}}$ and through $\mathbf{p}$, or at distance d from the origin,

$$\mathbf{\Pi}_{\mathcal{C}^k} = \hat{\mathbf{n}}_{\mathcal{E}^k} + (\mathbf{p}_{\mathcal{E}^k} \cdot \hat{\mathbf{n}}_{\mathcal{E}^k})\mathbf{e}_{\infty k} = \hat{\mathbf{n}}_{\mathcal{E}^k} + d\mathbf{e}_{\infty k} \tag{59}$$

are the CGA 1-versor reflection operators (*reflector*), and lead to the corresponding paired DCGA IPNS 2-blade plane entity $\mathbf{\Pi}_{\mathcal{D}} = \mathbf{\Pi}_{\mathcal{C}^1}\mathbf{\Pi}_{\mathcal{C}^2}$ that acts as the proper DCGA 2-versor reflector.

The DCGA 2-versor *inversor* $\mathbf{S}$ is the DCGA 2-blade standard sphere $\mathbf{S}$ (48). That means for $k = 1, 2$, the CGA IPNS sphere vectors $\mathbf{S}_{\mathcal{C}^k}$, centered at point $\mathbf{P}_{\mathcal{C}^k}$ and through point $\mathbf{Q}_{\mathcal{C}^k}$, or with radius r,

$$\mathbf{S}_{\mathcal{C}^k} = \mathbf{P}_{\mathcal{C}^k} + (\mathbf{Q}_{\mathcal{C}^k} \cdot \mathbf{P}_{\mathcal{C}^k})\mathbf{e}_{\infty k} = \mathbf{P}_{\mathcal{C}^k} - \tfrac{1}{2}r^2\mathbf{e}_{\infty k} \tag{60}$$

are the CGA 1-versor inversion operators (*inversor*), and lead to the corresponding DCGA IPNS sphere 2-blade entity $\mathbf{S}_{\mathcal{D}} = \mathbf{S}_{\mathcal{C}^1}\mathbf{S}_{\mathcal{C}^2}$ that acts as the proper DCGA 2-versor inversor.

The 4-versors T, R, D can be factored into products of $\mathbf{\Pi}$ and $\mathbf{S}$ entities as successive reflections or inversions in two planes or two spheres.

By *versor outermorphism* [22], the DCGA versors (58) for translation $T_{\mathcal{D}}$, rotation $R_{\mathcal{D}}$, and dilation $D_{\mathcal{D}}$ can operate on any DCGA entity $\mathbf{\Upsilon} \in \{A_{\mathcal{D}}, \mathbf{\Omega}, \mathbf{\Omega}\wedge A_{\mathcal{D}}\}$ or its DCGA dual $\mathbf{\Upsilon}^{*\mathcal{D}} = \mathbf{\Upsilon}\mathbf{I}_{\mathcal{D}}^{-1}$. The inversion of any DCGA entity $\mathbf{\Upsilon}$ in a DCGA bivector *standard sphere* $\mathbf{S} = \mathbf{S}_{\mathcal{C}^1}\mathbf{S}_{\mathcal{C}^2}$ is expressed by $\mathbf{S}\mathbf{\Upsilon}\mathbf{S}^{\sim}$. The reflection of any DCGA entity $\mathbf{\Upsilon}$ in a DCGA bivector *standard plane* $\mathbf{\Pi} = \mathbf{\Pi}_{\mathcal{C}^1}\mathbf{\Pi}_{\mathcal{C}^2}$, is expressed by $\mathbf{\Pi}\mathbf{\Upsilon}\mathbf{\Pi}^{\sim}$. The DCGA versors (58) can be factored into inversion and reflection operations, which in turn can be factored into products of vectors.

3.4.1. Discussion of anisotropic dilation. No versor for anisotropic dilation was found in $\mathcal{G}_{8,2}$ DCGA, since geometrically reflections in planes and spheres only map spheres to spheres, and thus cannot produce ellipsoids. However, in $\mathcal{G}_{4,8}$ Double Conformal Space-Time Algebra (DCSTA) [8], or in the similar extension $\mathcal{G}_{8,4}$ Double Conformal "Time-Space Algebra" (DCTSA) that differs only by some $\pm$ sign changes, an anisotropic dilation operation applicable to any bivector quadric entity $\mathbf{Q}$ can be formed using a 4-versor *hyperbolic rotor (boost)* B operation $\mathbf{Q}' = B\mathbf{Q}B^{\sim}$ with imaginary hyperbolic angle (rapidity) $\varphi\sqrt{-1}$ followed by a grade-10 pseudoscalar projection operation $\mathbf{Q}'' = (\mathbf{Q}' \cdot \mathbf{I}_{\mathcal{D}})\mathbf{I}_{\mathcal{D}}^{-1}$ into the $\mathcal{G}_{2,8}$ Double Conformal Space Algebra (DCSA) subalgebra, or into the very similar $\mathcal{G}_{8,2}$ DCGA subalgebra, which has been the subject of this paper.

3.5. DCGA intersection entity

Since any DCGA IPNS *standard* entity $\mathbf{X}_{\mathcal{D}} = A_{\mathcal{D}}$ (1) can be used as reflection or inversion operator on any other DCGA IPNS entity $\mathbf{\Upsilon}$, it can be shown, in particular, that their DCGA IPNS *intersection* entity are obtained by $\mathbf{\Upsilon}\wedge A_{\mathcal{D}}$

with even grade $4 \le g \le 10$. However, the wedge of any two general DCGA IPNS entities $\mathbf{\Upsilon}_1 \wedge \mathbf{\Upsilon}_2$ is geometrically not yet well interpreted. In particular, the wedge of two DCGA IPNS bivector quadrics $\mathbf{Q}_1 \wedge \mathbf{Q}_2$, where we saw that $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are generally not simple blades, can not straightforwardly be interpreted as intersection entity, in analogy to the meet product of CGA [15]. In general, it is possible to intersect by means of the outer product any DCGA IPNS entity $\mathbf{\Upsilon}$ with any DCGA IPNS standard entity $A_{\mathcal{D}}$, including the standard sphere $\mathbf{S}_{\mathcal{D}} = \mathbf{S}_{\mathcal{C}^1}\mathbf{S}_{\mathcal{C}^2}$, plane $\mathbf{\Pi}_{\mathcal{D}} = \mathbf{\Pi}_{\mathcal{C}^1}\mathbf{\Pi}_{\mathcal{C}^2}$, line $\mathbf{L}_{\mathcal{D}} = \mathbf{\Pi}_{\mathcal{D}_1} \wedge \mathbf{\Pi}_{\mathcal{D}_2}$, and circle $\mathbf{C}_{\mathcal{D}} = \mathbf{S}_{\mathcal{D}} \wedge \mathbf{\Pi}_{\mathcal{D}}$, as the DCGA IPNS intersection entity $\mathbf{\Upsilon} \wedge A_{\mathcal{D}}$ (61) of even grade $4 \le g \le 10$.

The DCGA IPNS k-vector *intersection* entity $\mathbf{X}$ of even grade $4 \le k \le 10$ is defined as

$$\mathbf{X}_{\langle 4 \le k \le 10 \rangle} = \begin{cases} \bigwedge_{i=1}^{2 \le n \le 5} \mathbf{B}_i & : \ \mathbf{B}_i \in \mathcal{S} = \{\mathbf{S}, \mathbf{\Pi}\} \\ \mathbf{\Omega} \wedge \bigwedge_{i=1}^{1 \le n \le 4} \mathbf{B}_i & : \ \mathbf{\Omega} \notin \mathcal{S}, \mathbf{B}_i \in \mathcal{S}. \end{cases} \tag{61}$$

The argument for (61) is the following. For $\mathbf{\Omega}$ and $\mathbf{B}_i$, $1 \le i \le n \le 4$, all linearly independent, we have $T_D \cdot \mathbf{X} = 0$, iff separately $T_D \cdot \mathbf{\Omega} = 0$, and $T_D \cdot \mathbf{B}_i = 0$, $1 \le i \le n \le 4$.

3.6. DCGA differential elements

The DCGA bivector *differential elements* are defined as

$$D_x = 2T_x T_{x^2}^{-1}, \quad D_y = 2T_y T_{y^2}^{-1}, \quad D_z = 2T_z T_{z^2}^{-1}. \tag{62}$$

With the commutator product $\times$, a unit magnitude linear combination of the differential elements forms an $\mathbf{n}$-direction derivative operator as

$$\partial_{\mathbf{n}} = \frac{\partial}{\partial \mathbf{n}} = D_{\mathbf{n}} \times = (n_x D_x + n_y D_y + n_z D_z) \times . \tag{63}$$

Any DCGA IPNS bivector entity $\mathbf{\Omega}$ can be differentiated as $\partial_{\mathbf{n}} \mathbf{\Omega} = D_{\mathbf{n}} \times \mathbf{\Omega}$.

For example, we have with (40)

$$D_x \times \mathbf{E} = 2T_x T_{x^2}^{-1} \times \mathbf{E} = 2\frac{T_x}{a^2}, \tag{64}$$

and thus

$$(D_x \times \mathbf{E}) \cdot \mathbf{T}_{\mathcal{D}} = 2\frac{T_x}{a^2} \cdot \mathbf{T}_{\mathcal{D}} = \frac{\partial}{\partial x}(\mathbf{E} \cdot \mathbf{T}_{\mathcal{D}}). \tag{65}$$

The paper [6] offers more details on the differential operators, including pseudo-integral operators.

3.7. DCGA conic section entities

By intersecting any DCGA IPNS bivector *quadric surface* $\mathbf{Q}$ with a DCGA 2-blade *standard plane* $\mathbf{\Pi}$, a DCGA quadvector *conic* entity $\boldsymbol{\kappa}$ is formed. The quadric surface $\mathbf{Q}$ can be a cone $\mathbf{K}$ and *conic sections* can be formed[7]. A conic entity $\boldsymbol{\kappa}$ can be projected orthographically or perspectively onto a

[7] The topic of conics in DCGA would certainly deserve a more detailed study. Due to page limitations of the current paper, we only indicate how conics can be studied in DCGA.

DCGA 2-blade *standard plane* $\mathbf{\Pi}$. The orthographic projection $\boldsymbol{\kappa}_{\text{ortho}}$ of a DCGA IPNS quadvector *conic* entity $\boldsymbol{\kappa} = \mathbf{Q} \wedge \mathbf{\Pi}_{\boldsymbol{\kappa}}$, obtained by application of (61), onto a DCGA IPNS 2-blade *standard plane* $\mathbf{\Pi}$ is defined as

$$\boldsymbol{\kappa}_{\text{ortho}} = (\boldsymbol{\kappa} \cdot \mathbf{\Pi})\mathbf{\Pi}^{-1} \tag{66}$$

which is the algebraic projection of $\boldsymbol{\kappa}$ onto $\mathbf{\Pi}$.

The perspective projection $\boldsymbol{\kappa}_{\text{persp}}$ of a DCGA IPNS quadvector *conic* entity $\boldsymbol{\kappa}$ onto a DCGA IPNS 2-blade *standard plane* $\mathbf{\Pi}$ from the viewpoint $\mathbf{p} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ represented by a DCGA IPNS 2-blade *standard sphere* $\mathbf{S}$ with center $\mathbf{P}_{\mathcal{D}} = \mathcal{D}(\mathbf{p})$ and radius $r = 1$ can be defined as

$$\boldsymbol{\kappa}_{\text{persp}} = (((\boldsymbol{\kappa} \cdot \mathbf{S})\mathbf{S}^{-1}) \cdot \mathbf{S}) \wedge \mathbf{\Pi} = ((\mathbf{S}\boldsymbol{\kappa}\mathbf{S}^{-1}) \cdot \mathbf{S}) \wedge \mathbf{\Pi} = \mathbf{K}_{\mathbf{p}} \wedge \mathbf{\Pi} \tag{67}$$

where $\mathbf{K}_{\mathbf{p}}$ is the DCGA bivector *cone* of the perspective projection with vertex or eye point at $\mathbf{p}$. The radius r of $\mathbf{S}$ is arbitrary, but $r = 1$ is a good choice. For conics in DCGA see also [7, 9].

If one is only interested in conics in two dimensions, then the DCGA $\mathcal{G}_{6,2} = Cl(6,2)$ for the Euclidean plane, would be fully sufficient. The above $\boldsymbol{\kappa} = \mathbf{Q} \wedge \mathbf{\Pi}_{\boldsymbol{\kappa}}$ represents a conic generally oriented and positioned in three-dimensional Euclidean space. And as we just saw above, the perspective projections use the formalism of CGA, since by pairing (doubling) CGA points, lines, circles, planes and spheres in DCGA, operations of CGA with these geometric entities also become available in DCGA.

3.8. DCGA (bi-CGA) IPNS k-blade union entities

The wedge of any CGA1 IPNS r-blade entity $\mathbf{A}_r$ with any other CGA2 IPNS s-blade entity $\mathbf{B}_s$ forms the DCGA (bi-CGA) IPNS ($k = r + s$)-blade *union* entity $\mathbf{U} = \mathbf{A}_r \wedge \mathbf{B}_s$. For $k = 2$, examples include the *toroid* $\mathbf{O}$ with $R = 0$, where each can be written as a linear combination of the extraction elements T_s that factors into the 2-blade wedge of a CGA1 IPNS 1-blade *sphere* $\mathbf{A}_1 = \mathbf{S}_{\mathcal{C}^1_1}$ and a CGA2 IPNS 1-blade *sphere* $\mathbf{B}_1 = \mathbf{S}_{\mathcal{C}^2_2}$; other IPNS 2-blade *union* entity examples are similar.

In general, a linear combination of the extraction elements T_s forms an IPNS *2-vector Darboux cyclide* entity $\mathbf{\Omega}$ that cannot *in general* be factored into a 2-blade except in *specific* cases where $\mathbf{\Omega} = \mathbf{A}_1 \wedge \mathbf{B}_1$. The DCGA (bi-CGA) IPNS $2r$-blade *standard entities* are the union entities $\mathbf{U}$ where $\mathbf{A}_r$ and $\mathbf{B}_r$ are *corresponding entities* of the same grade r in CGA1 and CGA2, respectively; the toroid $\mathbf{O}$ with $R = 0$ forms a DCGA IPNS 2-blade standard sphere $\mathbf{S}$, a union entity representing a sphere implicit surface function F^2 of multiplicity 2. For $2 < k \leq 8$, the IPNS k-blade union entity $\mathbf{U}$ is not a 2-blade nor expressible as a linear combination of the extraction elements T_s, but represents the geometric union of the two geometric surfaces or curves represented by $\mathbf{A}_r$ and $\mathbf{B}_s$.

By outermorphism, any DCGA versor V operates correctly on any union entity[8] $\mathbf{U}$ as $\mathbf{U}' = V\mathbf{U}V^{\sim}$. The Geometric Algebra Computing software *Gaalop* [14] can be used to visualize any DCGA entity, including any DCGA

[8] Note, that for odd k and odd versor parity, a sign change has to be taken into account.

(bi-CGA) IPNS k-blade *union* entity $\mathbf{U}$. DCGA computing and visualization using Gaalop is (due to page limitations) discussed in detail, with code samples, in the DCGA original research preprint working paper [5].

4. Conclusion

We have briefly reviewed the concepts of conformal geometric algebra (CGA), in order to introduce the necessary notation for constructing double CGA (DCGA). Next, we introduced DCGA, showed how to describe points, form geometric entities of CGA in DCGA by outer products of points, formulated Darboux cyclides as bivectors in DCGA, and described versor operations, intersections, differentiation operators, and conic sections. We also provided technical details on certain types of Darboux cyclides as bivectors in DCGA, see Appendix A.

Double Conformal / Darboux Cyclide Geometric Algebra (DCGA) $\mathcal{G}_{8,2}$ is certainly interesting for further future research and for present applications.

We note, that DCGA introduces the algebraic differential operators[9] D_x, D_y, D_z for entity analysis or other purposes. Some applications to distance geometry, for computing distances between points or geometric surfaces, may in the future also become possible using DCGA. Furthermore, DCGA introduces a new representation of conic sections $\boldsymbol{\kappa} = \mathbf{Q} \wedge \mathbf{\Pi}_{\boldsymbol{\kappa}}$ as the intersection of a quadric entity $\mathbf{Q}$ and a plane entity $\mathbf{\Pi}_{\boldsymbol{\kappa}}$ and also provides operations for orthographic and perspective projections of the conics onto a view plane $\mathbf{\Pi}$, which may have computer graphics applications or other computational geometry applications.

An extension of $\mathcal{G}_{8,2}$ DCGA, the Double Conformal "Time-Space" Algebra (DCTSA) $\mathcal{G}_{8,4}$, extends the DCGA spatial entities to entities in spacetime with a new time derivative operator $D_t = D_w/c$ and a new versor B for hyperbolic rotation (boost) that may support spacetime physics, the boost of quadrics into moving quadrics at velocities with corresponding special relativity length contractions, and the anisotropic dilation of quadrics. The $\mathcal{G}_{4,8}$ Double Conformal Space-Time Algebra (DCSTA) [8] is another possible extension. Extension of $\mathcal{G}_{8,2}$ DCGA to a $\mathcal{G}_{12,3}$ *Triple* or $\mathcal{G}_{16,4}$ *Quadruple Conformal Geometric Algebra* may be theoretically feasible, and may allow for general cubic and quartic surface entities in Euclidean 3D space. DCGAs $\mathcal{G}_{2p+2,2q+2}$ for hypersurfaces in $\mathbb{R}^{p,q}$ are also possible.

[9]Due page limitations, we did not elaborate further on the methods and applications of these differential operators. But we think it is an important feature, which needs to be made known.

Appendix A. Detailed DCGA surface representation bivector entities

A.1. DCGA IPNS ellipsoid in general location

The implicit quadric equation for a principal axes-aligned ellipsoid is

$$(x-p_x)^2/r_x^2 + (y-p_y)^2/r_y^2 + (z-p_z)^2/r_z^2 - 1 = 0 \tag{68}$$

where $\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$ is the position (shifted origin or center) of the ellipsoid, and r_x, r_y, r_z are the semi-diameters (often denoted a, b, c). Expanding the squares, the equation can be written as

$$\frac{-2p_x x}{r_x^2} + \frac{-2p_y y}{r_y^2} + \frac{-2p_z z}{r_z^2} + \left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} + \frac{z^2}{r_z^2}\right) + \left(\frac{p_x^2}{r_x^2} + \frac{p_y^2}{r_y^2} + \frac{p_z^2}{r_z^2} - 1\right) = 0. \tag{69}$$

The DCGA IPNS bivector *ellipsoid* surface entity $\mathbf{E}$ is defined in terms of T_s (Table 1) as

$$\mathbf{E} = -\frac{2p_x T_x}{r_x^2} - \frac{2p_y T_y}{r_y^2} - \frac{2p_z T_z}{r_z^2} + \frac{T_{x^2}}{r_x^2} + \frac{T_{y^2}}{r_y^2} + \frac{T_{z^2}}{r_z^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_y^2}{r_y^2} + \frac{p_z^2}{r_z^2} - 1\right) T_1. \tag{70}$$

An ellipsoid in general orientation is obtained by applying the rotor sandwich operation of Section 3.4 to $\mathbf{E}$.

A.2. DCGA IPNS line

The DCGA IPNS 4-blade standard *line* 1D surface entity $\mathbf{L} = \mathbf{L}_{\mathcal{C}^1} \wedge \mathbf{L}_{\mathcal{C}^2} = \mathbf{L}_{\mathcal{C}^1}\mathbf{L}_{\mathcal{C}^2}$ is defined as

$$\mathbf{L} = (\hat{\mathbf{d}}^*_{\mathcal{E}^1} - (\mathbf{p}_{\mathcal{E}^1} \cdot \hat{\mathbf{d}}^*_{\mathcal{E}^1})\mathbf{e}_{\infty 1})(\hat{\mathbf{d}}^*_{\mathcal{E}^2} - (\mathbf{p}_{\mathcal{E}^2} \cdot \hat{\mathbf{d}}^*_{\mathcal{E}^2})\mathbf{e}_{\infty 2}). \tag{71}$$

This is the wedge of the line as represented in CGA1 with the same line as represented in CGA2. It could also be called a bi-CGA IPNS line entity. The unit vector $\hat{\mathbf{d}}_{\mathcal{E}^k}$ is the direction of the line, and $\mathbf{p}_{\mathcal{E}^k}$ is any point on the line. The unit bivector $\hat{\mathbf{d}}^*_{\mathcal{E}^k} = \hat{\mathbf{d}}_{\mathcal{E}^k}\mathbf{I}^{-1}_{\mathcal{E}^k}$ is orthogonal to the line.

A.3. DCGA IPNS plane and line

The DCGA IPNS 2-blade standard *plane* surface entity $\mathbf{\Pi}$ is defined as

$$\mathbf{\Pi} = \mathbf{\Pi}_{\mathcal{C}^1} \wedge \mathbf{\Pi}_{\mathcal{C}^2} = \mathbf{\Pi}_{\mathcal{C}^1}\mathbf{\Pi}_{\mathcal{C}^2} = (\hat{\mathbf{n}}_{\mathcal{E}^1} + d\mathbf{e}_{\infty 1})(\hat{\mathbf{n}}_{\mathcal{E}^2} + d\mathbf{e}_{\infty 2}). \tag{72}$$

This is the wedge of the plane as represented in CGA1 with the same plane as represented in CGA2. It could also be called a bi-CGA IPNS plane entity. The vector $\hat{\mathbf{n}}_{\mathcal{E}^k}$ is a unit vector perpendicular (normal) to the plane, and the scalar d is the signed distance of the plane from the origin toward $\hat{\mathbf{n}}_{\mathcal{E}^k}$. The plane bivector of equation (72) represents the DCGA dual of the OPNS 8-blade plane of (36).

The DCGA IPNS line $\mathbf{L}$ can also be defined as the intersection of two DCGA IPNS planes by

$$\mathbf{L} = \mathbf{\Pi}_1 \wedge \mathbf{\Pi}_2. \tag{73}$$

The line 4-blade of equation (73) represents the DCGA dual of the OPNS 6-blade line of (37).

A.4. DCGA IPNS circle

A circle is the intersection of a sphere and plane. We can intersect a DCGA IPNS 2-blade standard plane $\mathbf{\Pi}$ with either a DCGA IPNS 2-blade standard sphere $\mathbf{S}$ or with a spherical DCGA IPNS bivector ellipsoid $\mathbf{E}$ and get two different IPNS quadvector circle entities. The first can be intersected again with any other DCGA IPNS entity, but the second can only be intersected again with another DCGA IPNS standard entity.

The DCGA IPNS 4-blade standard *circle* 1D surface entity $\mathbf{C}$ is defined by

$$\mathbf{C} = \mathbf{S} \wedge \mathbf{\Pi} = \mathbf{C}_{\mathcal{C}^1} \wedge \mathbf{C}_{\mathcal{C}^2}. \tag{74}$$

The circle 4-blade of equation (74) represents the DCGA dual of the OPNS 6-blade circle of (38).

A.5. DCGA IPNS elliptic cylinder

An axes-aligned elliptic cylinder is the limit of an ellipsoid as one of the semi-diameters approaches ∞. The limit eliminates the terms of the cylinder axis from the implicit ellipsoid equation. The z-axis aligned cylinder takes $r_z \to \infty$, reducing the ellipsoid equation to

$$(x - p_x)^2/r_x^2 + (y - p_y)^2/r_y^2 - 1 = 0 \tag{75}$$

where $\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$ is the position of the related ellipsoid, and r_x, r_y, r_z are semi-diameters.

The DCGA IPNS bivector *z-axis aligned cylinder* surface entity $\mathbf{H}^{||z}$ is defined as

$$\mathbf{H}^{||z} = \tfrac{-2p_xT_x}{r_x^2} + \tfrac{-2p_yT_y}{r_y^2} + \tfrac{T_{x^2}}{r_x^2} + \tfrac{T_{y^2}}{r_y^2} + \left(\tfrac{p_x^2}{r_x^2} + \tfrac{p_y^2}{r_y^2} - 1\right)T_1. \tag{76}$$

The presence of the term $T_{x^2}/r_x^2 + T_{y^2}/r_y^2$ means that the elliptic cylinder $\mathbf{H}^{||z}$, located at the origin ($p_x = p_y = 0$), is manifestly not a blade. The application of a translation versor, to move the cylinder into general position, cannot change this property.

A.6. DCGA IPNS elliptic cone

An axis-aligned elliptic cone is an axis-aligned cylinder that is linearly scaled along the axis. The implicit quadric equation for an z-axis aligned cone is

$$(x - p_x)^2/r_x^2 + (y - p_y)^2/r_y^2 - (z - p_z)^2/r_z^2 = 0 \tag{77}$$

where $\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$ is the position of the cone apex, and r_x, r_y, r_z are semi-diameters. The DCGA IPNS bivector *z-axis aligned elliptic cone* surface entity $\mathbf{K}^{||z}$ is defined as

$$\mathbf{K}^{||z} = 2\left(\tfrac{p_zT_z}{r_z^2} - \tfrac{p_yT_y}{r_y^2} - \tfrac{p_xT_x}{r_x^2}\right) + \tfrac{T_{x^2}}{r_x^2} + \tfrac{T_{y^2}}{r_y^2} - \tfrac{T_{z^2}}{r_z^2} + \left(\tfrac{p_x^2}{r_x^2} + \tfrac{p_y^2}{r_y^2} - \tfrac{p_z^2}{r_z^2}\right)T_1. \tag{78}$$

The presence of the term $T_{x^2}/r_x^2 + T_{y^2}/r_y^2 - T_{z^2}/r_z^2$ means that the elliptic cone $\mathbf{K}^{||z}$ is manifestly not a blade.

A.7. DCGA IPNS hyperbolic paraboloid

The hyperbolic paraboloid has a saddle-like shape. Its z-axis aligned implicit quadric equation is

$$x^2/r_x^2 - y^2/r_y^2 - z/r_z = 0. \tag{79}$$

The DCGA IPNS bivector *z-axis aligned hyperbolic paraboloid* surface entity $\mathbf{M}$ is defined as

$$\mathbf{M} \quad = \quad T_{x^2}/r_x^2 - T_{y^2}/r_y^2 - T_z/r_z. \tag{80}$$

The presence of the term $T_{x^2}/r_x^2 - T_{y^2}/r_y^2$ means that the hyperbolic paraboloid $\mathbf{M}$ is manifestly not a blade.

A.8. DCGA IPNS hyperboloid of one sheet

The hyperboloid of one sheet has an hourglass-like shape. Its z-axis aligned implicit equation is

$$x^2/r_x^2 + y^2/r_y^2 - z^2/r_z^2 - 1 = 0. \tag{81}$$

The DCGA IPNS bivector *z-axis aligned hyperboloid of one sheet* surface entity $\boldsymbol{\Sigma}$ is defined as

$$\boldsymbol{\Sigma} \quad = \quad T_{x^2}/r_x^2 + T_{y^2}/r_y^2 - T_{z^2}/r_z^2 - T_1. \tag{82}$$

The presence of the term $T_{x^2}/r_x^2 + T_{y^2}/r_y^2 - T_{z^2}/r_z^2$ means that the hyperboloid of one sheet $\boldsymbol{\Sigma}$ is manifestly not a blade.

A.9. DCGA IPNS hyperboloid of two sheets

The hyperboloid of two sheets has two dish-like sheets. Its z-axis aligned implicit quadric equation is

$$-x^2/r_x^2 - y^2/r_y^2 + z^2/r_z^2 - 1 = 0. \tag{83}$$

The DCGA IPNS bivector *z-axis aligned hyperboloid of two sheets* surface entity $\boldsymbol{\Xi}$ is defined as

$$\boldsymbol{\Xi} \quad = \quad -T_{x^2}/r_x^2 - T_{y^2}/r_y^2 + T_{z^2}/r_z^2 - T_1. \tag{84}$$

The presence of the term $-T_{x^2}/r_x^2 - T_{y^2}/r_y^2 + T_{z^2}/r_z^2$ means that the hyperboloid of two sheets $\boldsymbol{\Xi}$ is manifestly not a blade.

A.10. DCGA IPNS parabolic cylinder

The implicit quadric equation for a z-axis aligned parabolic cylinder is

$$\frac{x^2}{r_x^2} - \frac{y}{r_y} = 0. \tag{85}$$

The DCGA IPNS bivector *z-axis aligned parabolic cylinder* surface entity $\mathbf{B}^{||z}$ is defined as

$$\mathbf{B}^{||z} \quad = \quad \frac{T_{x^2}}{r_x^2} - \frac{T_y}{r_y}. \tag{86}$$

Inspection of the bivectors T_{x^2}/r_x^2 and $-T_y/r_y$ in Table 1 shows manifestly, that the parabolic cylinder $\mathbf{B}^{||z}$ is not a blade.

A.11. DCGA IPNS hyperbolic cylinder

The implicit quadric equation for a z-axis aligned hyperbolic cylinder is

$$\frac{x^2}{r_x^2} - y^2/r_y^2 - 1 = 0. \tag{87}$$

The DCGA IPNS bivector *z-axis aligned hyperbolic cylinder* surface entity $\mathbf{J}^{||z}$ is defined as

$$\mathbf{J}^{||z} = \frac{T_{x^2}}{r_x^2} - \frac{T_{y^2}}{r_y^2} - T_1. \tag{88}$$

The presence of the term $T_{x^2}/r_x^2 - T_{y^2}/r_y^2$ means that the hyperbolic cylinder $\mathbf{J}^{||z}$ is manifestly not a blade.

A.12. DCGA IPNS parallel planes pair

The implicit quadric equation for a pair of parallel planes perpendicular to the x-axis is

$$(x - p_{x1})(x - p_{x2}) = 0. \tag{89}$$

Each solution, $x = p_{x1}$ and $x = p_{x2}$, represents a plane. Expanding the equation gives

$$x^2 - (p_{x1} + p_{x2})x + p_{x1}p_{x2} = 0. \tag{90}$$

The DCGA IPNS bivector *parallel x-planes pair* entity $\mathbf{\Pi}^{\perp x}$ is defined as

$$\mathbf{\Pi}^{\perp x} = T_{x^2} - (p_{x1} + p_{x2})T_x + p_{x1}p_{x2}T_1. \tag{91}$$

As the construction of the parallel planes pair indicates, the bivector $\mathbf{\Pi}^{\perp x}$ can indeed be factorized as a blade into $-(\mathbf{e}_1+p_1\mathbf{e}_{\infty 1})(\mathbf{e}_6+p_2\mathbf{e}_{\infty 2})$, if the operator equivalence of $p_1\mathbf{e}_{\infty 1}\mathbf{e}_6 + p_2\mathbf{e}_1\mathbf{e}_{\infty 2}$ with $\frac{1}{2}(p_1+p_2)(\mathbf{e}_{\infty 1}\mathbf{e}_6 + \mathbf{e}_1\mathbf{e}_{\infty 2})$, as discussed in Section 3.3, is taken into account.

A.13. DCGA IPNS non-parallel planes pair

The implicit quadric equation for a pair of non-parallel planes that are parallel to the z-axis is

$$\frac{x^2}{r_x^2} - \frac{y^2}{r_y^2} = 0. \tag{92}$$

The DCGA IPNS bivector *z-axis aligned non-parallel planes pair* entity $\mathbf{X}^{||z}$ is defined as

$$\mathbf{X}^{||z} = \frac{T_{x^2}}{r_x^2} - \frac{T_{y^2}}{r_y^2}. \tag{93}$$

The presence of the term $T_{x^2}/r_x^2 - T_{y^2}/r_y^2$ means that the non-parallel planes pair $\mathbf{X}^{||z}$ is manifestly not a blade.

A.14. DCGA IPNS parabola

The DCGA IPNS quadvector *xy-plane parabola* 1D surface entity $\boldsymbol{\rho}^{||xy}$ is defined as

$$\boldsymbol{\rho}^{||xy} = \boldsymbol{\Pi}^{z=0} \wedge \mathbf{B}^{||z} \tag{94}$$

where the DCGA IPNS 2-blade standard *plane* $\boldsymbol{\Pi}^{z=0}$ is for plane $z = 0$, and the DCGA IPNS bivector *parabolic cylinder* $\mathbf{B}^{||z}$ is as previously defined and directly represents a parabola in the xy-plane.

Note that the wedge product of the non-blade entity $\mathbf{B}^{||z}$ of Section A.10, with the completely orthogonal blade $\boldsymbol{\Pi}^{z=0} = \mathbf{e}_3\mathbf{e}_8$, cannot not produce a blade. Therefore, the parabola $\boldsymbol{\rho}^{||xy}$ is manifestly not a blade.

A.15. DCGA IPNS hyperbola

The DCGA IPNS quadvector *xy-plane hyperbola* 1D surface entity $\boldsymbol{\eta}^{||xy}$ is defined as

$$\boldsymbol{\eta}^{||xy} = \boldsymbol{\Pi}^{z=0} \wedge \mathbf{J}^{||z} \tag{95}$$

where the DCGA IPNS 2-blade standard *plane* $\boldsymbol{\Pi}^{z=0}$ is for plane $z = 0$, and the DCGA IPNS bivector *hyperbolic cylinder* $\mathbf{J}^{||z}$ is as previously defined and directly represents a hyperbola in the xy-plane.

Note that the wedge product of the non-blade entity $\mathbf{J}^{||z}$ of Section A.11, with the completely orthogonal blade $\boldsymbol{\Pi}^{z=0} = \mathbf{e}_3\mathbf{e}_8$, cannot not produce a blade. Therefore, the hyperbola $\boldsymbol{\eta}^{||xy}$ is manifestly not a blade.

A.16. DCGA IPNS horned Dupin cyclide

The *horned Dupin cyclide* is a variation of the *Dupin cyclide*, formed by swapping μ and c in the implicit surface equation of the Dupin cyclide. The parameters a, b, c, μ are defined as

$$a = R,\ \ \mu = (1/2)(r_1 + r_2),\ \ c = (1/2)(r_1 - r_2),\ \ b^2 = a^2 - \mu^2. \tag{96}$$

The DCGA IPNS bivector *horned Dupin cyclide* surface entity $\boldsymbol{\Gamma}$ is defined as

$$\boldsymbol{\Gamma} = T_{\mathbf{t}^4} + 2T_{\mathbf{t}^2}(b^2 - c^2) - 4a^2T_{x^2} - 4b^2T_{y^2} + 8ac\mu T_x + ((b^2 - c^2)^2 - 4c^2\mu^2)T_1.$$

The DCGA IPNS horned Dupin cyclide $\boldsymbol{\Gamma}$ has the same related center points as for $\boldsymbol{\Phi}$. The entity $\boldsymbol{\Gamma}$ can be a *horned ring cyclide* $(r_1 + r_2) < 2R$, *horned spindle cyclide* $(r_1 + r_2) > 2R$, *horned (tangent) spheres* $(r_1 + r_2) = 2R$, *horned ring torus* $(r_1 = r_2) < R$, *horned spindle torus* $(r_1 = r_2) > R$, or *horned spheres* $(r_1 = r_2) = R$. All DCGA versors can be applied to $\boldsymbol{\Gamma}$, to transform it into general positions and scale.

Note that the discussion of the non-blade character at the end of Section 3.3.7 applies in particular to the horned Dupin cyclide as well.

Acknowledgments

We thank the organizers of the CGI 2016 conference in Heraklion, Crete, Greece, and in particular the organizers of the GACSE 2016 workshop day at CGI 2016. We further deeply thank the anonymous reviewers for their time consuming detailed study of our manuscript and very many suggestions for improvement.

References

[1] Rafael Alves and Carlile Lavor. *Geometric Algebra to Model Uncertainties in the Discretizable Molecular Distance Geometry Problem.* Advances in Applied Clifford Algebras, Volume 27, Issue 1, pages 439–452, (2017). DOI: 10.1007/s00006-016-0653-2.

[2] Alan Bromborsky. *Geometric Algebra Module for SymPy.* (2016). Software: `https://github.com/brombo/galgebra`.

[3] Chris Doran and Anthony Lasenby. *Geometric Algebra for Physicists.* Cambridge: Cambridge University Press, (2007).

[4] Leo Dorst, Daniel Fontijne, and Stephen Mann. *Geometric Algebra for Computer Science (Revised Edition): An Object-Oriented Approach to Geometry.* The Morgan Kaufmann Series in Computer Graphics. Elsevier, (2009).

[5] Robert Benjamin Easter. *$\mathcal{G}_{8,2}$ Geometric Algebra, DCGA.* Original research working paper. (2015). Vixra.org preprint: 1508.0086.

[6] Robert Benjamin Easter. *Differential Operators in the $\mathcal{G}_{8,2}$ Geometric Algebra, DCGA.* Original research working paper. (2015). Vixra.org preprint: 1512.0303.

[7] Robert Benjamin Easter. *Conic and Cyclidic Sections in the $\mathcal{G}_{8,2}$ Geometric Algebra, DCGA.* Original research working paper. (2015). Vixra.org preprint: 1511.0182.

[8] Robert Benjamin Easter. *Double Conformal Space-Time Algebra.* Original research working monograph. (2016). Vixra.org preprint: 1602.0114.

[9] Robert Benjamin Easter and Eckhard Hitzer. *Conic and Cyclidic Sections in Double Conformal Geometric Algebra $\mathcal{G}_{8,2}$.* In proceedings of SSI 2016, Session SS11, 6-8 December 2016, Ohtsu, Shiga, Japan, pages 866–871. `http://www.sice.or.jp/org/SSI2016/program.php#SS11`. Vixra.org preprint: 1612.0221.

[10] Michael George Eastwood and Peter Wolfram Michor. *Some Remarks on the Plücker Relations.* Rendiconti del Circolo Matematico di Palermo, Volume II-63, pages 85–88, (2000). Arxiv.org preprint: math/9905090.

[11] Ron Goldman and Stephen Mann. *R(4,4) As a Computational Framework for 3-Dimensional Computer Graphics.* Advances in Applied Clifford Algebras, Volume 25, Issue 1, pages 113–149, (2015). DOI: 10.1007/s00006-014-0480-2.

[12] David Hestenes. *New Foundations for Classical Mechanics.* Volume 99 of *Fundamental Theories of Physics.* 2nd edition. Dordrecht: Kluwer, (1999).

[13] David Hestenes and Garret Sobczyk. *Clifford Algebra to Geometric Calculus, A Unified Language for Mathematics and Physics.* Volume 5 of *Fundamental Theories of Physics.* Dordrecht: Kluwer, (1984).

[14] Dietmar Hildenbrand. *Foundations of Geometric Algebra Computing.* Volume 8 of *Geometry and Computing.* New York: Springer, (2013). DOI: 10.1007/978-3-642-31794-1.

[15] Eckhard Hitzer. *Conic Sections and Meet Intersections in Geometric Algebra.* In Hongbo Li, Peter J. Olver, and Gerald Sommer (Editors), Computer Algebra and Geometric Algebra with Applications: 6th International Workshop, IWMM 2004 Shanghai, China, May 19-21, 2004 and International Workshop, GIAE 2004 Xian, China, May 24-28, 2004 Revised Selected Papers. Lecture Notes in Computer Science, Volume 3519. pages 350–362. Berlin, Heidelberg, New York: Springer, (2005). DOI: 10.1007/11499251_25.

[16] Eckhard Hitzer, Kanta Tachibana, Sven Buchholz and Isseki Yu. *Carrier Method for the General Evaluation and Control of Pose, Molecular Conformation, Tracking, and the Like.* Advances in Applied Clifford Algebras, Volume 19, Issue 2, pages 339–364, (2009). DOI: 10.1007/s00006-009-0160-9.

[17] Eckhard Hitzer. *Crystal planes and reciprocal space in Clifford geometric algebra.* Mathematical Methods in the Applied Sciences, Volume 34, Issue 12, pages 1421–1429, (2011). DOI: 10.1002/mma.1442.

[18] Eckhard Hitzer. *Introduction to Clifford's Geometric Algebra.* SICE Journal of Control, Measurement, and System Integration, Volume 51, Number 4, pages 338–350, (April 2012). Arxiv.org preprint: 1306.1660.

[19] Eckhard Hitzer, Tohru Nitta, and Yasuaki Kuroe. *Applications of Clifford's Geometric Algebra.* Advances in Applied Clifford Algebras, Volume 23, Issue 2, pages 377-404, (2013). DOI: 10.1007/s00006-013-0378-4. Arxiv.org preprint: 1305.5663.

[20] Hongbo Li. *Invariant Algebras and Geometric Reasoning.* Singapore: World Scientific, (2008). DOI: 10.1142/9789812770110.

[21] Helmut Pottmann, Ling Shi, and Mikhail Skopenkov. *Darboux cyclides and webs from circles.* Computer Aided Geometric Design, Volume 29, Issue 1, pages 77–97, (2012). DOI: 10.1016/j.cagd.2011.10.002. Arxiv.org preprint: 1106.1354.

[22] Christian Perwass. *Geometric Algebra with Applications in Engineering.* Volume 4 of *Geometry and Computing.* Berlin, Heidelberg: Springer, (2009). DOI: 10.1007/978-3-540-89068-3.

[23] Bodo Rosenhahn. *Pose Estimation Revisited.* PhD thesis, Christian-Albrechts-Univ. zu Kiel, September 2003.

[24] Michael Schrott and Boris Odehnal. *Ortho-Circles of Dupin Cyclides.* Journal for Geometry and Graphics, Volume 10, Number 1, pages 73–98, (2006).

[25] SymPy Development Team. *SymPy: Python library for symbolic mathematics.* (2016). Software: `http://sympy.org`.

[26] Gerald Sommer (Editor). *Geometric Computing with Clifford Algebras: Theoretical Foundations and Applications in Computer Vision and Robotics.* Berlin: Springer, (2001). DOI: 10.1007/978-3-662-04621-0.

Robert Benjamin Easter
Bangkok, Thailand
e-mail: `reaster2015@gmail.com`

Eckhard Hitzer
International Christian University, Osawa 3-10-2, Mitaka, Tokyo 181-8585, Japan
e-mail: `hitzer@icu.ac.jp`