Depth-integrated characteristics of nonlinear water waves

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Abstract

This exposition has the following main objects in view. (1) All main depth-integrated time-dependent and time-averaged characteristics, – as the velocity potential, velocity, pressure, momentum flux density tensor, volumetric kinetic, potential, and total energies, Poynting (energy flux density) vector, radiation (wave) stress tensor, etc, – of the ideal (inviscid, incompressible, and irrotational) fluid flow in an imaginary wave-perturbed infinite water layer with an arbitrary shaped bed and with a free upper boundary surface, and also the pertinent depth-integrated time-dependent and time-averaged differential continuity equations, – as those of the mass density, energy density, and momentum flux density (Euler's and Bernoulli's equations), etc, – are rigorously deduced from the respective basic local (bulk and surface) characteristics and from the respective bulk continuity equations, with allowance for the corresponding exact kinematic boundary conditions at the upper (free) and bottom surfaces and also with allowance for the corresponding exact dynamic boundary condition at the free surface, which follows from the basic Bernoulli equation. (2) The recursive asymptotic perturbation method with respect to powers of ka that has been developed recently by the present author for the local characteristics and bulk continuity equations of the ideal fluid flow in the presence of a priming (seeding) progressive, or standing, monochromatic gravity water wave (PPPMGWW or PSPMGWW) of a wave number $k>0$ and of wave amplitude $a>0$ in an imaginary infinite water layer of a uniform depth $d>0$ is extended to flow's momentary and time-averaged (TA), depth-integrated (DI) characteristics and to their continuity equations, particularly to the 3×3 radiation, or wave, stress tensor (RST). (3) The extended recursive method is applied to

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PPPMGWW’s and PSPMGWW’s with the purpose to obtain their main TADI characteristics in terms of elementary functions. (4) The first non-vanishing asymptotic approximation of a characteristic, particularly that of the 3×3-TADIRST, of a PPPMGWW or PSPMGWW is generalized to a priming progressive, or standing, quasi-pane (PPQP or PSQP) MGWW. (5) The longshore wave–induced sediment transport rate, expressed by the so-called CERC (Coastal Engineering Research Council) formula, is briefly discussed in its relation to the $\langle x,y\rangle$-component of the 3×3-TADIRST of the pertinent PPQPMGWW. (6) The presently common 2×2-TADIRST’s of progressive and standing water waves, which have been deduced by various writers from intuitive considerations and have been canonized about 55 years ago, are revised in accordance with the 3×3 ones of the recursive asymptotic theory.

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1. Introduction

In my resent paper Iosilevskii [2017], to be cited henceforth as I, a recursive asymptotic theory has been developed for *local* (*bulk* and *surface*) *characteristics* of an *ideal* (*inviscid*, *incompressible*, and *irrotational*) *nonlinear gravity wave motion* on an infinite water (liquid) layer of a constant depth *d* with respect to powers of a dimensionless real-valued scaling parameter ‘*ka*’, where $k>0$ is the wave number and $a>0$ the amplitude of a *priming* (*seeding*) *progressive*, or *standing*, *plane monochromatic gravity water wave* (briefly *PPPMGWW* or *PSPMGWW* respectively). The method, by which the nonlinear water wave problem has been treated in I from scratch, can be regarded as a peculiar instance of the general *perturbation method*, which is known as the *Liouville-Green* (*LG*) *method* in mathematics and as the *Wentzel-Kramers-Brillouin* (*WKB*) *method* in physics. Broadly speaking, this exposition is a continuation of I, in which the recursive asymptotic theory, developed in I, is extended to any *momentary* or *time-averaged depth-integrated characteristic* and *depth-integrated continuity equation*, *of a nonlinear gravity wave motion* induced by a PPPMGWW or PSPMGWW. To be more specific, the plan of this study, outlined in its Abstract, will be followed closely.

The notation of paper I retain, and it will, as a rule, be used throughout the exposition without any further comments. This applies particularly to the conventional set-theoretic notation (see, e.g., Halmos [1960] and to the Special Quotation Method (SQM), which is used for distinguishing between use and mention of graphic symbols; both are important parts of the underlying language of I. Still, for the reader’s convenience, I shall, from time to time, recall some elements of the notation of I. In referring to numbered articles (as sections, subsections, equations, corollaries, definitions, etc) of paper I, the numeral name of an article will be preceded by the Roman numeral ‘I’. For instance, “(I.6.14)” or “Corollary I.6.1” stands for equation (6.14) or Corollary 6.1 in I. Whenever confusion can result, the end of an article as a comment, preliminary remark, proof, etc will be marked by a heavy dot ‘•’, – just as in I.

2. Basic depth-integrated continuity equations

2.1. Preliminaries

Definition 2.1 (a definition schema**).** Let '$F(t,\underline{x})$' be a real-valued functional form which, along with its first-order partial derivatives, is defined for each $\langle t,\underline{x}_2\rangle \in R\times \underline{E}_2$ and for each $z\in[-h(\underline{x}_2),\mathrm{Z}(t,\underline{x}_2)]$ (i.e. for each $\langle t,\underline{x}\rangle \in \mathbf{D}_{\mathrm{ff}}^{\mathrm{cc}}(-h,\mathrm{Z})$, in accordance with Definition I.3.2). Then

$$\widehat{F} \doteq \widehat{F}(t,\underline{x}_2) \doteq \int_{-h(\underline{x}_2)}^{\mathrm{Z}(t,\underline{x}_2)} F(t,\underline{x})\,dz \doteq \int_{-h}^{\mathrm{Z}} F\,dz\,, \tag{2.1}$$

the understanding being that 'F' is a placeholder (ellipsis), which should be replaced by a specific base letter (as 'Φ', 'V', 'E', 'P', etc) with some or no labels. The functional form '$\widehat{F}(t,\underline{x}_2)$' (or the associated function $\widehat{F}$ of the form) is said to be the *depth-integrated* of the functional form '$F(t,\underline{x})$' (or, correspondingly, of the function $\widehat{F}$).•

Comment 2.1. When standing alone, the place-holding symbol '$\widehat{F}$' is ambiguous, because it either is an *abbreviation* of the respective functional form '$\widehat{F}(t,\underline{x}_2)$', as stated by the train of definitions (2.1), or it is a *functional constant* that denotes the *associated function* $\widehat{F}$ of the functional form '$\widehat{F}(t,\underline{x}_2)$'**.** It is hoped that this ambiguity will be solved by the context, in which that symbol occurs. •

Lemma 2.1: Let '$z_1(t,\underline{x}_2)$' and '$z_2(t,\underline{x}_2)$' be given real-valued functional forms, which, along with their first-order partial derivatives, are defined for each $\langle t,\underline{x}_2\rangle \in R\times \underline{E}_2$. Let '$F(t,\underline{x})$' be a real-valued functional form which, along with its first-order partial derivatives, is defined for each $\langle t,\underline{x}_2\rangle \in R\times \underline{E}_2$ and for each $z\in[z_1(t,\underline{x}_2),z_2(t,\underline{x}_2)]$, the understanding being that $z_2(t,\underline{x}_2) > z_1(t,\underline{x}_2)$. Given $\langle t,\underline{x}_2\rangle \in R\times \underline{E}_2$, let

$$\hat{F} \doteq \hat{F}(t,\underline{x}_2;z_1,z_2) \doteq \hat{F}(t,\underline{x}_2;z_1(t,\underline{x}_2),z_2(t,\underline{x}_2)) \doteq \int_{z_1(t,\underline{x}_2)}^{z_2(t,\underline{x}_2)} F(t,\underline{x})\,dz \doteq \int_{z_1}^{z_2} F\,dz\,, \tag{2.2}$$

subject to the pertinent version of Comment 2.1. Then

$$\frac{\partial\hat{F}}{\partial t} = \int_{z_1}^{z_2} \frac{\partial F}{\partial t}\,dz + (F)_{z=z_2}\frac{\partial z_2}{\partial t} - (F)_{z=z_1}\frac{\partial z_1}{\partial t}\,, \tag{2.3}$$

$$\nabla_j \hat{F} = \int_{z_1}^{z_2} \nabla_j F \, dz + (F)_{z=z_2} \nabla_j z_2 - (F)_{z=z_1} \nabla_j z_1 \text{ for each } j \in \{1,2\}. \tag{2.4}$$

Proof: The lemma immediately follows from the Leibniz rule of differentiation of a definite integral with variable limits, and also from the chain rule of differentiation of a composite function•

Corollary 2.1.

$$\hat{F} \doteqdot \hat{F}(t, \underline{x}_2) \doteqdot \hat{F}(t, \underline{x}_2; -h, \mathrm{Z}) \doteqdot \hat{F}(t, \underline{x}_2; -h(\underline{x}_2), \mathrm{Z}(t, \underline{x}_2)) \tag{2.5}$$

and hence

$$\frac{\partial \hat{F}}{\partial t} = \int_{-h}^{\mathrm{Z}} \frac{\partial F}{\partial t} dz + (F)_{z=\mathrm{Z}} \frac{\partial \mathrm{Z}}{\partial t}, \tag{2.6}$$

$$\nabla_j \hat{F} = \int_{-h}^{\mathrm{Z}} \nabla_j F \, dz + (F)_{z=\mathrm{Z}} \nabla_j \mathrm{Z} + (F)_{z=-h} \nabla_j h \text{ for each } j \in \{1,2\}. \tag{2.7}$$

Proof: The corollary immediately follows from Definition 2.1 by Lemma 2.1 at $z_1 = -h$ and $z_2 = \mathrm{Z}$, the understanding being that the function h is independent of 't'.•

Definition 2.2. If a real-valued functional form **'**$G(t)$**'**, depending on the time-valued variable 't' and perhaps on some other variables, is integrable on any finite interval of time continuum X (from Greek 'Χρόνος' \xrónos\, meaning *time*) then

$$\overline{G} \doteqdot \overline{G(t)}^t \doteqdot \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} G(t) dt \tag{2.8}$$

which is basically the same definition as (I.10.28). Consequently, if

$$G(t) = \dot{H}(t) \doteqdot \frac{\partial H(t)}{\partial t}. \tag{2.9}$$

while the functional form is bounded on X, then it follows from (2.8) that

$$\overline{\dot{H}} \doteqdot \overline{\dot{H}(t)}^t \doteqdot \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{\partial H(t)}{\partial t} dt = \lim_{T \to \infty} \frac{1}{T} \left[H\left(\frac{T}{2}\right) - H\left(-\frac{T}{2}\right) \right] = 0. \tag{2.10}$$ •

2.2. The depth-integrated continuity equation for the mass density

Theorem 2.1. In accordance with Definition 2.1, let

$$\hat{V}_j(t,\underline{x}_2) \doteqdot \int_{-h(\underline{x}_2)}^{Z(t,\underline{x}_2)} V_j\,(t,\underline{x})dz \text{ for each } j \in \omega_{1,3}. \tag{2.11}$$

Then for each for each $\langle t,\underline{x}_2\rangle \in R\times\underline{E}_2$:

$$\frac{\partial Z(t,\underline{x}_2)}{\partial t} + \sum_{j=1}^{2} \nabla_j \hat{V}_j(t,\underline{x}_2) = 0, \tag{2.12}$$

which is the depth-integrated of equation (I.4.3), i.e. of this one:

$$\sum_{j=1}^{3} \nabla_j V_j(t,\underline{x}_2) = 0, \tag{2.12$_1$}$$

Proof: It follows from (2.12$_1$) that

$$\int_{-h}^{Z} \sum_{j=1}^{3} \nabla_j V_j\; dz = \sum_{j=1}^{2} \int_{-h}^{Z} \nabla_j V_j\; dz + \int_{-h}^{Z} \nabla_3 V_3\; dz = \sum_{j=1}^{2} \int_{-h}^{Z} \nabla_j V_j\; dz + (V_3)_{z=Z} - (V_3)_{z=-h} = 0. \tag{2.13}$$

At the same time, equation (2.7) with $F \doteqdot V_j$ yields

$$\sum_{j=1}^{2} \nabla_j \hat{V}_j = \sum_{j=1}^{2} \int_{-h}^{Z} \nabla_j V_j\; dz + \sum_{j=1}^{2} (V_j)_{z=Z} \nabla_j Z + \sum_{j=1}^{2} (V_j)_{z=-h} \nabla_j h. \tag{2.13$_1$}$$

By (2.13), equation (2.13$_1$) can be rewritten thus:

$$\sum_{j=1}^{2} \nabla_j \hat{V}_j = \left(\sum_{j=1}^{2} V_j \nabla_j Z - V_3 \right)_{z=Z} + \left(\sum_{j=1}^{2} V_j \nabla_j h + V_3 \right)_{z=-h}. \tag{2.13$_2$}$$

Owing to equations (I.4.70) and (I.4.71), namely these ones

$$\left(\sum_{j=1}^{2} V_j \nabla_j h + V_3 \right)_{z=-h} = 0, \tag{2.13$_3$}$$

$$\frac{\partial Z}{\partial t} + \left(\sum_{j=1}^{2} V_j \nabla_j Z - V_3 \right)_{z=Z} = 0, \tag{2.13$_4$}$$

which are the kinematic boundary conditions at the bottom and upper surfaces of the water layer respectively, equation (2.13$_2$) immediately turns into (2.12). QED.•

Comment 2.2. Equation (2.12) can be rewritten as

$$\frac{\partial \hat{\rho}_0(t,\underline{x}_2)}{\partial t} + \rho_0 \sum_{j=1}^{2} \nabla_j \hat{V}_j(t,\underline{x}_2) = 0, \tag{2.14}$$

where for each $\langle t, \underline{x}_2 \rangle \in R \times \underline{E}_2$:

$$\hat{\rho}_0(t, \underline{x}_2) \doteqdot \rho_0 \int_{-h(\underline{x}_2)}^{Z(t,\underline{x}_2)} dz = \rho_0 [Z(t, \underline{x}_2) + h(\underline{x}_2)]; \tag{2.15}$$

$\hat{\rho}_0(t, \underline{x}_2)$ is the water mass column per unit area of the XY-plane at a temporo-spatial point $\langle t, \underline{x}_2 \rangle$.•

Comment 2.3. Under the natural assumption that, given $\underline{x}_2 \in \underline{E}_2$, $Z(t, \underline{x}_2)$ is bounded for each $t \in R$, it follows from the pertinent instance of (2.10) that

$$\overline{\dot{Z}(t, \underline{x}_2)}^t \doteqdot \overline{\frac{\partial Z(t, \underline{x}_2)}{\partial t}}^t \doteqdot \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{\partial Z(t, \underline{x}_2)}{\partial t} dt = 0. \tag{2.16}$$

Hence, it follows from (2.12) that

$$\sum_{j=1}^{2} \nabla_j \bar{\hat{V}}_j(\underline{x}_2) = 0 \tag{2.17}$$

subject to

$$\bar{\hat{V}}_j(\underline{x}_2) \doteqdot \overline{\hat{V}_j(t, \underline{x}_2)}^t \doteqdot \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \hat{V}_j(t, \underline{x}_2) dt, \tag{2.18}$$

which is the pertinent instance of (2.8).•

2.3. The depth-integrated continuity equation for the energy density

Theorem 2.2. In accordance with Definition 2.1, let

$$\hat{E}_k(t, \underline{x}_2) = \int_{-h(\underline{x}_2)}^{Z(t,\underline{x}_2)} E_k(t, \underline{x}) dz, \tag{2.19}$$

$$\hat{E}_p(t, \underline{x}_2) \doteqdot \int_{-h(\underline{x}_2)}^{Z(t,\underline{x}_2)} E_p(z) dz = \rho_0 g \int_{-h(\underline{x}_2)}^{Z(t,\underline{x}_2)} z\, dz = \frac{1}{2} \rho_0 g \left[Z^2(t, \underline{x}_2) - h^2(\underline{x}_2) \right], \tag{2.20}$$

$$\hat{E}(t, \underline{x}_2) \doteqdot \int_{-h(\underline{x}_2)}^{Z(t,\underline{x}_2)} E(t, \underline{x}) dz = \hat{E}_k(t, \underline{x}_2) + \hat{E}_p(t, \underline{x}_2), \tag{2.21}$$

$$\hat{Q}_j(t, \underline{x}_2) \doteqdot \int_{-h(\underline{x}_2)}^{Z(t,\underline{x}_2)} Q_j(t, \underline{x}) dz \text{ for each } j \in \omega_{1,3}, \tag{2.22}$$

where

$$E_k \doteqdot E_k(t, \underline{x}) \doteqdot \frac{1}{2} \rho_0 [\underline{\nabla} \Phi(t, \underline{x})]^2 = \frac{1}{2} \rho_0 [\underline{V}(t, \underline{x})]^2, \tag{2.19$_1$}$$

$$E_{\mathrm{p}} \stackrel{\rightharpoonup}{=} E_{\mathrm{p}}(z) \stackrel{\rightharpoonup}{=} -P_{\mathrm{hs}}(z) = \rho_0 gz\,, \tag{2.20$_1$}$$

$$E \stackrel{\rightharpoonup}{=} E(t,\underline{x}) \stackrel{\rightharpoonup}{=} E_{\mathrm{k}}(t,\underline{x}) + E_{\mathrm{p}}(z) = E_{\mathrm{k}}(t,\underline{x}) + \rho_0 gz\,, \tag{2.21$_1$}$$

$$\begin{aligned} Q_j &\stackrel{\rightharpoonup}{=} Q_j(t,\underline{x}) \stackrel{\rightharpoonup}{=} V_j\left(P + E_{\mathrm{k}} + \rho_0 gz\right) = V_j\left(P + E_{\mathrm{k}} + E_{\mathrm{p}}\right) = V_j\left(P + E\right) \\ &\stackrel{\rightharpoonup}{=} V_j\left(P_0(t) - \rho_0 \frac{\partial \Phi}{\partial t}\right) = Q_{*j}(t,\underline{x}) + P_0(t)V_j \ \text{ for each } j \in \omega_{1,3}, \end{aligned} \tag{2.22$_1$}$$

in accordance with (I.4.40)–(I.4.42), and (I.4.61) or (I.4.61a) respectively; $\Phi \stackrel{\rightharpoonup}{=} \Phi(t,\underline{x})$ is the velocity potential and $P_0(t)$ is defined by (I.4.47)–(I.4.49).

Then for each $\langle t,\underline{x}_2\rangle \in R \times \underline{E}_2$:

$$\frac{\partial \widehat{E}_{\mathrm{k}}(t,\underline{x}_2)}{\partial t} + \sum_{j=1}^{2} \nabla_j \widehat{Q}_j(t,\underline{x}_2) = -\left[\rho_0 g \mathrm{Z}(t,\underline{x}_2) + P_0(t)\right]\frac{\partial \mathrm{Z}(t,\underline{x}_2)}{\partial t}\,, \tag{2.23}$$

$$\frac{\partial \widehat{E}(t,\underline{x}_2)}{\partial t} + \sum_{j=1}^{2} \nabla_j \widehat{Q}_j(t,\underline{x}_2) = -P_0(t)\frac{\partial \mathrm{Z}(t,\underline{x}_2)}{\partial t}\,; \tag{2.23a}$$

(2.23) is the depth-integrated of equation (I.4.60), i.e. of this one

$$\frac{\partial E_{\mathrm{k}}}{\partial t} + \sum_{i=1}^{3} \nabla_i Q_i = 0\,. \tag{2.23$_1$}\bullet$$

Proof: It immediately follows from (2.23$_1$) that

$$\begin{aligned} \int_{-h}^{\mathrm{Z}} \left(\frac{\partial E_{\mathrm{k}}}{\partial t} + \sum_{j=1}^{3} \nabla_j Q_j\right) dz &= \int_{-h}^{\mathrm{Z}} \frac{\partial E_{\mathrm{k}}}{\partial t}\, dz + \sum_{j=1}^{2} \int_{-h}^{\mathrm{Z}} \nabla_j Q_j\, dz + \int_{-h}^{\mathrm{Z}} \nabla_3 Q_3\, dz \\ &= \int_{-h}^{\mathrm{Z}} \frac{\partial E_{\mathrm{k}}}{\partial t}\, dz + \sum_{j=1}^{2} \int_{-h}^{\mathrm{Z}} \nabla_j Q_j\, dz + \left(Q_3\right)_{z=\mathrm{Z}} - \left(Q_3\right)_{z=-h} = 0. \end{aligned} \tag{2.23$_2$}$$

At the same time, equation (2.6) with $F \stackrel{\rightharpoonup}{=} E_{\mathrm{k}}$ and equation (2.7) with $F \stackrel{\rightharpoonup}{=} Q_j$ yield

$$\int_{-h}^{\mathrm{Z}} \frac{\partial E_{\mathrm{k}}}{\partial t}\, dz = \frac{\partial \widehat{E}_{\mathrm{k}}}{\partial t} - \left(E_{\mathrm{k}}\right)_{z=\mathrm{Z}} \frac{\partial \mathrm{Z}}{\partial t}\,, \tag{2.23$_4$}$$

$$\sum_{j=1}^{2} \int_{-h}^{\mathrm{Z}} \nabla_j Q_j\, dz = \sum_{j=1}^{2} \nabla_j \widehat{Q}_j - \sum_{j=1}^{2} \left(Q_j\right)_{z=\mathrm{Z}} \nabla_j \mathrm{Z} - \sum_{j=1}^{2} \left(Q_j\right)_{z=-h} \nabla_j h\,, \tag{2.23$_5$}$$

respectively. Hence, the final equation of the train (2.23$_2$) can be written thus:

$$\frac{\partial \widehat{E}_{\mathrm{k}}}{\partial t} + \sum_{j=1}^{2} \nabla_j \widehat{Q}_j - R_{\mathrm{B}} - R_{\mathrm{T}} = 0\,, \tag{2.23$_6$}$$

where

$$R_{\mathrm{B}} \stackrel{\rightharpoonup}{=} \left(\sum_{j=1}^{2} Q_j \nabla_j h + Q_3\right)_{z=-h} = \left[\left(P_0(t) - \rho_0 \frac{\partial \Phi}{\partial t}\right)\left(\sum_{j=1}^{2} V_j \nabla_j h + V_3\right)\right]_{z=-h} = 0\,, \tag{2.23$_7$}$$

$$R_{\mathrm{T}} \stackrel{\rightharpoonup}{=} \left(E_{\mathrm{k}} \frac{\partial Z}{\partial t} + \sum_{j=1}^{2} Q_j \nabla_j Z - Q_3\right)_{z=Z} = \left[E_{\mathrm{k}} \frac{\partial Z}{\partial t} + \left(P_0(t) - \rho_0 \frac{\partial \Phi}{\partial t}\right)\left(\sum_{j=1}^{2} V_j \nabla_j Z - V_3\right)\right]_{z=Z}$$

$$= \left[\left(E_{\mathrm{k}} - P_0(t) + \rho_0 \frac{\partial \Phi}{\partial t}\right)\frac{\partial Z}{\partial t} + \left(P_0(t) - \rho_0 \frac{\partial \Phi}{\partial t}\right)\left(\frac{\partial Z}{\partial t} + \sum_{j=1}^{2} V_j \nabla_j Z - V_3\right)\right]_{z=Z} \tag{2.23$_8$}$$

$$= \left(E_{\mathrm{k}} - P_0(t) + \rho_0 \frac{\partial \Phi}{\partial t}\right)_{z=Z} \frac{\partial Z}{\partial t} = -\left[P_0(t) + P_{\mathrm{d}}(t, \underline{x})\right]_{z=Z} \frac{\partial Z}{\partial t} = -\left[\rho_0 g Z + P_0(t)\right]\frac{\partial Z}{\partial t}.$$

In developing the final result in the train of equations (2.23$_7$), use of the kinematic boundary condition (2.13$_3$) has been made. In developing the final result in the train of equations (2.23$_8$), use has been made of the kinematic boundary condition (2.13$_4$) and of the dynamic boundary condition (I.4.67), namely of this one:

$$\mathrm{Z}(t, \underline{x}_2) = -\frac{1}{g}\left[\frac{\partial \Phi(t, \underline{x})}{\partial t} + \frac{1}{\rho_0} E_{\mathrm{k}}(t, \underline{x})\right]_{z=\mathrm{Z}(t, \underline{x}_2)} = \frac{1}{\rho_0 g}\left[P_{\mathrm{d}}(t, \underline{x})\right]_{z=\mathrm{Z}(t, \underline{x}_2)}, \tag{2.23$_9$}$$

Equation (2.23$_6$) subject to (2.23$_7$) and (2.23$_6$) coincides with (2.23).

From (2.21) subject to (2.20), it follows that

$$\frac{\partial \widehat{E}(t, \underline{x}_2)}{\partial t} = \frac{\partial \widehat{E}_{\mathrm{k}}(t, \underline{x}_2)}{\partial t} + \frac{\partial \widehat{E}_{\mathrm{p}}(t, \underline{x}_2)}{\partial t} = \frac{\partial \widehat{E}_{\mathrm{k}}(t, \underline{x}_2)}{\partial t} + \rho_0 g Z(t, \underline{x}_2) \frac{\partial \mathrm{Z}(t, \underline{x}_2)}{\partial t}. \tag{2.24}$$

By (2.24), equation (2.23) turns into (2.23a). QED.•

Comment 2.4. Equation (2.20) can be rewritten as

$$\widehat{E}_{\mathrm{p}}(t, \underline{x}_2) = \widehat{E}_{\mathrm{pw}}(t, \underline{x}_2) + \widehat{E}_{\mathrm{pe}}(\underline{x}_2), \tag{2.25}$$

where

$$\widehat{E}_{\mathrm{pw}}(t, \underline{x}_2) \stackrel{\rightharpoonup}{=} \int_0^{\mathrm{Z}(t, \underline{x}_2)} E_{\mathrm{p}}(z)\, dz = \rho_0 g \int_0^{\mathrm{Z}(t, \underline{x}_2)} z\, dz = \frac{1}{2} \rho_0 g Z^2(t, \underline{x}_2), \tag{2.25w}$$

$$\widehat{E}_{\mathrm{pe}}(\underline{x}_2) \stackrel{\rightharpoonup}{=} \int_{-h(\underline{x}_2)}^{0} E_{\mathrm{p}}(z)\, dz = \rho_0 g \int_{-h(\underline{x}_2)}^{0} z\, dz = -\frac{1}{2} \rho_0 g h^2(\underline{x}_2). \tag{2.25e}$$

Consequently, equation (2.21) can be written as:

$$\widehat{E}(t, \underline{x}_2) = \widehat{E}_{\mathrm{k}}(t, \underline{x}_2) + \widehat{E}_{\mathrm{p}}(t, \underline{x}_2) = \widehat{E}_{\mathrm{k}}(t, \underline{x}_2) + \widehat{E}_{\mathrm{pw}}(t, \underline{x}_2) + \widehat{E}_{\mathrm{pe}}(\underline{x}_2) = \widehat{E}_{\mathrm{w}}(t, \underline{x}_2) + \widehat{E}_{\mathrm{pe}}(\underline{x}_2), \tag{2.26}$$

where

$$\widehat{E}_{\mathrm{w}}(t,\underline{x}_2) \stackrel{\rightharpoonup}{=} \widehat{E}_{\mathrm{k}}(t,\underline{x}_2) + \widehat{E}_{\mathrm{pw}}(t,\underline{x}_2). \tag{2.27}$$

According to (2.19)–(2.21), $\widehat{E}$, $\widehat{E}_{\mathrm{k}}$, and $\widehat{E}_{\mathrm{p}}$ are respectively the *depth-integrated total, kinetic,* and *potential energy densities* of the perturbed liquid layer; $\widehat{E}_{\mathrm{pw}}$, defined by (2.25w), and $\widehat{E}_{\mathrm{w}}$, defined by (2.27), can respectively be called the *depth-integrated wave-related total* and *potential energy densities* of the perturbed liquid layer; and $\widehat{E}_{\mathrm{pe}}$, defined by (2.25e). can be called the *depth-integrated equilibrium-related potential energy density* of the perturbed liquid layer. At the same time, it is understood that if the liquid layer is in equilibrium then its *depth-integrated potential energy density* equals $\widehat{E}_{\mathrm{pe}}$, whereas

$$\widehat{\Phi} = \widehat{E}_{\mathrm{w}} = \widehat{E}_{\mathrm{k}} = \widehat{E}_{\mathrm{p}w} = \widehat{V}_j = \widehat{Q}_j = \widehat{Q}_{*j} = \mathrm{Z} = C_0, \tag{2.28}$$

where C_0 is the null-valued constant function (see subsection 2.4 for greater detail).

By (2.25w) and (2.27), equation (2.23) can alternatively be written as:

$$\frac{\partial \widehat{E}_{\mathrm{w}}(t,\underline{x}_2)}{\partial t} + \sum_{j=1}^{2} \nabla_j \widehat{Q}_j(t,\underline{x}_2) = -P_0(t)\frac{\partial \mathrm{Z}(t,\underline{x}_2)}{\partial t}. \tag{2.23b}\bullet$$

Convention 2.1. In paper I, '$P_0(t)$' is an arbitrary real-valued functional form independent of '$\underline{x}$' and hence possibly depending only on 't' – the form, which had appeared in the process of integration of the Euler equation (I.4.38) subject to (I.4.39) so as to result in *its first integral* (I.4.46), being the pertinent *general unsteady Bernoulli equation*. Immediately after deducing the latter equation, '$P_0(t)$' was specified by definitions (I.4.47)–(I.4.49), according to which '$P_0(t)$' can take one of two values, namely (a) $P_0(t) \stackrel{\rightharpoonup}{=} 0$ if the part of space above the upper boundary surface $z = \mathrm{Z}(t,\underline{x}_2)$ of the liquid layer is *vacuous* or (b) $P_0(t) \stackrel{\rightharpoonup}{=} P_{\mathrm{a}}(t)$ if the above part of space is occupied with air producing a given *atmospheric pressure* $P_{\mathrm{a}}(t)$ *at* $z = \mathrm{Z}(t,\underline{x}_2)$. In the latter case, I have tacitly assumed that $P_{\mathrm{a}}(t)$ is the same at least for $z \in [\mathrm{Z}_{\mathrm{m}}, \mathrm{Z}_{\mathrm{M}}]$, where Z_{m} is the infimum and Z_{M} is the supremum of $\mathrm{Z}(t,\underline{x}_2)$; I have also neglected the surface tension of the liquid. Now, it is natural to assume that $P_{\mathrm{a}}(t)$ remains constant, P_{a}, within a span of time, in which some bulk characteristics of the fluid flow in the liquid layer significantly change. Consequently, I shall henceforth assume that '$P_0(t)$' is a two-valued constant 'P_0' such that $P_0 \stackrel{\rightharpoonup}{=} 0$ or $P_0 \stackrel{\rightharpoonup}{=} P_{\mathrm{a}}$. Alternatively, for the sake of definiteness and for avoidance of triviality, I may, without loss of generality, assume that the definition $P_0 \dot{\equiv} P_\mathrm{a}$ is the only one to hold,•

Comment 2.5. Besides (2.16), it is clear that

$$\overline{\frac{\partial \mathrm{Z}^2(t,\underline{x}_2)}{\partial t}}^{t} \dot{\equiv} \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{\partial \mathrm{Z}^2(t,\underline{x}_2)}{\partial t}dt = 0\,. \tag{2.29}$$

Under the evident assumption that the functions $\widehat{E}_\mathrm{k}$, $\widehat{E}$, and $\widehat{E}_\mathrm{w}$ are bounded, so that

$$\overline{\frac{\partial \widehat{E}_\mathrm{k}(t,\underline{x}_2)}{\partial t}}^{t} = \overline{\frac{\partial \widehat{E}_\mathrm{k}(t,\underline{x}_2)}{\partial t}}^{t} = \overline{\frac{\partial \widehat{E}_\mathrm{k}(t,\underline{x}_2)}{\partial t}}^{t} = 0\,, \tag{2.30}$$

averaging each one of equations (2.23), (2.23a), and (2.23b) subject to Convention 2.1 with respect to t yields:

$$\sum_{j=1}^{2}\nabla_j \overline{\widehat{Q}}_j(\underline{x}_2) = 0\,. \tag{2.31}$$ •

2.4. The depth-integrated Bernoulli equation

Definition 2.3. In accordance with Definition 2.1, let

$$\widehat{\Phi}(t,\underline{x}_2) \dot{\equiv} \int_{-h(\underline{x}_2)}^{\mathrm{Z}(t,\underline{x}_2)} \Phi(t,\underline{x})\,dz\,, \tag{2.32}$$

$$\widehat{\dot{\Phi}}(t,\underline{x}_2) \dot{\equiv} \int_{-h(\underline{x}_2)}^{\mathrm{Z}(t,\underline{x}_2)} \dot{\Phi}(t,\underline{x})\,dz = \int_{-h(\underline{x}_2)}^{\mathrm{Z}(t,\underline{x}_2)} \frac{\partial \Phi(t,\underline{x})}{\partial t}\,dz\,, \tag{2.33}$$

the understanding being that

$$\dot{\Phi}(t,\underline{x}) \dot{\equiv} \frac{\partial \Phi(t,\underline{x})}{\partial t}\,, \tag{2.34}$$

Besides (2.11), (2.19)–(2.21), (2.32), and (2.33), it also follows from Definition 2.1 that

$$\widehat{P}(t,\underline{x}_2) \dot{\equiv} \int_{-h(\underline{x}_2)}^{\mathrm{Z}(t,\underline{x}_2)} P(t,\underline{x})\,dz\,, \tag{2.35}$$

$$\widehat{P}_\mathrm{d}(t,\underline{x}_2) \dot{\equiv} \int_{-h(\underline{x}_2)}^{\mathrm{Z}(t,\underline{x}_2)} P_\mathrm{d}(t,\underline{x})\,dz\,, \tag{2.36}$$

$$\widehat{P}_0(t,\underline{x}_2) \dot{\equiv} P_0 \int_{-h(\underline{x}_2)}^{\mathrm{Z}(t,\underline{x}_2)} dz = P_0\left[\mathrm{Z}(t,\underline{x}_2) + h(\underline{x}_2)\right]; \tag{2.37}$$

where, in accordance with (I.4.50)–(I.4.52) subject to (I.4.40)–(I.4.42) (or (2.19_1)–(2.21_1)) and (I.4.47)–(I.4.49) and also subject to Convention 2.1,

$$\begin{aligned}P(t,\underline{x}) &= P_0 - \rho_0\dot{\Phi}(t,\underline{x}) - E(t,\underline{x}) = P_0 - \rho_0\dot{\Phi}(t,\underline{x}) - E_{\mathrm{k}}(t,\underline{x}) - E_{\mathrm{p}}(z)\\ &= P_0 + P_{\mathrm{d}}(t,\underline{x}) + P_{\mathrm{hs}}(z) = P_0 - \rho_0\frac{\partial\Phi(t,\underline{x})}{\partial t} - \frac{1}{2}\rho_0\left[\underline{\nabla}\Phi(t,\underline{x})\right]^2 - \rho_0 gz,\end{aligned} \tag{2.35$_1$}$$

the understanding being that $P(t,\underline{x})$ is the *total pressure* at a temporo-spatial point $(t,\underline{x})$ of the perturbed liquid layer, $P_{\mathrm{d}}(t,\underline{x})$, defined by (I.4.52), i.e. as:

$$P_{\mathrm{d}}(t,\underline{x}) \stackrel{=}{=} -\rho_0\dot{\Phi}(t,\underline{x}) - E_{\mathrm{k}}(t,\underline{x}) = -\rho_0\frac{\partial\Phi(t,\underline{x})}{\partial t} - \frac{1}{2}\rho_0\left[\underline{\nabla}\Phi(t,\underline{x})\right]^2, \tag{2.36$_1$}$$

is the *dynamic pressure* at that point, and $P_{\mathrm{hs}}(z)$, defined by (I.4.41), i.e. by (2.20_1), is the *hydrostatic pressure* at each temporo-spatial point $(t,\underline{x})$ of the perturbed liquid layer, which is located in the horizontal plane with the applicate *z*. Each equation of the train (2.35_1), whose left-hand side is $P(t,\underline{x})$, is a version of the *unsteady Bernoulli equation for the fluid flow in the perturbed liquid layer* – the equation being the *first integral of the Euler equation (I.4.24)*. Equation (2.37) is analogous to (2.15).

By (2.19_1)–(2.21_1) and (2.33), definitions (2.35) and (2.36) subject to (2.35_1) and (2.36_1) become

$$\begin{aligned}\widehat{P}(t,\underline{x}_2) &= \widehat{P}_0(t,\underline{x}_2) - \rho_0\dot{\widehat{\Phi}}(t,\underline{x}_2) - \widehat{E}(t,\underline{x}_2)\\ &= P_0\left[\mathrm{Z}(t,\underline{x}_2) + h(\underline{x}_2)\right] - \rho_0\dot{\widehat{\Phi}}(t,\underline{x}_2) - E_{\mathrm{k}}(t,\underline{x}_2) - \frac{1}{2}\rho_0\left[\mathrm{Z}^2(t,\underline{x}_2) - h^2(\underline{x}_2)\right]\\ &= P_0\left[\mathrm{Z}(t,\underline{x}_2) + h(\underline{x}_2)\right] + \widehat{P}_{\mathrm{d}}(t,\underline{x}_2) - \frac{1}{2}\rho_0\left[\mathrm{Z}^2(t,\underline{x}_2) - h^2(\underline{x}_2)\right],\end{aligned} \tag{2.38}$$

$$\widehat{P}_{\mathrm{d}}(t,\underline{x}_2) = -\rho_0\dot{\widehat{\Phi}}(t,\underline{x}_2) - E_{\mathrm{k}}(t,\underline{x}_2). \tag{2.39}$$

the understanding being that the train of equations (2.38) presents three versions of *the depth-integrated Bernoulli equation,* whereas equation (2.39) defines *the depth-integrated dynamic pressure in the perturbed liquid layer*.•

2.5. The depth-integrated momentum flux density tenor and the momentary and time-averaged radiation (wave) stress tensors

Definition 2.4. In accordance with Definition 2.1, let for each $i \in \omega_{1,3}$, for each $j \in \omega_{1,3}$ and for each $(t,\underline{x}_2) \in R \times \underline{E}_2$:

$$\widehat{S}_{ij}(t,\underline{x}_2) \doteqdot \int_{-h(\underline{x}_2)}^{Z(t,\underline{x}_2)} S_{ij}(t,\underline{x})\,dz\,, \tag{2.40}$$

$$\widehat{E}_{ij}(t,\underline{x}_2) \doteqdot \int_{-h(\underline{x}_2)}^{Z(t,\underline{x}_2)} E_{\mathrm{ij}}(t,\underline{x})\,dz\,, \tag{2.41}$$

where, in accordance with (I.4.55), (I.4.56), and Convention 2.1, for each $z \in (-h(\underline{x}_2), Z(t,\underline{x}_2)]$:

$$\begin{aligned} S_{ij}(t,\underline{x}) &\doteqdot \left[P_0 - \rho_0\dot{\Phi}(t,\underline{x})\right]\delta_{ij} + \left[E_{ij}(t,\underline{x}) - E_{\mathrm{k}}(t,\underline{x})\delta_{ij}\right] - E_{\mathrm{p}}(z)(\delta_{ij} - \delta_{i3}\delta_{j3}) \\ &= \left[P_0 - \rho_0\dot{\Phi}(t,\underline{x})\right]\delta_{ij} + \left[E_{ij}(t,\underline{x}) - E_{\mathrm{k}}(t,\underline{x})\delta_{ij}\right] - \rho_0 g z(\delta_{ij} - \delta_{i3}\delta_{j3}), \end{aligned} \tag{2.40$_1$}$$

$$E_{ij}(t,\underline{x}) \doteqdot \rho_0 V(t,\underline{x})_i V_j(t,\underline{x}) = \rho_0\left[\nabla_i \Phi(t,\underline{x})\right]\left[\nabla_j \Phi(t,\underline{x})\right], \tag{2.41$_1$}$$

subject to (2.19$_1$), (2.20$_1$), and (2.34). The 3×3–tensor $S_{ij}(t,\underline{x})$ thus defined is called the *bulk momentum flux density tensor of the perturbed liquid layer at a given temporo-spatial point* $(t,\underline{x})$.

By (2.40$_1$), definition (2.40) yields

$$\begin{aligned} \widehat{S}_{ij}(t,\underline{x}_2) &= \left\{\widehat{P}_0(t,\underline{x}_2) - \rho_0\widehat{\dot{\Phi}}(t,\underline{x}_2)\right\}\delta_{ij} + \left[\widehat{E}_{ij}(t,\underline{x}_2) - \widehat{E}_{\mathrm{k}}(t,\underline{x}_2)\delta_{ij}\right] - \widehat{E}_{\mathrm{p}}(t,\underline{x}_2)(\delta_{ij} - \delta_{i3}\delta_{j3}) \\ &= \left\{P_0\left[Z(t,\underline{x}_2) + h(\underline{x}_2)\right] - \rho_0\widehat{\dot{\Phi}}(t,\underline{x}_2)\right\}\delta_{ij} + \left[\widehat{E}_{ij}(t,\underline{x}_2) - \widehat{E}_{\mathrm{k}}(t,\underline{x}_2)\delta_{ij}\right] \\ &\quad - \frac{1}{2}\rho_0 g\left[Z^2(t,\underline{x}_2) - h^2(\underline{x}_2)\right](\delta_{ij} - \delta_{i3}\delta_{j3}) \end{aligned} \tag{2.42}$$

subject to (2.19), (2.20), (2.33), (2.37), and (2.41). The 3×3–tensor $\widehat{S}_{ij}(t,\underline{x}_2)$ thus defined is called the *depth-integrated momentum flux density tensor of the perturbed liquid layer at a given instant* $t \in R$ *along the given vertical line* $L(\underline{x}_2)$*, defined as*:

$$L(\underline{x}_2) \doteqdot \{z \mid z \in (h(\underline{x}_2), Z(t,\underline{x}_2)]\} \subset \underline{E} \tag{2.43}$$

at a given point $\underline{x}_2 \in \underline{E}_2$.•

Comment 2.6. In accordance with the instance of definition (2.6) with $F \doteqdot \dot{\Phi}$, it follows from (2.32) that

$$\begin{aligned} \dot{\widehat{\Phi}}(t,\underline{x}_2) &\doteqdot \frac{\partial\widehat{\Phi}(t,\underline{x}_2)}{\partial t} = \frac{\partial}{\partial t}\int_{-h(\underline{x}_2)}^{Z(t,\underline{x}_2)} \Phi(t,\underline{x})\,dz = \int_{-h(\underline{x}_2)}^{Z(t,\underline{x}_2)} \frac{\partial\Phi(t,\underline{x})}{\partial t}\,dz + \left[\Phi(t,\underline{x})\right]_{z=Z(t,\underline{x}_2)} \frac{\partial Z(t,\underline{x}_2)}{\partial t} \\ &= \widehat{\dot{\Phi}}(t,\underline{x}_2) + \left[\Phi(t,\underline{x})\right]_{z=Z(t,\underline{x}_2)} \dot{Z}(t,\underline{x}_2), \end{aligned} \tag{2.44}$$

whence

$$\widehat{\dot{\Phi}}(t,\underline{x}_2) = \dot{\widehat{\Phi}}(t,\underline{x}_2) - \left[\Phi(t,\underline{x})\right]_{z=Z(t,\underline{x}_2)} \dot{Z}(t,\underline{x}_2), \tag{2.45}$$

the understanding being that

$$\dot{Z}(t,\underline{x}) \doteqdot \frac{\partial Z(t,\underline{x})}{\partial t}. \tag{2.46}$$ •

Corollary 2.2. 1) Here follow the individual components of the 3×3-tensor $\widehat{S}_{ij}(t,\underline{x}_2)$ versus the respective components of 3×3-tensor $S_{ij}(t,\underline{x})$:

$$\begin{aligned}
&\widehat{S}_{ij} = \widehat{S}_{ji} = \left(\widehat{P}_0 - \rho_0\widehat{\dot{\Phi}}\right)\delta_{ij} + \left(\widehat{E}_{ij} - \widehat{E}_{\mathrm{k}}\delta_{ij}\right) - \widehat{E}_{\mathrm{p}}\delta_{ij} \\
&\qquad = \left[P_0(Z+h) - \rho_0\widehat{\dot{\Phi}}\right]\delta_{ij} + \left(\widehat{E}_{ij} - \widehat{E}_{\mathrm{k}}\delta_{ij}\right) - \frac{1}{2}\rho_0 g\left(Z^2 - h^2\right)\delta_{ij},\ i \in \omega_{1,2},\ j \in \omega_{1,2}, \quad \text{(a)} \\
&\widehat{S}_{i3} = \widehat{S}_{3i} = \widehat{E}_{i3} = \widehat{E}_{3i},\ i \in \omega_{1,2},\ j \doteqdot 3, \quad \text{(b)} \\
&\widehat{S}_{33} = \widehat{P}_0 - \rho_0\widehat{\dot{\Phi}} - \widehat{E}_{\mathrm{k}} + \widehat{E}_{33} = P_0(Z+h) - \rho_0\widehat{\dot{\Phi}} - \widehat{E}_{\mathrm{k}} + \widehat{E}_{33},\ i \doteqdot j \doteqdot 3. \quad \text{(c)}
\end{aligned} \tag{2.42′}$$

$$\begin{aligned}
&S_{ij} = S_{ji} = \left(P_0 - \rho_0\dot{\Phi}\right)\delta_{ij} + \left(E_{ij} - E_{\mathrm{k}}\delta_{ij}\right) - E_{\mathrm{p}}\delta_{ij} \\
&\qquad = \left(P_0 - \rho_0\dot{\Phi}\right)\delta_{ij} + \left(E_{ij} - E_{\mathrm{k}}\delta_{ij}\right) - \rho_0 g z\delta_{ij},\ i \in \omega_{1,2},\ j \in \omega_{1,2}, \quad \text{(a)} \\
&S_{i3} = S_{3i} = E_{i3} = E_{3i},\ i \in \omega_{1,2},\ j \doteqdot 3, \quad \text{(b)} \\
&S_{33} = P_0 - \rho_0\dot{\Phi} - E_{\mathrm{k}} + E_{33},\ i \doteqdot j \doteqdot 3. \quad \text{(c)}
\end{aligned} \tag{2.40$_{1'}$}$$

2) It is understood that $h(\underline{x}_2) \geq h_{\mathrm{m}}$ in (2.42′) and that $z \geq -h_{\mathrm{m}}$ in (2.40$_{1'}$), where h_{m} is the infimum of all values of '$h(\underline{x}_2)$'. Therefore, $\widehat{S}_{ij}(t,\underline{x}_2)$, defined by (2.42′,a), $\widehat{S}_{33}(t,\underline{x}_2)$, defined by (2.42′,c), and $S_{ij}(t,\underline{x})$, defined by (2.40$_{1'}$,a), become unbounded as $h_{\mathrm{m}} \to \infty$. That is to say, $\widehat{S}_{ij}(t,\underline{x}_2)$, defined by (2.42), and $S_{ij}(t,\underline{x})$, defined by (2.40$_1$), do not exist in the case of infinitely deep water.•

Definition 2.5. In agreement with Comment I.4.2, if the liquid layer is *in equilibrium* (*at rest*), so that for each $(t,\underline{x}_2) \in R \times \underline{E}_2$ and each $z \in (-h(\underline{x}_2),0]$:

$$\Phi(t,\underline{x}) \doteqdot 0\ ,\ \underline{V}(t,\underline{x}) \doteqdot \underline{0} = (0,0,0),\ Z(t,\underline{x}_2) \doteqdot \underline{0}\,, \tag{2.47}$$

then equations (2.40$_1$) and (2.42) become

$$S_{\mathrm{e}ij}(t,\underline{x}) \doteqdot P_0\delta_{ij} - \rho_0 g z\left(\delta_{ij} - \delta_{i3}\delta_{j3}\right) = P_0 - E_{\mathrm{p}}(z)\left(\delta_{ij} - \delta_{i3}\delta_{j3}\right), \tag{2.40$_1$e}$$

$$\begin{aligned}
&\widehat{S}_{\mathrm{e}ij}(t,\underline{x}_2) \doteqdot \int_{-h(\underline{x}_2)}^{0} S_{\mathrm{e}ij}(t,\underline{x})\,dz \\
&= \widehat{P}_{0\mathrm{e}}(\underline{x}_2)\delta_{ij} - \widehat{E}_{\mathrm{pe}}(\underline{x}_2)\left(\delta_{ij} - \delta_{i3}\delta_{j3}\right) = P_0 h(\underline{x}_2)\delta_{ij} + \frac{1}{2}\rho_0 g h^2(\underline{x}_2)\left(\delta_{ij} - \delta_{i3}\delta_{j3}\right),
\end{aligned} \tag{2.42e}$$

the understanding being that $\widehat{E}_{\mathrm{pe}}(\underline{x}_2)$ is defined by definition (2.25e) in Comment 2.4, whereas

$$\widehat{P}_{0\mathrm{e}}(\underline{x}_2) \doteqdot P_0 \int_{-h(\underline{x}_2)}^{0} dz = P_0 h(\underline{x}_2); \tag{2.37e}$$

It goes without saying that the 3×3–tensor $S_{\mathrm{e}ij}(t,\underline{x})$, defined by (2.40$_1$e), is the *bulk momentum flux density tensor of the equilibrious liquid layer at a given temporo-spatial point* $(t,\underline{x})$, whereas the 3×3–tensor $\widehat{S}_{\mathrm{e}ij}(t,\underline{x}_2)$, defined by (2.42e), is the *depth-integrated one at a given instant* $t \in R$ *along the given vertical line* $L_{\mathrm{e}}(\underline{x}_2)$, *defined as:*

$$L_{\mathrm{e}}(\underline{x}_2) \doteqdot \{z | z \in (-h(\underline{x}_2), 0]\} \subset \underline{E} \tag{2.43e}$$

at a given point $\underline{x}_2 \in \underline{E}_2$.•

Definition 2.6. In accordance with (2.40), (2.42), and (2.42e), the 3×3-tensor $\widehat{S}_{\mathrm{w}ij}(t,\underline{x}_2)$, which is defined for each $i \in \omega_{1,3}$, for each $j \in \omega_{1,3}$, and for each $(t,\underline{x}_2) \in R \times \underline{E}_2$ as:

$$\begin{aligned}
&\widehat{S}_{\mathrm{w}ij}(t,\underline{x}_2) \doteqdot \widehat{S}_{ij}(t,\underline{x}_2) - \widehat{S}_{\mathrm{e}ij}(\underline{x}_2) \\
&= \left[P_0 Z(t,\underline{x}_2) - \rho_0 \hat{\dot{\Phi}}(t,\underline{x}_2)\right]\delta_{ij} + \left[\widehat{E}_{ij}(t,\underline{x}_2) - \widehat{E}_{\mathrm{k}}(t,\underline{x}_2)\delta_{ij}\right] - \frac{1}{2}\rho_0 g Z^2(t,\underline{x}_2)(\delta_{ij} - \delta_{i3}\delta_{j3}) \\
&= \left[\widehat{P}_{0\mathrm{w}}(t,\underline{x}_2) - \rho_0 \hat{\dot{\Phi}}(t,\underline{x}_2)\right]\delta_{ij} + \left[\widehat{E}_{ij}(t,\underline{x}_2) - \widehat{E}_{\mathrm{k}}(t,\underline{x}_2)\delta_{ij}\right] - \widehat{E}_{\mathrm{pw}}(t,\underline{x}_2)(\delta_{ij} - \delta_{i3}\delta_{j3}),
\end{aligned} \tag{2.48}$$

is called the *full*, or 3×3, *momentary radiation*, or *wave-related, stress tensor at a given instant* $t \in R$ *along the given vertical line* $L(\underline{x}_2)$, *defined by (2.43) at a given point* $\underline{x}_2 \in \underline{E}_2$; the subscript 'w' in '$\widehat{S}_{\mathrm{w}ij}$', '$\widehat{P}_{0\mathrm{w}}$', and '$\widehat{E}_{\mathrm{pw}}$' is an abbreviation for "*wave*". In writing (2.48), I have tacitly made use of definition (2.25w) of Comment 2.4, whereas:

$$\widehat{P}_{0\mathrm{w}}(\underline{x}_2) \doteqdot P_0 \int_0^{Z(t,\underline{x}_2)} dz = P_0 Z(t,\underline{x}_2), \tag{2.37w}$$

which is analogous to (2.37e). In contrast to $\widehat{S}_{ij}(t,\underline{x}_2)$, defined by (2.42), $\widehat{S}_{\mathrm{w}ij}(t,\underline{x}_2)$, defined by (2.48), exists in the case of infinitely deep water.•

Definition 2.7. 1) In accordance with the pertinent instances of (2.8), it follows from (2.48) that for each $i \in \omega_{1,3}$ for each $j \in \omega_{1,3}$ and for each $\underline{x}_2 \in \underline{E}_2$:

$$\begin{aligned}
&\overline{\widehat{S}}_{\mathrm{w}ij}(\underline{x}_2) \doteqdot \overline{\widehat{S}}_{ij}(\underline{x}_2) - \widehat{S}_{\mathrm{e}ij}(\underline{x}_2) \\
&\doteqdot \left[P_0 \overline{Z}(\underline{x}_2) - \rho_0 \overline{\hat{\dot{\Phi}}}(\underline{x}_2)\right]\delta_{ij} + \left[\overline{\widehat{E}}_{ij}(\underline{x}_2) - \overline{\widehat{E}}_{\mathrm{k}}(\underline{x}_2)\delta_{ij}\right] - \frac{1}{2}\rho_0 g \overline{Z^2}(\underline{x}_2)(\delta_{ij} - \delta_{i3}\delta_{j3}) \\
&= \left[\overline{\widehat{P}}_{0\mathrm{w}}(\underline{x}_2) - \rho_0 \overline{\hat{\dot{\Phi}}}(\underline{x}_2)\right]\delta_{ij} + \left[\overline{\widehat{E}}_{ij}(\underline{x}_2) - \overline{\widehat{E}}_{\mathrm{k}}(\underline{x}_2)\delta_{ij}\right] - \overline{\widehat{E}}_{\mathrm{pw}}(\underline{x}_2)(\delta_{ij} - \delta_{i3}\delta_{j3}),
\end{aligned} \tag{2.49}$$

the understanding being that

$$\overline{\dot{\hat{\Phi}}}(\underline{x}_2) \doteq \overline{\dot{\hat{\Phi}}(t,\underline{x}_2)}^t = \overline{\dot{\hat{\Phi}}(t,\underline{x}_2)}^t - \overline{[\Phi(t,\underline{x})]_{z=Z(t,\underline{x}_2)}\dot{Z}(t,\underline{x}_2)}^t, \tag{2.50}$$

by the pertinent instance of (2.6). The 3×3–tensor $\overline{\hat{S}}_{\mathrm{w}ij}(\underline{x}_2)$, defined by (2.49), is called the *full*, or *3×3, time-averaged depth-integrated radiation,* or *wave-related, stress tensor of the perturbed liquid along the given vertical line* $L(\underline{x}_2)$*, defined by (2.43) at a given point* $\underline{x}_2 \in \underline{E}_2$.

2) The 2×2–tensor $\overline{\hat{\mathcal{S}}}{}^{2\times 2}_{\mathrm{w}ij}(\underline{x}_2)$, defined as

$$\overline{\hat{\mathcal{S}}}{}^{2\times 2}_{\mathrm{w}ij}(\underline{x}_2) \doteq \overline{\hat{S}}_{\mathrm{w}ij}(\underline{x}_2) \text{ for each } i \in \omega_{1,2} \text{ for each } j \in \omega_{1,2} \text{ and for each } \underline{x}_2 \in \underline{E}_2, \tag{2.51}$$

is called the *abridged*, or *2×2, time-averaged depth-integrated radiation,* or *wave-related, stress tensor of the perturbed liquid along the given vertical line* $L(\underline{x}_2)$*, defined by (2.43) at a given point* $\underline{x}_2 \in \underline{E}_2$.•

Definition 2.8. Definitions (2.48), (2.49), and (2.51) can be provided with the following *wordy interpretations.*

1) For each $i \in \omega_{1,3}$, for each $j \in \omega_{1,3}$, and for each $(t,\underline{x}_2) \in R \times \underline{E}_2$, the 3×3-tensor $\hat{S}_{\mathrm{w}ij}(t,\underline{x}_2)$ is *the excess of the momentary depth-integrated momentum flux density 3×3-tensor* $\hat{S}_{ij}(t,\underline{x}_2)$ *due to the presence of waves as compared to that in equilibrium,* $\hat{S}_{\mathrm{e}ij}(\underline{x}_2)$.

2) For each $i \in \omega_{1,3}$, for each $j \in \omega_{1,3}$, and for each $\underline{x}_2 \in \underline{E}_2$, the 3×3-tensor $\overline{\hat{S}}_{\mathrm{w}ij}(\underline{x}_2)$ is *the excess of the time-averaged depth-integrated momentum flux density 3×3-tensor* $\overline{\hat{S}}_{ij}(\underline{x}_2)$ *due to the presence of waves as compared to that in equilibrium,* $\hat{S}_{\mathrm{e}ij}(\underline{x}_2)$.

3) For each $i \in \omega_{1,2}$ for each $j \in \omega_{1,2}$ and for each $\underline{x}_2 \in \underline{E}_2$, the 2×2-tensor $\overline{\hat{\mathcal{S}}}{}^{2\times 2}_{\mathrm{w}ij}(\underline{x}_2)$ is *the excess of the abridged,* or *2×2, time-averaged depth-integrated momentum flux density tensor* $\overline{\hat{S}}_{ij}(\underline{x}_2)$ *due to the presence of waves as compared to that in equilibrium,* $\hat{S}_{\mathrm{e}ij}(\underline{x}_2)$.•

Comment 2.7. Longuet-Higgins and Stewart [1960; 1961, p. 575 f; 1962; 1964, pp. 532, 535, and 536 f] seem to have been the first writers to introduce the notion of *radiation stress* for gravity water waves and to calculate the latter from intuitive considerations for a progressive wave in the first non-vanishing approximation with respect to *ka*, By "radiation stress" they meant the time-averaged depth-integrated wave-related 2×2 horizontal momentum flux density tensor, which they denoted by 'S_{ij}' (not to be confused with my homograph 'S_{ij}') subject to $i \in \omega_{1,2}$ and $j \in \omega_{1,2}$ and which they characterized in their paper of 1964 wordily as "*the excess flow of momentum due to the presence of waves*". This description will be referred to as *Longuet-Higgins and Stewart's wordy definiens of their 2×2 time-averaged depth-integrated radiation*, or *wave, stress tensor* (*2×2 TADIRST*) and be denoted by 'S_{ij}^{LS}'. Since then, in the literature on water wave dynamics, there appeared several different *intuitive* logographic (formulary) definitions, – e.g. those of Phillips [1977, p. 62, equation (3.6.12)], Mei [1989, p. 457, equation (2.25)],], and Dingemans [1997, part 1, p. 193, equation (2.446), or p. 211, equation (2.501)]. – of a 2×2 time-averaged depth-integrated tensor, which is equivocally denoted by 'S_{ij}' subject to $i \in \omega_{1,2}$ and $j \in \omega_{1,2}$ and which its author or authors call by the generic name "*radiation stress tensor*", thus regarding its definiens as an adequate *logographic interpretand* of the above wordy definiens. It is, however, amazing that all those different definitions result, as stated by their authors, in the same expression for the 2×2 TADIRST as that obtained initially by Longuet-Higgins and Stewart, in spite of the fact that the latter expression is *debatable*. In this connection, it is worthy to emphasize that the 2×2 TADIRST is always calculated as an *approximate* and tacitly *asymptotic* quantity of the order of $(ka)^{2,}$ and simultaneously as an *approximate* and tacitly *asymptotic* quantity of first order with respect to an unspecified small parameter of mild depth variation, so that intuition is unreliable basis for dealing with such calculations. At the same time, in the absence of rigorous syntactic rules of calculation, one can easily overlook some contributions of the required order $(ka)^2$. I shall therefore discuss the present situation in the matter of radiation stresses after I deduce a comparable concrete expression for the 2×2 TADIRST from the recursive asymptotic theory in question.•

2.6. The depth-integrated continuity equations for the momentum flux density and for the momentary radiation (wave) stress

Theorem 2.3. For each $i \in \omega_{1,3}$ and each $\langle t, \underline{x}_2 \rangle \in R \times \underline{E}_2$:

$$\begin{aligned} &\rho_0 \frac{\partial \widehat{\mathcal{N}}_i(t, \underline{x}_2)}{\partial t} + \sum_{j=1}^{2} \nabla_j \widehat{S}_{ij}(t, \underline{x}_2) \\ &= \left[P_0 + \left(P_{\mathrm{d}}(t, \underline{x})\right)_{z=-h(\underline{x}_2)} + \rho_0 g h(\underline{x}_2)\right] \nabla_i h(\underline{x}_2) + P_0 \nabla_i Z(t, \underline{x}_2) \\ &\quad - \left[\rho_0 g Z(t, \underline{x}_2) - \left(P_{\mathrm{d}}(t, \underline{x})\right)_{z=-h(\underline{x}_2)}\right] \delta_{i3}, \end{aligned} \tag{2.52}$$

which is a tautological equation, being the *depth-integrated one of the tautological continuity equation for the momentum flux density (I.4.53)* or *(I.4.57)*, i.e. *of the equation*

$$\rho_0 \frac{\partial V_i(t,\underline{x})}{\partial t} + \sum_{j=1}^{3} \frac{\partial S_{ij}(t,\underline{x})}{\partial x_j} = 0 \qquad (2.52_1)$$

subject to (I.4.55) and (I.4.56), i.e. *subject to* (2.40_1) *and* (2.41_1), in the perturbed liquid layer. Owing to Corollary 2.2(2), equation (2.52) does not hold as $h_{\mathrm{m}} \to \infty$

Proof: It immediately follows that for each $i \in \omega_{1,3}$:

$$\int_{-h}^{Z} \sum_{j=1}^{3} \nabla_j S_{ij}\, dz = \sum_{j=1}^{2} \int_{-h}^{Z} \nabla_j S_{ij}\, dz + \int_{-h}^{Z} \nabla_3 S_{i3}\, dz = \sum_{j=1}^{2} \int_{-h}^{Z} \nabla_j S_{ij}\, dz + (S_{i3})_{z=Z} - (S_{i3})_{z=-h}\,. \qquad (2.52_2)$$

Hence, by (2.52_1) and (2.52_2), it follows that

$$\int_{-h}^{Z} \left(\rho_0 \frac{\partial V_i}{\partial t} + \sum_{j=1}^{3} \nabla_j S_{ij} \right) dz = \int_{-h}^{Z} \left(\rho_0 \frac{\partial V_i}{\partial t} + \sum_{j=1}^{2} \nabla_j S_{ij} \right) dz + (S_{i3})_{z=Z} - (S_{i3})_{z=-h} = 0\,. \qquad (2.52_3)$$

At the same time, equation (2.6) with $F \stackrel{\rightharpoonup}{=} V_i$ and equation (2.7) with $F \stackrel{\rightharpoonup}{=} S_{ij}$ yield

$$\int_{-h}^{Z} \frac{\partial V_i}{\partial t}\, dz = \frac{\partial \widehat{V}_i}{\partial t} - (V_i)_{z=Z} \frac{\partial Z}{\partial t}, \qquad (2.52_4)$$

$$\int_{-h}^{Z} \sum_{j=1}^{2} \nabla_j S_{ij}\, dz = \sum_{j=1}^{2} \nabla_j \widehat{S}_{ij} - \sum_{j=1}^{2} (S_{ij})_{z=Z} \nabla_j Z - \sum_{j=1}^{2} (S_{ij})_{z=-h} \nabla_j h\,, \qquad (2.52_5)$$

respectively. Hence, the final equation of the train (2.52_3) can be written thus:

$$\rho_0 \frac{\partial \widehat{V}_i}{\partial t} + \sum_{j=1}^{2} \nabla_j \widehat{S}_{ij} - R_{\mathrm{B}i} - R_{\mathrm{T}i} = 0\,, \qquad (2.52_6)$$

where, in analogy with (2.18_7) and (2.18_8),

$$\begin{aligned} R_{\mathrm{B}i} &\stackrel{\rightharpoonup}{=} \left(\sum_{j=1}^{2} S_{ij} \nabla_j h + S_{i3} \right)_{z=-h} \\ &= \left[\left(P_0 - \rho_0 \dot{\Phi} - E_{\mathrm{k}} + \rho_0 g h\right) \nabla_i h + \sum_{j=1}^{2} E_{ij} \nabla_j h + E_{i3} + \left(P_0 - \rho_0 \dot{\Phi} - E_{\mathrm{k}}\right) \delta_{i3} \right]_{z=-h} \\ &= \left[P_0 + (P_{\mathrm{d}})_{z=-h} + \rho_0 g h\right] \nabla_i h + \left[P_0 + (P_{\mathrm{d}})_{z=-h}\right] \delta_{i3}, \end{aligned} \qquad (2.52_7)$$

$$R_{\mathrm{T}i} \fallingdotseq \left(V_i \frac{\partial Z}{\partial t} + \sum_{j=1}^{2} S_{ij} \nabla_j Z - S_{i3} \right)_{z=Z}$$

$$= \left[\left(P_0 - \rho_0 \dot{\Phi} - E_{\mathrm{k}} - \rho_0 g Z \right) \nabla_i Z + V_i \frac{\partial Z}{\partial t} + \sum_{j=1}^{2} E_{ij} \nabla_j Z - E_{i3} - \left(P_0 - \rho_0 \dot{\Phi} - E_{\mathrm{k}} \right) \delta_{i3} \right]_{z=Z} \tag{2.52$_8$}$$

$$= \left[P_0 + \left(P_{\mathrm{d}} \right)_{z=Z} - \rho_0 g Z \right] \nabla_i Z - \left[P_0 + \left(P_{\mathrm{d}} \right)_{z=Z} \right] \delta_{i3} = P_0 \nabla_i Z - \left(P_0 + \rho_0 g Z \right) \delta_{i3}.$$

In developing the equation trains (2.52$_7$) and (2.52$_8$), use has been made of the equations:

$$\sum_{j=1}^{2} E_{ij} \nabla_j h + E_{i3} = \left[V_i \left(\sum_{j=1}^{2} V_j \nabla_j h + V_3 \right) \right]_{z=-h} = 0\,, \tag{2.52$_9$}$$

$$V_i \frac{\partial Z}{\partial t} + \sum_{j=1}^{2} E_{ij} \nabla_j Z - E_{i3} = \left[V_i \left(\frac{\partial Z}{\partial t} + \sum_{j=1}^{2} V_j \nabla_j Z - V_3 \right) \right]_{z=Z} = 0\,, \tag{2.52$_{10}$}$$

which hold owing to the kinematic boundary conditions (2.13$_3$) and (2.13$_4$) respectively. Also, in developing the final result in the train (2.52$_8$), use has been made of the dynamic boundary condition (2.23$_9$) subject to (2.36$_1$) . By (2.52$_7$) and (2.52$_8$), it follows that

$$\begin{aligned} R_{\mathrm{B}i} + R_{\mathrm{T}i} &= \left[P_0 + \left(P_{\mathrm{d}} \right)_{z=-h} + \rho_0 g h \right] \nabla_i h + \left[P_0 + \left(P_{\mathrm{d}} \right)_{z=-h} \right] \delta_{i3} + P_0 \nabla_i Z - \left(P_0 + \rho_0 g Z \right) \delta_{i3} \\ &= \left[P_0 + \left(P_{\mathrm{d}} \right)_{z=-h} + \rho_0 g h \right] \nabla_i h + P_0 \nabla_i Z - \left[\rho_0 g Z - \left(P_{\mathrm{d}} \right)_{z=-h} \right] \delta_{i3} \\ &= \left[P_0 + \left(P_{\mathrm{d}} \right)_{z=-h} + \rho_0 g h \right] \nabla_i h + P_0 \nabla_i Z - \left[\left(P_{\mathrm{d}} \right)_{z=Z} - \left(P_{\mathrm{d}} \right)_{z=-h} \right] \delta_{i3}. \end{aligned} \tag{2.52$_{11}$}$$

Equation (2.52$_6$) subject to (2.52$_{11}$) coincides with (2.52). QED.•

Corollary 2.3. a) For each $i \in \omega_{1,2}$, equations (2.42) and (2.52) become

$$\begin{aligned} \widehat{S}_{ij}\left(t, \underline{x}_2\right) = \left\{ P_0 \left[Z\left(t, \underline{x}_2\right) + h\left(\underline{x}_2\right) \right] - \rho_0 \widehat{\dot{\Phi}}\left(t, \underline{x}_2\right) \right\} \delta_{ij} + \left[\widehat{E}_{ij}\left(t, \underline{x}_2\right) - \widehat{E}_{\mathrm{k}}\left(t, \underline{x}_2\right) \delta_{ij} \right] \\ - \frac{1}{2} \rho_0 g \left[Z^2\left(t, \underline{x}_2\right) - h^2\left(\underline{x}_2\right) \right] \delta_{ij}, \end{aligned} \tag{2.42a}$$

$$\begin{aligned} \rho_0 \frac{\partial \widehat{V}_i\left(t, \underline{x}_2\right)}{\partial t} + \sum_{j=1}^{2} \nabla_j \widehat{S}_{ij}\left(t, \underline{x}_2\right) \\ = P_0 \nabla_i Z\left(t, \underline{x}_2\right) + \left[P_0 + \left(P_{\mathrm{d}}\left(t, \underline{x}\right) \right)_{z=-h\left(\underline{x}_2\right)} + \rho_0 g h\left(\underline{x}_2\right) \right] \nabla_i h\left(\underline{x}_2\right) \end{aligned} \tag{2.52a}$$

for each $\left\langle t, \underline{x}_2 \right\rangle \in R \times \underline{E}_2$, while equation (2.40$_1$) becomes

$$S_{ij}\left(t, \underline{x}\right) \fallingdotseq \left[P_0 - \rho_0 \dot{\Phi}\left(t, \underline{x}\right) \right] \delta_{ij} + \left[E_{ij}\left(t, \underline{x}\right) - E_{\mathrm{k}}\left(t, \underline{x}\right) \delta_{ij} \right] - \rho_0 g z \delta_{ij} \tag{2.40$_1$a}$$

for each $\left\langle t, \underline{x}_2 \right\rangle \in R \times \underline{E}_2$ and each $z \in \left[-h\left(\underline{x}_2\right), Z\left(t, \underline{x}_2\right) \right]$.

b) For $i \fallingdotseq 3$, equations (2.42) and (2.52) become

$$\hat{S}_{3j}(t,\underline{x}_2) = \left\{P_0\left[Z(t,\underline{x}_2)+h(\underline{x}_2)\right]-\rho_0\dot{\hat{\Phi}}(t,\underline{x}_2)\right\}\delta_{3j} + \left[\hat{E}_{3j}(t,\underline{x}_2)-\hat{E}_{\mathrm{k}}(t,\underline{x}_2)\delta_{3j}\right] \\ -\frac{1}{2}\rho_0 g\left[Z^2(t,\underline{x}_2)-h^2(\underline{x}_2)\right]\delta_{i3}(1-\delta_{j3}), \tag{2.42b}$$

$$\rho_0\frac{\partial \hat{V}_3(t,\underline{x}_2)}{\partial t}+\sum_{j=1}^{2}\nabla_j\hat{S}_{3j}(t,\underline{x}_2) = -\rho_0 g Z(t,\underline{x}_2)+\left[P_{\mathrm{d}}(t,\underline{x})\right]_{z=-h(\underline{x}_2)} \tag{2.52b}$$

for each $\langle t,\underline{x}_2\rangle \in R\times\underline{E}_2$, while equation (2.40$_1$) becomes

$$S_{3j}(t,\underline{x}) \doteq \left[P_0-\rho_0\dot{\Phi}(t,\underline{x})\right]\delta_{3j}+\left[E_{3j}(t,\underline{x})-E_{\mathrm{k}}(t,\underline{x})\delta_{3j}\right]-\rho_0 g z\delta_{i3}(1-\delta_{j3}) \tag{2.40$_1$b}$$

for each $\langle t,\underline{x}_2\rangle \in R\times\underline{E}_2$ and each $z\in\left[-h(\underline{x}_2), Z(t,\underline{x}_2)\right]$.

c) If

$$h = C_d\text{, i.e. } h(\underline{x}_2)=d \text{ for each } \underline{x}_2\in\underline{E}_2, \tag{2.53}$$

where 'd' is a constant so that $\nabla_i h(\underline{x}_2)=0$, then for each $i\in\omega_{1,3}$ and each $\langle t,\underline{x}_2\rangle \in R\times\underline{E}_2$, equations (2.42) and (2.52) become

$$\hat{S}_{ij}(t,\underline{x}_2) = \left\{\hat{P}_0(t,\underline{x}_2)-\rho_0\dot{\hat{\Phi}}(t,\underline{x}_2)\right\}\delta_{ij}+\left[\hat{E}_{ij}(t,\underline{x}_2)-\hat{E}_{\mathrm{k}}(t,\underline{x}_2)\delta_{ij}\right]-\hat{E}_{\mathrm{p}}(t,\underline{x}_2)(\delta_{ij}-\delta_{i3}\delta_{j3}) \\ = \left\{P_0\left[Z(t,\underline{x}_2)+d\right]-\rho_0\dot{\hat{\Phi}}(t,\underline{x}_2)\right\}\delta_{ij}+\left[\hat{E}_{ij}(t,\underline{x}_2)-\hat{E}_{\mathrm{k}}(t,\underline{x}_2)\delta_{ij}\right] \\ -\frac{1}{2}\rho_0 g\left[Z^2(t,\underline{x}_2)-d^2\right](\delta_{ij}-\delta_{i3}\delta_{j3}), \tag{2.42c}$$

$$\rho_0\frac{\partial \hat{V}_i(t,\underline{x}_2)}{\partial t}+\sum_{j=1}^{2}\nabla_j\hat{S}_{ij}(t,\underline{x}_2) = P_0\nabla_i Z(t,\underline{x}_2)-\left[\rho_0 g Z(t,\underline{x}_2)-\left(P_{\mathrm{d}}(t,\underline{x})\right)_{z=-h(\underline{x}_2)}\right]\delta_{i3}. \tag{2.52c}$$

while equation (2.40$_1$) remains unaltered.•

Theorem 2.4. For each $i\in\omega_{1,3}$ and each $\langle t,\underline{x}_2\rangle \in R\times\underline{E}_2$:

$$\rho_0\frac{\partial \hat{V}_i(t,\underline{x}_2)}{\partial t}+\sum_{j=1}^{2}\nabla_j\hat{S}_{\mathrm{w}ij}(t,\underline{x}_2) \\ = \left(P_{\mathrm{d}}(t,\underline{x})\right)_{z=-h(\underline{x}_2)}\nabla_i h(\underline{x}_2)+P_0\nabla_i Z(t,\underline{x}_2)-\left[\rho_0 g Z(t,\underline{x}_2)-\left(P_{\mathrm{d}}(t,\underline{x})\right)_{z=-h(\underline{x}_2)}\right]\delta_{i3}, \tag{2.53}$$

which is a tautological continuity equation, being the *full momentary depth-integrated radiation stress-tensor* $\hat{S}_{\mathrm{w}ij}(t,\underline{x}_2)$, defined by (2.48). In contrast to equation (2.52), equation (2.53) exists in the case of infinitely deep water (cf. Definition 2.6).

Proof: It immediately follows from (2.42e) that

$$\sum_{j=1}^{2}\nabla_j\hat{S}_{\mathrm{e}ij}(t,\underline{x}_2) = P_0\nabla_i h(\underline{x}_2)+\frac{1}{2}\rho_0 g\nabla_i h^2(\underline{x}_2). \tag{2.54}$$

Subtraction equation (2.54) from (2.52) yields (2.53).•

Corollary 2.4. It has been shown in paper I that all wave-related bulk characteristics of fluid flow in the perturbed liquid layer decrease exponentially as e^{kz} (k>0, z<0) with |z| increasing (see (I.7.32) and (I.7.78)). Therefore, it is natural to assume that $\widehat{V}_i(t,\underline{x}_2)$ (e.g.) is bounded and that hence

$$\overline{\frac{\partial \widehat{V}_i(t,\underline{x}_2)}{\partial t}}^{t} = 0\,, \tag{2.55}$$

in accordance with Definition 2.2. Consequently, averaging of equations (2.52) and (2.53) with respect to t yields:

$$\begin{aligned}\sum_{j=1}^{2}\nabla_j \overline{\widehat{S}}_{ij}(\underline{x}_2) = {}&\left[P_0 + \left(\overline{P}_{\mathrm{d}}(\underline{x})\right)_{z=-h(\underline{x}_2)} + \rho_0 g h(\underline{x}_2)\right]\nabla_i h(\underline{x}_2) + P_0 \nabla_i \overline{Z}(\underline{x}_2)\\ &- \left[\rho_0 g \overline{Z}(\underline{x}_2) - \left(\overline{P}_{\mathrm{d}}(\underline{x})\right)_{z=-h(\underline{x}_2)}\right]\delta_{i3},\end{aligned} \tag{2.56}$$

$$\sum_{j=1}^{2}\nabla_j \overline{\widehat{S}}_{\mathrm{w}ij}(\underline{x}_2) = \left(\overline{P}_{\mathrm{d}}(\underline{x})\right)_{z=-h(\underline{x}_2)}\nabla_i h(\underline{x}_2) + P_0 \nabla_i \overline{Z}(\underline{x}_2) - \left[\rho_0 g \overline{Z}(\underline{x}_2) - \left(\overline{P}_{\mathrm{d}}(\underline{x})\right)_{z=-h(\underline{x}_2)}\right]\delta_{i3}\,, \tag{2.57}$$

respectively.•

3. Asymptotic power expansions of the depth-integrated functional forms

3.1. A general algorithm for the asymptotic power expansion of a depth-integrated wave-related functional form

Definition 3.1 (a modification of Definition 2.1**).** 1) Unless stated otherwise, I shall henceforth assume in accordance with (I.4.73) that $h = C_d$, i.e. $h(\underline{x}_2) = d$ for each $\underline{x}_2 \in \underline{E}_2$, where '$d$' is a constant, while C_d is the constant funct*ion of* $\underline{x}_2$, every value of which equals d. Accordingly, $\varepsilon \doteqdot ka$, where k>0 is the wave number and a>0 is the amplitude of a *priming* (*seeding*) *progressive*, or *standing*, *plane monochromatic gravity water wave* (briefly *PPPMGWW* or *PSPMGWW* respectively).

2) Let **'**$F(t;\underline{x},\varepsilon)$**'** be a real-valued functional form which, along with its first-order partial derivatives, is defined for each $\langle t,\underline{x}_2\rangle \in R \times \underline{E}_2$ and for each $z \in [-d, Z(t,\underline{x}_2)]$ (i.e. for each $\langle t,\underline{x}\rangle \in \mathbf{D}_{\mathrm{ff}}^{\mathrm{cc}}(-C_d, \mathrm{Z})$, in accordance with Definition I.3.2), and also for each $\varepsilon \in [0,1)$. Then

$$\widehat{F} \doteqdot \widehat{F}(t;\underline{x}_2,\varepsilon) \doteqdot \int_{-h(\underline{x}_2)}^{\mathrm{Z}(t;\underline{x}_2,\varepsilon)} F(t;\underline{x},\varepsilon)dz \doteqdot \int_{-d}^{\mathrm{Z}} F dz = \widehat{F}' + \widehat{F}'', \tag{3.1}$$

$$\widehat{F}' \doteqdot \widehat{F}'(t;\underline{x}_2,\varepsilon) \doteqdot \int_{-h(\underline{x}_2)}^{0} F(t;\underline{x},\varepsilon)dz \doteqdot \int_{-d}^{0} F dz, \tag{3.2}$$

$$\widehat{F}'' \doteqdot \widehat{F}''(t;\underline{x}_2,\varepsilon) \doteqdot \int_{0}^{\mathrm{Z}(t;\underline{x}_2,\varepsilon)} F(t;\underline{x},\varepsilon)dz \doteqdot \int_{0}^{\mathrm{Z}} F dz, \tag{3.3}$$

the understanding being that **'***F***'** is, as before, a placeholder (ellipsis), which should be replaced by a specific base letter (as 'Φ', '*V*', '*E*', '*P*', etc) with some or no labels. The functional form **'**$\widehat{F}(t;\underline{x}_2,\varepsilon)$**'** (or the associated function $\widehat{F}$ of the form) is, as before, said to be the *depth-integrated* of the functional form '$F(t;\underline{x},\varepsilon)$' (or, correspondingly, of the function $\widehat{F}$).•

Corollary 3.1. 1) In accordance with Hypothesis I.4.2, the functional form $\widehat{F}''(t;\underline{x}_2,\varepsilon)$ as specified by (3.3) can, with the help of the Leibnitz rule (cf. Lemma 2.1), be expanded into the Maclaurin series with respect to $\mathrm{Z}(t;\underline{x}_2,\varepsilon)$, so that

$$\widehat{F}''(t;\underline{x}_2,\varepsilon)=\sum_{m=0}^{\infty}\frac{1}{(m+1)!}F^{(m)}(t;\underline{x}_2,\varepsilon)\mathrm{Z}^{m+1}(t;\underline{x}_2,\varepsilon), \tag{3.4}$$

where

$$F^{(m)}(t;\underline{x}_2,\varepsilon)\doteqdot\left[\frac{\partial^m F(t;\underline{x},\varepsilon)}{\partial z^m}\right]_{z=0} \text{ for each } m\in\omega_0,\ F^{(0)}(t;\underline{x}_2,\varepsilon)\doteqdot[F(t;\underline{x},\varepsilon)]_{z=0} \tag{3.5}$$

(cf. (I.4.85) and (I.4.86)). At the same time, by Hypothesis I.5.1(1), for each $\langle t,\underline{x}_2\rangle\in R\times\underline{E}_2$ and for each $\varepsilon\in[0,1)$:

$$\mathrm{Z}(t;\underline{x}_2,\varepsilon)\sim\mathrm{Z}_{[\infty,1]}(t;\underline{x}_2,\varepsilon)\doteqdot\sum_{n=1}^{\infty}\varepsilon^n\zeta_{(n)}(t;\underline{x}_2,\varepsilon), \tag{3.6}$$

whereas by Definition I.5.1, subject to Hypothesis I.5.1(2), given a bulk characteristic of the wave-related fluid flow $F(t;\underline{x},\varepsilon)$, there exists a natural number $\nu\in\omega_0$ such that for each $\langle t,\underline{x}_2\rangle\in R\times\underline{E}_2$, for each $z\in(-d,\mathrm{Z}(t;\underline{x}_2,\varepsilon)]$, and for each $\varepsilon\in[0,1)$:

$$F(t;\underline{x},\varepsilon)\sim F_{[\infty,\nu]}(t;\underline{x},\varepsilon)\doteqdot\sum_{n=\nu}^{\infty}\varepsilon^n f_{(n)}(t,\underline{x}). \tag{3.7}$$

As a consequence, it follows from definition (3.2) that

$$\widehat{F}'(t;\underline{x}_2,\varepsilon)\sim\widehat{F}_{[\infty,\nu]}(t;\underline{x}_2,\varepsilon)\doteqdot\sum_{l=\nu}^{\infty}\varepsilon^l\widehat{f}'_{(l)}(t,\underline{x}_2), \tag{3.8}$$

where

$$\widehat{f}'_{(l)}(t,\underline{x}_2)\doteqdot\int_{-d}^{0}f_{(l)}(t,\underline{x})\,dz \text{ for each } l\in\omega_\nu. \tag{3.9}$$

By (3.7), it also follows from definition (3.5) that

$$F^{(m)}(t;\underline{x}_2,\varepsilon)\sim F^{(m)}_{[\infty,\nu]}(t;\underline{x}_2,\varepsilon)\doteqdot\sum_{n=\nu}^{\infty}\varepsilon^n f^{(m)}_{(n)}(t,\underline{x}_2), \tag{3.10}$$

where for each $n\in\omega_\nu$:

$$f^{(m)}_{(n)}(t,\underline{x}_2)\doteqdot\left[\frac{\partial^m f_{(n)}(t,\underline{x})}{\partial z^m}\right]_{z=0} \text{ for each } m\in\omega_0,\ f^{(0)}_{(n)}(t,\underline{x}_2)\doteqdot\left[f_{(n)}(t,\underline{x})\right]_{z=0}. \tag{3.11}$$

2) On the other hand, the variants of (I.6.14) (subject to the abbreviation $\mathrm{Z}(t,\underline{x}_2)\doteqdot\mathrm{Z}(t;\underline{x}_2,\varepsilon)$) and (I.6.15) with '$m$+1' in place of '$m$', yield

$$\mathrm{Z}^{m+1}(t;\underline{x}_2,\varepsilon)\sim\sum_{l_{m+1}=m+1}^{\infty}\varepsilon^{l_{m+1}}\zeta^{<m+1>}_{(l_{m+1})}(t,\underline{x}_2) \text{ for each } m\in\omega_0, \tag{3.12}$$

where

$$\zeta^{<m+1>}_{(l_{m+1})}\left(t,\underline{x}_2\right) \stackrel{\rightarrow}{=} \sum_{l_m=m}^{l_{m+1}-1}\sum_{l_{m-1}=m-1}^{l_m-1}\ldots\sum_{l_2=2}^{l_3-1}\sum_{l_1=1}^{l_2-1}\zeta_{(l_m-l_{m-1})}\left(t,\underline{x}_2\right)\zeta_{(l_{m-1}-l_{m-2})}\left(t,\underline{x}_2\right)\ldots\zeta_{(l_2-l_1)}\left(t,\underline{x}_2\right)\zeta_{(l_1)}\left(t,\underline{x}_2\right) \tag{3.13}$$

$$\text{for each } m\in\omega_0 \text{ and for each } l_{m+1}\in\omega_{m+1},$$

the understanding being that

$$\zeta^{<m+1>}_{(l)}\left(t,\underline{x}_2\right) \stackrel{\rightarrow}{=} 0 \text{ for each } m\in\omega_0 \text{ and for each } l\in\omega_{0,m}, \tag{3.13$_0$}$$

$$\zeta^{<1>}_{(l_1)}\left(t,\underline{x}_2\right) \stackrel{\rightarrow}{=} \zeta_{(l_1)}\left(t,\underline{x}_2\right) \text{ for } m \stackrel{\rightarrow}{=} 0 \text{ and for each } l_1\in\omega_1, \tag{3.14}$$

$$\zeta^{<2>}_{(l_2)}\left(t,\underline{x}_2\right) \stackrel{\rightarrow}{=} \sum_{l_1=1}^{l_2-1}\zeta_{(l_2-l_1)}\left(t,\underline{x}_2\right)\zeta_{(l_1)}\left(t,\underline{x}_2\right) \text{ for } m \stackrel{\rightarrow}{=} 1 \text{ and for each } l_2\in\omega_2, \tag{3.15}$$

$$\zeta^{<3>}_{(l_3)}\left(t,\underline{x}_2\right) \stackrel{\rightarrow}{=} \sum_{l_2=2}^{l_3-1}\sum_{l_1=1}^{l_2-1}\zeta_{(l_3-l_2)}\left(t,\underline{x}_2\right)\zeta_{(l_2-l_1)}\left(t,\underline{x}_2\right)\zeta_{(l_1)}\left(t,\underline{x}_2\right) \tag{3.16}$$

$$\text{for } m \stackrel{\rightarrow}{=} 2 \text{ and for each } l_3\in\omega_3,$$

in accordance with the pertinent variants of (I.6.15$_0$) and (I.6.16)–(I.6.18). In turn, equation (3.15) for each $l_2\in\omega_{2,4}$, e.g., yields

$$\zeta^{<2>}_{(2)} = \zeta^2_{(1)},\ \zeta^{<2>}_{(3)} = 2\zeta_{(1)}\zeta_{(2)},\ \zeta^{<2>}_{(4)} = 2\zeta_{(1)}\zeta_{(3)} + \zeta^2_{(2)}, \tag{3.17}$$

whereas equation (3.16) for each $l_3\in\omega_{3,5}$, e.g., yields

$$\zeta^{<3>}_{(3)} = \zeta^3_{(1)},\ \zeta^{<3>}_{(4)} = 3\zeta_{(2)}\zeta^2_{(1)},\ \zeta^{<3>}_{(5)} = 3\zeta_{(1)}\left(\zeta_{(1)}\zeta_{(3)} + \zeta^2_{(2)}\right), \tag{3.18}$$

in agreement with (I.6.19) and (I.6.20). Also, in general, given $m\in\omega_0$, it follows from (6.13) that

$$\zeta^{<m+1>}_{(m+1)}\left(t,\underline{x}_2\right) = \zeta^{m+1}_{(1)}\left(t,\underline{x}_2\right) \stackrel{\rightarrow}{=} \left[\zeta_{(l1}\left(t,\underline{x}_2\right)\right]^{m+1}, \tag{3.19}$$

in agreement with the pertinent variant of (I.6.21).

3) By(3.10) and (3.12), the asymptotic expansion of the functional form $\widehat{F}''\left(t;\underline{x}_2,\varepsilon\right)$, defined by (3.4) , in powers of ε can be constructed in analogy with asymptotically expanding either one of the functional forms $\mathrm{A_d}\left(t,\underline{x}_2\right)$ and $\mathrm{A_k}\left(t,\underline{x}_2\right)$ (abbreviations of $\mathrm{A_d}\left(t;\underline{x}_2,\varepsilon\right)$ and $\mathrm{A_k}\left(t;\underline{x}_2,\varepsilon\right)$) in subsections I.6.1 and.I.6.2. The above asymptotic expansion is the object of the following theorem.•

Theorem 3.1. For each $\left\langle t,\underline{x}_2\right\rangle\in R\times\underline{E}_2$ and each $\varepsilon\in[0,1)$:

$$\widehat{F}''\left(t;\underline{x}_2,\varepsilon\right) \sim \widehat{F}''_{[\infty,\nu+1]}\left(t;\underline{x}_2,\varepsilon\right) \stackrel{\rightarrow}{=} \sum_{l=\nu+1}^{\infty}\varepsilon^l\,\widehat{f}''_{(l)}\left(t,\underline{x}_2\right), \tag{3.20}$$

where

$$\widehat{f}''_{(l)}(t,\underline{x}_2)=\sum_{q=\nu}^{l-1}\frac{1}{(l-q)!}\sum_{n=\nu}^{q} f_{(n)}^{(l-q-1)}(t,\underline{x}_2)\zeta_{(l-n)}^{<l-q>}(t,\underline{x}_2) \text{ for each } l\in\omega_{\nu+1}. \tag{3.21}$$

Note. In writing equations (3.20) and (3.21), it is tacitly assumed that, given $\nu\in\omega_0$, for each $\langle t,\underline{x}_2\rangle\in R\times\underline{E}_2$ and each $z\in(-h(\underline{x}_2),\mathrm{Z}(t;\underline{x}_2,\varepsilon)]$:

$$f_{(n)}(t,\underline{x})\doteqdot 0 \text{ for each } n\in\omega_{0,\nu-1} \tag{3.20$_0$}$$

(cf. (3.13$_0$)). Therefore, without altering the final result, the lower limits of summation with respect to q and n in equation (3.20) can be any natural numbers of the set $\omega_{0,\nu}$ and, consequently, the lower limit of summation with respect to l in equation (3.21) can be any natural numbers of the set $\omega_{0,\nu+1}$.

Proof: 1) By (3.10) and (3.12), and in analogy with (I.6.22) or (I.6.23), it follows that for each $m\in\omega_0$:

$$\begin{aligned}F^{(m)}(t;\underline{x}_2,\varepsilon)\mathrm{Z}^{m+1}(t;\underline{x}_2,\varepsilon)&\sim\left[\sum_{n=\nu}^{\infty}\varepsilon^n f_{(n)}^{(m)}(t,\underline{x}_2)\right]\left[\sum_{l_{m+1}=m+1}^{\infty}\varepsilon^{l_{m+1}}\zeta_{(l_{m+1})}^{<m+1>}(t,\underline{x}_2)\right]\\&=\sum_{l_{m+1}=m+1}^{\infty}\sum_{n=\nu}^{\infty}\varepsilon^{l_{m+1}+n} f_{(n)}^{(m)}(t,\underline{x}_2)\zeta_{(l_{m+1})}^{<m+1>}(t,\underline{x}_2).\end{aligned} \tag{3.22}$$

The final expression in (3.22) can be developed further in analogy with item b of the proof of Theorem I.5.2 as follows. Let $l\doteqdot l_{m=1}+n$, so that $l\in\omega_{m+1+\nu}$, because $l=m+1+\nu$ when $l_{m+1}=m+1$ and $n=\nu$. If 'l' is employed as a new variable of summation instead of '$l_{m=1}$', so that $l_{m=1}=l-n$, then the domain of values of the variable 'n' is determined by the conjunction of two relations: (i) $n\in\omega_\nu$, i.e. $\nu\le n<\infty$, and (ii) $n=l-m-1$ at $l_{m+1}=m+1$. Hence, $\nu\le n\le l-m-1$, i.e. $n\in\omega_{\nu,l-m-1}$. Therefore, relation (3.22) becomes

$$F^{(m)}(t;\underline{x}_2,\varepsilon)\mathrm{Z}^{m+1}(t;\underline{x}_2,\varepsilon)\sim\sum_{l=m+1+\nu}^{\infty}\sum_{n=\nu}^{l-m-1}\varepsilon^l f_{(n)}^{(m)}(t,\underline{x}_2)\zeta_{(l-n)}^{<m+1>}(t,\underline{x}_2) \text{ for each } m\in\omega_0 \tag{3.23}$$

(cf. (I.6.24) and (I.6.25)). Hence, equation (3.4) yields

$$\widehat{F}''(t;\underline{x}_2,\varepsilon)\sim\sum_{m=0}^{\infty}\frac{1}{(m+1)!}\sum_{l=m+1+\nu}^{\infty}\sum_{n=\nu}^{l-m-1}\varepsilon^l f_{(n)}^{(m)}(t,\underline{x}_2)\zeta_{(l-n)}^{<m+1>}(t,\underline{x}_2) \tag{3.24}$$

(cf. (I.6.26)–(I.6.31)).

2) Let 'q', defined as $q\doteqdot l-m-1$, be a new variable of summation to be employed in (3.24) instead of 'm'. Therefore, (i) $q=\nu$ when $l=m+1+\nu$ and (ii) $q\doteqdot l-1$ when m=0, so that $q\in\omega_{\nu`,l-1}$.

At the same time, since $l = m+1+q$, therefore $l = \nu+1$ if m=0 and q=ν, so that $l \in \omega_{\nu+1}$. Also, $m = l-q-1$. Hence, (3.24) reduces to

$$\widehat{F}''(t;\underline{x}_2,\varepsilon) \sim \sum_{l=\nu+1}^{\infty} \varepsilon^l \sum_{q=\nu}^{l-1} \frac{1}{(l-q)!} \sum_{n=\nu}^{q} f_{(n)}^{(l-q-1)}(t,\underline{x}_2) \zeta_{(l-n)}^{<l-q>}(t,\underline{x}_2) \tag{3.25}$$

(cf. (I.6.32) and (I.6.33)), which is equivalent to (3.20) subject to (3.21). QED.•

Corollary 3.2. By (3.8) and (3.16), it follows from (3.1)–(3.3) that relation (3.7) holds with

$$\widehat{f}_{(\nu)}(t,\underline{x}_2) = \widehat{f}'_{(\nu)}(t,\underline{x}_2), \tag{3.26}$$

$$\widehat{f}_{(l)}(t,\underline{x}_2) = \widehat{f}'_{(l)}(t,\underline{x}_2) + \widehat{f}''_{(l)}(t,\underline{x}_2) \text{ for each } l \in \omega_{\nu+1}, \tag{3.27}$$

subject to subject to (3.9) and (3.21).•

Corollary 3.3.

$$\begin{aligned} \widehat{f}''_{(\nu+1)}(t,\underline{x}_2) &= \sum_{q=\nu}^{\nu} \frac{1}{(\nu+1-q)!} \sum_{n=\nu}^{q} f_{(n)}^{(\nu-q)}(t,\underline{x}_2) \zeta_{(\nu+1-n)}^{<\nu+1-q>}(t,\underline{x}_2) \\ &= f_{(\nu)}^{(0)}(t,\underline{x}_2) \zeta_{(1)}^{<1>}(t,\underline{x}_2) = \left[f_{(\nu)}(t,\underline{x}) \right]_{z=0} \zeta_{(1)}(t,\underline{x}_2), \end{aligned} \tag{3.28}$$

$$\begin{aligned} \widehat{f}''_{(\nu+2)}(t,\underline{x}_2) &= \sum_{q=\nu}^{\nu+1} \frac{1}{(\nu+2-q)!} \sum_{n=\nu}^{q} f_{(n)}^{(\nu+1-q)}(t,\underline{x}_2) \zeta_{(\nu+2-n)}^{<\nu+2-q>}(t,\underline{x}_2) \\ &= \frac{1}{2!} f_{(\nu)}^{(1)}(t,\underline{x}_2) \zeta_{(2)}^{<2>}(t,\underline{x}_2) + \frac{1}{1!} \sum_{n=\nu}^{\nu+1} f_{(n)}^{(0)}(t,\underline{x}_2) \zeta_{(\nu+2-n)}^{<1>}(t,\underline{x}_2) \\ &= \frac{1}{2} f_{(\nu)}^{(1)}(t,\underline{x}_2) \zeta_{(1)}^{2}(t,\underline{x}_2) + f_{(\nu)}^{(0)}(t,\underline{x}_2) \zeta_{(2)}^{<1>}(t,\underline{x}_2) + f_{(\nu+1)}^{(0)}(t,\underline{x}_2) \zeta_{(1)}^{<1>}(t,\underline{x}_2) \\ &= \frac{1}{2} f_{(\nu)}^{(1)}(t,\underline{x}_2) \zeta_{(1)}^{2}(t,\underline{x}_2) + f_{(\nu)}^{(0)}(t,\underline{x}_2) \zeta_{(2)}(t,\underline{x}_2) + f_{(\nu+1)}^{(0)}(t,\underline{x}_2) \zeta_{(1)}(t,\underline{x}_2). \end{aligned} \tag{3.29}$$

Proof: Equations (3.28) and (3.2) are instances of (3.21) at $l \doteqdot \nu+1$ and at $l \doteqdot \nu+2$ respectively. In developing the final expressions in (3.28) and (3.29), use of (3.14) at $l_1 \in \{1,2\}$, of (3.15) at $l_1 \in \{1,2\}$, and of (3.19) at $m \in \{0,1\}$ have been made.•

3.2. Asymptotic power expansions of specific depth-integrated functional forms

Corollary 3.4. 1) All pertinent logographic operata, i.e. equations and logographic terms, of the previous subsection apply with each one of the following triples of functional variables or base symbols of functional variables:

$$\left\langle \Phi, \phi_{(l)i}, 1 \right\rangle, \left\langle \dot{\Phi}, \dot{\phi}_{(l)i}, 1 \right\rangle, \left\langle V_i, v_{(l)i}, 1 \right\rangle, \left\langle E_{\mathrm{k}}, e_{\mathrm{k}(l)}, 2 \right\rangle, \left\langle Q_{*i}, q_{*(l)i}, 2 \right\rangle, \left\langle Q_i, q_{(l)i}, 1 \right\rangle,$$
$$\left\langle P_{\mathrm{d}}, p_{\mathrm{d}(l)}, 1 \right\rangle, \left\langle S_{ij}, s_{(l)ij}, 1 \right\rangle, \left\langle E_{ij}, e_{(l)ij}, 2 \right\rangle, \text{for each } i \in \omega_{1,3} \text{ and each } j \in \omega_{1,3}, \tag{3.30}$$

in place of the triple of placeholders

$$\left\langle F, f_{(l)}, \nu \right\rangle. \tag{3.31}$$

2) According to the above item 1, the subject matter of the previous subsection applies to any quintet of functions

$$\left\langle \hat{F}, \hat{f}_{(l)}; F, f_{(l)}, \nu \right\rangle, \tag{3.32}$$

or actually to the quintet of functional forms, of which those functions are associate, and which are based on a certain well-defined basic asymptotic power series (3.7) for the pertinent bulk characteristic $F(t; \underline{x}_2, \varepsilon)$.

3) All basic asymptotic power expansions in the range of the place-holding (abstract) relation (3.7), except that for $\dot{\Phi}(t, \underline{x})$, defined by (2.34), are given in subsection I.5.4. At the same time, it immediately follows from (I.5.8) by (2.34) that

$$\dot{\Phi}(t, \underline{x}) \doteq \dot{\Phi}(t; \underline{x}, \varepsilon) \sim \dot{\Phi}_{[\infty,1]}(t; \underline{x}, \varepsilon) \doteq \sum_{n=1}^{\infty} \varepsilon^n \dot{\phi}_{(n)}(t, \underline{x}), \tag{3.33}$$

subject to

$$\dot{\phi}_{(n)}(t, \underline{x}) \doteq \frac{\partial \phi_{(n)}(t, \underline{x})}{\partial t} \text{ for each } n \in \omega_1, \tag{3.34}\bullet$$

Comment 3.1. In accordance with the *dynamic pressure* $P_{\mathrm{d}}(t, \underline{x})$, defined by (I.4.52) or (2.36_1), $\rho_0 \dot{\Phi}(t, \underline{x})$ can be called the *volumetric dynamic energy* and be, accordingly, denoted by '$E_{\mathrm{d}}(t, \underline{x})$', i.e.

$$E_{\mathrm{d}}(t, \underline{x}) \doteq \rho_0 \dot{\Phi}(t, \underline{x}). \tag{3.35}$$

Consequently, definition (I.4.52) or (2.36_1) turns into the equation

$$P_{\mathrm{d}}(t, \underline{x}) = -E_{\mathrm{d}}(t, \underline{x}) - E_{\mathrm{k}}(t, \underline{x}) = -\rho_0 \dot{\Phi}(t, \underline{x}) - \frac{1}{2} \rho_0 \left[\underline{\nabla} \Phi(t, \underline{x}) \right]^2. \tag{3.36}$$

In principle, the velocity potential $\Phi(t, \underline{x})$ of liquid flow is defined with an accuracy to an arbitrary time-dependent functional form $\Phi_0(t)$ (cf. Comment I.4.2), so that .both $P_{\mathrm{d}}(t, \underline{x})$ and $E_{\mathrm{d}}(t, \underline{x})$ are defined with an accuracy to $\rho_0 \dot{\Phi}_0(t)$. However, in paper I and in this exposition I have from the very beginning tacitly assumed that $\Phi_0(t) \doteqdot 0$.•

Corollary 3.5. By the pertinent instance of (3.7) subject to (3.26) and (3.27), it immediately follows from (3.33) subject to (3.3) that

$$\widehat{\dot{\Phi}}\left(t,\underline{x}_2\right) \doteqdot \widehat{\dot{\Phi}}\left(t;\underline{x}_2,\varepsilon\right) \sim \widehat{\dot{\Phi}}_{[\infty,1]}\left(t;\underline{x}_2,\varepsilon\right) \doteqdot \sum_{l=1}^{\infty} \varepsilon^l \widehat{\dot{\phi}}_{(l)}\left(t,\underline{x}_2\right), \tag{3.37}$$

where

$$\widehat{\dot{\phi}}_{(1)}(t,\underline{x}_2) = \widehat{\dot{\phi}}'_{(1)}\left(t,\underline{x}_2\right) \text{ for } l \doteqdot 1, \tag{3.38}$$

$$\widehat{\dot{\phi}}_{(l)}\left(t,\underline{x}_2\right) = \widehat{\dot{\phi}}'_{(l)}\left(t,\underline{x}_2\right) + \widehat{\dot{\phi}}''_{(l)}\left(t,\underline{x}_2\right) \text{ for each } l \in \omega_2\,, \tag{3.39}$$

subject to

$$\widehat{\dot{\phi}}'_{(l)}\left(t,\underline{x}_2\right) = \int_{-d}^{0} \dot{\phi}_{(l)}(t,\underline{x})\, dz \text{ for each } l \in \omega_1\,, \tag{3.40}$$

$$\widehat{\dot{\phi}}''_{(l)}\left(t,\underline{x}_2\right) = \sum_{q=1}^{l-1} \frac{1}{(l-q)!} \sum_{n=1}^{q} \dot{\phi}_{(n)}^{(l-q-1)}(t,\underline{x}_2)\zeta_{(l-n)}^{<l-q>}(t,\underline{x}_2) \text{ for each } l \in \omega_2\,. \tag{3.41}$$

Particularly, equation (3.41) yields:

$$\widehat{\dot{\phi}}''_{(2)}\left(t,\underline{x}_2\right) = \dot{\phi}_{(1)}^{(0)}(t,\underline{x}_2)\zeta_{(1)}^{<1>}(t,\underline{x}_2) = \dot{\phi}_{(1)}^{(0)}(t,\underline{x}_2)\zeta_{(1)}(t,\underline{x}_2) = -g\zeta_{(1)}^{2}(t,\underline{x}_2) \text{ at } l \doteqdot 2\,, \tag{3.41$_1$}$$

$$\begin{aligned}
\widehat{\dot{\phi}}''_{(3)}\left(t,\underline{x}_2\right) &= \sum_{q=1}^{2} \frac{1}{(3-q)!} \sum_{n=1}^{q} \dot{\phi}_{(n)}^{(2-q)}(t,\underline{x}_2)\zeta_{(3-n)}^{<3-q>}(t,\underline{x}_2) \\
&= \frac{1}{2!}\dot{\phi}_{(1)}^{(1)}(t,\underline{x}_2)\zeta_{(2)}^{<2>}(t,\underline{x}_2) + \frac{1}{1!}\sum_{n=1}^{2}\dot{\phi}_{(n)}^{(0)}(t,\underline{x}_2)\zeta_{(3-n)}^{<1>}(t,\underline{x}_2) \\
&= \frac{1}{2}\dot{\phi}_{(1)}^{(1)}(t,\underline{x}_2)\zeta_{(2)}^{<2>}(t,\underline{x}_2) + \dot{\phi}_{(1)}^{(0)}(t,\underline{x}_2)\zeta_{(2)}^{<1>}(t,\underline{x}_2) + \dot{\phi}_{(2)}^{(0)}(t,\underline{x}_2)\zeta_{(1)}^{<1>}(t,\underline{x}_2) \\
&= \frac{1}{2}\dot{\phi}_{(1)}^{(1)}(t,\underline{x}_2)\zeta_{(1)}^{2}(t,\underline{x}_2) + \dot{\phi}_{(1)}^{(0)}(t,\underline{x}_2)\zeta_{(2)}(t,\underline{x}_2) + \dot{\phi}_{(2)}^{(0)}(t,\underline{x}_2)\zeta_{(1)}(t,\underline{x}_2) \text{ at } l \doteqdot 3.
\end{aligned} \tag{3.41$_2$}$$
•

In developing the final result in the train (3.41$_1$), use of (I.6.52$_1$) has been made.

Corollary 3.6 (analogous to Corollary 3.5**).** By the pertinent instance of (3.7) subject to (3.26) and (3.27), it immediately follows from (I.5.43) subject to (I.5.44) that

$$\widehat{V}_i\left(t,\underline{x}_2\right) \doteqdot \widehat{V}_i\left(t;\underline{x}_2,\varepsilon\right) \sim \widehat{V}_{i[\infty,2]}\left(t;\underline{x}_2,\varepsilon\right) \doteqdot \sum_{l=1}^{\infty} \varepsilon^l \widehat{v}_{(l)i}(t,\underline{x}_2) \text{ for each } i \in \omega_{1,3}\,, \tag{3.42}$$

where

$$\hat{v}_{(1)i}(t,\underline{x}_2)=\hat{v}'_{(1)i}(t,\underline{x}_2) \text{ for } l \stackrel{.}{=} 1, \tag{3.43}$$

$$\hat{v}_{(l)i}(t,\underline{x}_2)=\hat{v}'_{(l)i}(t,\underline{x}_2)+\hat{v}''_{(l)i}(t,\underline{x}_2) \text{ for each } l\in\omega_2, \tag{3.44}$$

subject to

$$\hat{v}'_{(l)i}(t,\underline{x}_2)=\int_{-d}^{0} v_{(l)i}(t,\underline{x})\,dz \text{ for each } l\in\omega_1, \tag{3.45}$$

$$\hat{v}''_{(l)i}(t,\underline{x}_2)=\sum_{q=1}^{l-1}\frac{1}{(l-q)!}\sum_{n=1}^{q} v_{(n)i}^{(l-q-1)}(t,\underline{x}_2)\zeta_{(l-n)}^{<l-q>}(t,\underline{x}_2) \text{ for each } l\in\omega_2 \tag{3.46}$$

(cf. (3.37)–(3.41)). Particularly, equation (3.46) yields:

$$\hat{v}''_{(2)i}(t,\underline{x}_2)=v_{(1)i}^{(0)}(t,\underline{x}_2)\zeta_{(1)}^{<1>}(t,\underline{x}_2)=\left[v_{(1)i}(t,\underline{x}_2)\right]_{z=0}\zeta_{(1)}(t,\underline{x}_2) \text{ at } l \stackrel{.}{=} 2, \tag{3.46$_1$}$$

$$\begin{aligned}\hat{v}''_{(3)i}(t,\underline{x}_2)&=\sum_{q=1}^{2}\frac{1}{(3-q)!}\sum_{n=\nu}^{q} v_{(n)i}^{(2-q)}(t,\underline{x}_2)\zeta_{(3-n)}^{<3-q>}(t,\underline{x}_2)\\ &=\frac{1}{2!}v_{(1)i}^{(1)}(t,\underline{x}_2)\zeta_{(2)}^{<2>}(t,\underline{x}_2)+\frac{1}{1!}\sum_{n=1}^{2}v_{(n)i}^{(0)}(t,\underline{x}_2)\zeta_{(3-n)}^{<1>}(t,\underline{x}_2)\\ &=\frac{1}{2}v_{(1)i}^{(1)}(t,\underline{x}_2)\zeta_{(2)}^{<2>}(t,\underline{x}_2)+v_{(1)i}^{(0)}(t,\underline{x}_2)\zeta_{(2)}^{<1>}(t,\underline{x}_2)+v_{(2)i}^{(0)}(t,\underline{x}_2)\zeta_{(1)}^{<1>}(t,\underline{x}_2)\\ &=\frac{1}{2}v_{(1)i}^{(1)}(t,\underline{x}_2)\zeta_{(1)}^{2}(t,\underline{x}_2)+v_{(1)i}^{(0)}(t,\underline{x}_2)\zeta_{(2)}(t,\underline{x}_2)+v_{(2)i}^{(0)}(t,\underline{x}_2)\zeta_{(1)}(t,\underline{x}_2) \text{ at } l \stackrel{.}{=} 3.\end{aligned} \tag{3.46$_2$}$$

(cf. (3.41$_1$) and (3.41$_2$)).•

Corollary 3.7. By the pertinent instance of (3.7) subject to (3.26) and (3.27), it immediately follows from (I.5.49) subject to (I.5.50) that

$$\hat{E}_{\mathrm{k}}(t,\underline{x}_2)\stackrel{.}{=}\hat{E}_{\mathrm{k}}(t;\underline{x}_2,\varepsilon)\sim\hat{E}_{\mathrm{k}[\infty,2]}(t;\underline{x}_2,\varepsilon)\stackrel{.}{=}\sum_{l=2}^{\infty}\varepsilon^{l}\hat{e}_{\mathrm{k}(l)}(t,\underline{x}_2), \tag{3.47}$$

where

$$\hat{e}_{\mathrm{k}(2)}(t,\underline{x}_2)=\hat{e}'_{\mathrm{k}(2)}(t,\underline{x}_2)=\int_{-d}^{0}e_{\mathrm{k}(2)}(t,\underline{x})dz=\frac{1}{2}\rho_9\int_{-d}^{0}\sum_{i=1}^{3}\left[v_{(1)i}(t,\underline{x})\right]^2dz \text{ for } l \stackrel{.}{=} 2, \tag{3.48}$$

$$\hat{e}_{\mathrm{k}(l)}(t,\underline{x}_2)=\hat{e}'_{\mathrm{k}(l)}(t,\underline{x}_2)+\hat{e}''_{\mathrm{k}(l)}(t,\underline{x}_2) \text{ for each } l\in\omega_3, \tag{3.49}$$

subject to

$$\hat{e}'_{\mathrm{k}(l)}(t,\underline{x}_2)=\int_{-d}^{0}e_{\mathrm{k}(l)}(t,\underline{x})dz=\frac{1}{2}\rho_9\int_{-d}^{0}\sum_{m=1}^{l-1}\sum_{i=1}^{3}v_{(m)i}(t,\underline{x})v_{(l-m)i}(t,\underline{x})dz \text{ for each } l\in\omega_2, \tag{3.50}$$

$$\hat{e}''_{\mathrm{k}(l)}(t,\underline{x}_2)=\sum_{q=2}^{l-1}\frac{1}{(l-q)!}\sum_{n=2}^{q}e_{\mathrm{k}(n)}^{(l-q-1)}(t,\underline{x}_2)\zeta_{(l-n)}^{<l-q>}(t,\underline{x}_2) \text{ for each } l\in\omega_3. \tag{3.51}$$

Particularly, equation (3.51) becomes

$$\hat{e}''_{\mathrm{k}(3)}(t,\underline{x}_2)=e^{(l0)}_{\mathrm{k}(2)}(t,\underline{x}_2)\zeta^{<1>}_{(1)}(t,\underline{x}_2)=\left[e_{\mathrm{k}(2)}(t,\underline{x}_2)\right]_{z=0}\zeta_{(1)}(t,\underline{x}_2)\ \text{ at }\ l_2\doteqdot 3, \tag{3.51$_1$}$$

$$\begin{aligned}\hat{e}''_{\mathrm{k}(4)}(t,\underline{x}_2)&=\sum_{q=2}^{3}\frac{1}{(4-q)!}\sum_{n=2}^{q}e^{(3-q)}_{\mathrm{k}(n)}(t,\underline{x}_2)\zeta^{<4-q>}_{(4-n)}(t,\underline{x}_2)\\&=\frac{1}{2!}e^{(1)}_{\mathrm{k}(2)}(t,\underline{x}_2)\zeta^{<2>}_{(2)}(t,\underline{x}_2)+\frac{1}{1!}\sum_{n=2}^{3}e^{(0)}_{\mathrm{k}(n)}(t,\underline{x}_2)\zeta^{<1>}_{(4-n)}(t,\underline{x}_2)\\&=\frac{1}{2}e^{(1)}_{\mathrm{k}(2)}(t,\underline{x}_2)\zeta^{<2>}_{(2)}(t,\underline{x}_2)+e^{(0)}_{\mathrm{k}(2)}(t,\underline{x}_2)\zeta^{<1>}_{(2)}(t,\underline{x}_2)+e^{(0)}_{\mathrm{k}(3)}(t,\underline{x}_2)\zeta^{<1>}_{(1)}(t,\underline{x}_2)\\&=\frac{1}{2}e^{(1)}_{\mathrm{k}(2)}(t,\underline{x}_2)\zeta^{2}_{(1)}(t,\underline{x}_2)+e^{(0)}_{\mathrm{k}(2)}(t,\underline{x}_2)\zeta_{(2)}(t,\underline{x}_2)+e^{(0)}_{\mathrm{k}(3)}(t,\underline{x}_2)\zeta_{(1)}(t,\underline{x}_2)\text{ at } l\doteqdot 4.\end{aligned} \tag{3.51$_2$}\bullet$$

Comment 3.2. 1) In accordance with (I.4.41) or (2.20$_1$), $E_{\mathrm{p}}(z)$ has no asymptotic series in powers of ε or, in other words, it is of the order of ε^0. Therefore, the subject matter of subsection 3.1 is not applicable with 'E_{p}' in place of 'F'. Nevertheless, in accordance with (2.20), $\hat{E}_{\mathrm{p}}(t,\underline{x}_2)$ exists and it can be expanded into a *quasi-recursive* asymptotic series in powers of ε.

To be specific, comparison of (2.20), (2.25), (2.25w), and (2.25e) with (3.1)–(3.3) shows that

$$\hat{E}_{\mathrm{p}}(t,\underline{x}_2)=\hat{E}'_{\mathrm{p}}(\underline{x}_2)+\hat{E}''_{\mathrm{p}}(t,\underline{x}_2)=\hat{E}_{\mathrm{pe}}(\underline{x}_2)+\hat{E}_{\mathrm{pw}}(t,\underline{x}_2), \tag{3.52}$$

where

$$\hat{E}'_{\mathrm{p}}(\underline{x}_2)=\hat{E}_{\mathrm{pe}}(\underline{x}_2)=\int_{-d}^{0}E_{\mathrm{p}}(z)\,dz=\rho_0 g\int_{-d}^{0}z\,dz=-\frac{1}{2}\rho_0 g d^2, \tag{3.53}$$

$$\hat{E}''_{\mathrm{p}}(t,\underline{x}_2)=\hat{E}_{\mathrm{pw}}(t,\underline{x}_2)=\int_{0}^{\mathrm{Z}(t,\underline{x}_2)}E_{\mathrm{p}}(z)\,dz=\rho_0 g\int_{0}^{\mathrm{Z}(t,\underline{x}_2)}z\,dz=\frac{1}{2}\rho_0 g\mathrm{Z}^2(t,\underline{x}_2), \tag{3.54}$$

In this case, at $m\doteqdot 1$, relation (3.12) subject to (3.13) becomes

$$\mathrm{Z}^2(t;\underline{x}_2,\varepsilon)\sim\sum_{l_2=2}^{\infty}\varepsilon^{l_2}\zeta^{<2>}_{(l_2)}(t,\underline{x}_2), \tag{3.55}$$

subject to (3.15).and (3,17), Hence, (3.54) and (3.55) yield

$$\hat{E}''_{\mathrm{p}}(t,\underline{x}_2)=\hat{E}_{\mathrm{pw}}(t,\underline{x}_2)=\frac{1}{2}\rho_0 g\mathrm{Z}^2(t,\underline{x}_2)\sim\frac{1}{2}\rho_0 g\sum_{l=2}^{\infty}\varepsilon^{l}\sum_{m=1}^{l-1}\zeta_{(m)}(t,\underline{x}_2)\zeta_{(l-m)}(t,\underline{x}_2), \tag{3.56}$$

where I have set $l\doteqdot l_2$ and $m\doteqdot l_1$. This is a genuine power asymptotic expansion for $\hat{E}''_{\mathrm{p}}(t,\underline{x}_2)$ or $\hat{E}_{\mathrm{pw}}(t,\underline{x}_2)$. In accordance with (3.53) and (3.56), $\hat{E}_{\mathrm{p}}(t,\underline{x}_2)$, defined by equation (3.52), can be expanded into a power series in ε thus:

$$\widehat{E}_{\mathrm{p}}(t,\underline{x}_2)\sim\sum_{l=0}^{\infty}\varepsilon^l\widehat{e}_{\mathrm{p}(l)}(t,\underline{x}_2)\,, \tag{3.57}$$

where

$$\begin{gathered}\widehat{e}_{\mathrm{p}(0)}(t,\underline{x}_2)\doteqdot\widehat{E}'_{\mathrm{p}}(\underline{x}_2)=-\frac{1}{2}\rho_0 gd^2,\,\widehat{e}_{\mathrm{p}(1)}(t,\underline{x}_2)\doteqdot 0,\\ \widehat{e}_{\mathrm{p}(l)}(t,\underline{x}_2)\doteqdot\frac{1}{2}\rho_0 g\sum_{m=1}^{l-1}\zeta_{(m)}(t,\underline{x}_2)\zeta_{(l-m)}(t,\underline{x}_2)\text{ for each } l\in\omega_2.\end{gathered} \tag{3.58}$$

For instance, in accordance with (3.17$_1$), the second equation (3.58) at $l\doteqdot 2$ and at $l\doteqdot 3$ becomes

$$\widehat{e}_{\mathrm{p}(2)}(t,\underline{x}_2)=\frac{1}{2}\rho_0 g\zeta^2_{(1)}(t,\underline{x}_2)\,, \tag{3.58$_1$}$$

$$\widehat{e}_{\mathrm{p}(3)}(t,\underline{x}_2)=\rho_0 g\zeta_{(2)}(t,\underline{x}_2)\zeta_{(2)}(t,\underline{x}_2)\,, \tag{3.58$_2$}$$

respectively. Owing to the fact that the term of the order of ε^1 is absent in the series (3.57), the latter is not recursive, but it can be qualified *quasi*-recursive.•

Comment 3.3. In accordance with (3.35), multiplication of all terms of equations (3.37)–(3.41$_2$) by ρ_0 results in the equations pertinent to the recursive asymptotic expansion of $\widehat{E}_{\mathrm{d}}(t,\underline{x}_2)$. Particularly, comparison of (3.41$_1$) and (3.58$_1$) shows that

$$\widehat{e}_{\mathrm{d}(2)}(t,\underline{x}_2)=-2\widehat{e}_{\mathrm{p}(2)}(t,\underline{x}_2)\,. \tag{3.59}$$•

Comment 3.4. In accordance with (I.4.42) or (2.21$_1$), $E(t,\underline{x})$ has $E_{\mathrm{p}}(z)$ as one of its two summands, whereas $E_{\mathrm{k}}(t,\underline{x})$, being its second summand, has a recursive asymptotic power series in ε, which begins with a term of the order of ε^2, in accordance with (I.5.49) and (I.5.49). Therefore, the subject matter of subsection 3.1 is not applicable with '*E*' in place of '*F*' either. However, in accordance with (2.26), (2.27), (2.49), and (2.50), $\widehat{E}(t,\underline{x}_2)$, defined by (2.21), has the following *quasi*-recursive asymptotic expansion:

$$\widehat{E}(t,\underline{x}_2)\sim\sum_{l=0}^{\infty}\varepsilon^l\widehat{e}_{(l)}(t,\underline{x}_2)\,, \tag{3.60}$$

where

$$\begin{gathered}\widehat{e}_{(0)}(t,\underline{x}_2)\doteqdot\widehat{e}_{\mathrm{p}(0)}(t,\underline{x}_2)=0,\,\widehat{e}_{(2)}(t,\underline{x}_2)\doteqdot\widehat{e}'_{\mathrm{k}(2)}(t,\underline{x}_2)+\widehat{e}_{\mathrm{p}(2)}(t,\underline{x}_2),\\ \widehat{e}_{(l)}(t,\underline{x}_2)\doteqdot\widehat{e}_{\mathrm{k}(l)}(t,\underline{x}_2)+\widehat{e}_{\mathrm{p}(l)}(t,\underline{x}_2)=\widehat{e}'_{\mathrm{k}(l)}(t,\underline{x}_2)+\widehat{e}''_{\mathrm{k}(l)}(t,\underline{x}_2)+\widehat{e}_{\mathrm{p}(l)}(t,\underline{x}_2)\\ \text{for each } l\in\omega_3,\end{gathered} \tag{3.61}$$

subject to (3.48)–(3.51), (3.58), (3.58$_1$), and (3.58$_2$).•

Corollary 3.8 (analogous to Corollaries 3.5–3.7**).** By the pertinent instance of (3.7) subject to (3.26) and (3.27), it immediately follows from (I.5.64) subject to (I.5.65) that

$$\widehat{Q}_{*i}\left(t,\underline{x}_2\right) \doteqdot \widehat{Q}_{*i}\left(t;\underline{x}_2,\varepsilon\right) \sim \widehat{Q}_{*i[\infty,2]}\left(t;\underline{x}_2,\varepsilon\right) \doteqdot \sum_{l=2}^{\infty} \varepsilon^l \widehat{q}_{*(l)\mathrm{i}}(t,\underline{x}_2) \text{ for each } i \in \omega_{1,3}, \tag{3.62}$$

where

$$\widehat{q}_{*(2)\mathrm{i}}(t,\underline{x}_2) = \widehat{q}'_{*(2)\mathrm{i}}\left(t,\underline{x}_2\right) \text{ for } l \doteqdot 2, \tag{3.63}$$

$$\widehat{q}_{*(l)\mathrm{i}}\left(t,\underline{x}_2\right) = \widehat{q}'_{*(l)\mathrm{i}}\left(t,\underline{x}_2\right) + \widehat{q}''_{*(l)\mathrm{i}}\left(t,\underline{x}_2\right) \text{ for each } l \in \omega_3, \tag{3.64}$$

subject to

$$\begin{aligned} \widehat{q}'_{*(l)\mathrm{i}}\left(t,\underline{x}_2\right) &\doteqdot \int_{-d}^{0} q_{*(l)\mathrm{i}}(t,\underline{x})dz = -\rho_0 \int_{-d}^{0} \sum_{m=1}^{l-1} v_{(m)i}(t,\underline{x})\dot{\phi}_{(l-m)}(t,\underline{x})dz \\ &= -\rho_0 \int_{-d}^{0} \sum_{n=1}^{l-1} v_{(l-n)i}(t,\underline{x})\dot{\phi}_{(n)}(t,\underline{x})dz \text{ for each } l \in \omega_2, \end{aligned} \tag{3.65}$$

$$\widehat{q}''_{*(l)\mathrm{i}}\left(t,\underline{x}_2\right) = \sum_{q=2}^{l-1} \frac{1}{(l-q)!} \sum_{n=2}^{q} q^{(l-q-1)}_{*(n)i}(t,\underline{x}_2)\zeta^{<l-q>}_{(l-n)}(t,\underline{x}_2) \text{ for each } l \in \omega_3 \tag{3.66}$$

(cf. (3.47)–(3.51)). Particularly, equation (3.66) yields:

$$\widehat{q}''_{*(3)\mathrm{i}}\left(t,\underline{x}_2\right) = q^{(0)}_{*(2)i}(t,\underline{x}_2)\zeta^{<1>}_{(1)}(t,\underline{x}_2) = \left[q_{*(2)i}(t,\underline{x}_2)\right]_{z=0}\zeta_{(1)}(t,\underline{x}_2) \text{ at } l \doteqdot 3, \tag{3.66$_1$}$$

$$\begin{aligned} \widehat{q}''_{*(4)\mathrm{i}}\left(t,\underline{x}_2\right) &= \sum_{q=2}^{3} \frac{1}{(4-q)!} \sum_{n=2}^{q} q^{(3-q)}_{*(n)i}(t,\underline{x}_2)\zeta^{<4-q>}_{(4-n)}(t,\underline{x}_2) \\ &= \frac{1}{2!} q^{(1)}_{*(2)i}(t,\underline{x}_2)\zeta^{<2>}_{(2)}(t,\underline{x}_2) + \frac{1}{1!}\sum_{n=2}^{3} q^{(0)}_{*(n)i}(t,\underline{x}_2)\zeta^{<1>}_{(4-n)}(t,\underline{x}_2) \\ &= \frac{1}{2} q^{(1)}_{*(2)i}(t,\underline{x}_2)\zeta^{2}_{(1)}(t,\underline{x}_2) + q^{(0)}_{*(2)i}(t,\underline{x}_2)\zeta^{<1>}_{(2)}(t,\underline{x}_2) + q^{(0)}_{*(3)i}(t,\underline{x}_2)\zeta^{<1>}_{(1)}(t,\underline{x}_2) \\ &= \frac{1}{2} q^{(1)}_{*(2)i}(t,\underline{x}_2)\zeta^{2}_{(1)}(t,\underline{x}_2) + q^{(0)}_{*(2)i}(t,\underline{x}_2)\zeta_{(2)}(t,\underline{x}_2) + q^{(0)}_{*(3)i}(t,\underline{x}_2)\zeta_{(1)}(t,\underline{x}_2) \text{ at } l \doteqdot 4 \end{aligned} \tag{3.66$_2$}$$

(cf. (3.51$_1$) and (3.51$_2$)).•

Corollary 3.9 (analogous to Corollaries 3.5 and 3.6**).** By the pertinent instance of (3.7) subject to (3.26) and (3.27), it immediately follows from (I.5.66) subject to (I.5.67) that

$$\widehat{Q}_{i}\left(t,\underline{x}_2\right) \doteqdot \widehat{Q}_{i}\left(t;\underline{x}_2,\varepsilon\right) \sim \widehat{Q}_{i[\infty,1]}\left(t;\underline{x}_2,\varepsilon\right) \doteqdot \sum_{l=1}^{\infty} \varepsilon^l \widehat{q}_{(l)\mathrm{i}}(t,\underline{x}_2) \text{ for each } i \in \omega_{1,3}, \tag{3.67}$$

where

$$\widehat{q}_{(1)\mathrm{i}}(t,\underline{x}_2) = \widehat{q}'_{(1)\mathrm{i}}\left(t,\underline{x}_2\right) \text{ for } l \doteqdot 1, \tag{3.68}$$

$$\widehat{q}_{(l)\mathrm{i}}\left(t,\underline{x}_2\right)=\widehat{q}'_{(l)\mathrm{i}}\left(t,\underline{x}_2\right)+\widehat{q}''_{(l)\mathrm{i}}\left(t,\underline{x}_2\right) \text{ for each } l\in\omega_2\,, \tag{3.69}$$

subject to

$$\widehat{q}'_{(1)\mathrm{i}}(t,\underline{x}_2)\doteqdot P_0\widehat{v}'_{(1)i}(t,\underline{x}_2),\ \widehat{q}'_{(l)\mathrm{i}}(t,\underline{x}_2)=\widehat{q}'_{*(l)\mathrm{i}}(t,\underline{x}_2)+P_0\widehat{v}'_{(l)i}(t,\underline{x}_2) \text{ for each } l\in\omega_2\,, \tag{3.70}$$

$$\widehat{q}''_{(l)\mathrm{i}}\left(t,\underline{x}_2\right)=\sum_{q=1}^{l-1}\frac{1}{(l-q)!}\sum_{n=1}^{q}q_{(n)i}^{(l-q-1)}(t,\underline{x}_2)\zeta_{(l-n)}^{<l-q>}(t,\underline{x}_2) \text{ for each } l\in\omega_2 \tag{3.71}$$

(cf. (3.37)–(3.41) or (3.42)–(3.46)); ‘P_0’ is a constant in accordance with Convention 2.1. Particularly, equation (3.71) yields:

$$\widehat{q}''_{(2)i}\left(t,\underline{x}_2\right)=q_{(1)i}^{(0)}(t,\underline{x}_2)\zeta_{(1)}^{<1>}(t,\underline{x}_2)=\left[q_{(1)i}(t,\underline{x}_2)\right]_{z=0}\zeta_{(1)}(t,\underline{x}_2) \text{ at } l\doteqdot 2\,, \tag{3.71$_1$}$$

$$\begin{aligned}\widehat{q}''_{(3)i}\left(t,\underline{x}_2\right)&=\sum_{q=1}^{2}\frac{1}{(3-q)!}\sum_{n=1}^{q}q_{(n)i}^{(2-q)}(t,\underline{x}_2)\zeta_{(3-n)}^{<3-q>}(t,\underline{x}_2)\\&=\frac{1}{2!}q_{(1)i}^{(1)}(t,\underline{x}_2)\zeta_{(2)}^{<2>}(t,\underline{x}_2)+\frac{1}{1!}\sum_{n=1}^{2}q_{(n)i}^{(0)}(t,\underline{x}_2)\zeta_{(3-n)}^{<1>}(t,\underline{x}_2)\\&=\frac{1}{2}q_{(1)i}^{(1)}(t,\underline{x}_2)\zeta_{(2)}^{<2>}(t,\underline{x}_2)+q_{(1)i}^{(0)}(t,\underline{x}_2)\zeta_{(2)}^{<1>}(t,\underline{x}_2)+q_{(2)i}^{(0)}(t,\underline{x}_2)\zeta_{(1)}^{<1>}(t,\underline{x}_2)\\&=\frac{1}{2}q_{(1)i}^{(1)}(t,\underline{x}_2)\zeta_{(1)}^{2}(t,\underline{x}_2)+q_{(1)i}^{(0)}(t,\underline{x}_2)\zeta_{(2)}(t,\underline{x}_2)+q_{(2)i}^{(0)}(t,\underline{x}_2)\zeta_{(1)}(t,\underline{x}_2) \text{ at } l\doteqdot 3.\end{aligned} \tag{3.71$_2$}$$

(cf. (3.41$_1$) and (3.41$_2$) or (3.46$_1$) and (3.46$_2$)).•

Corollary 3.10 (analogous to Corollaries 3.7**).** By the pertinent instance of (3.7) subject to (3.26) and (3.27), it immediately follows from (I.5.52) subject to (I.5.53) that for each $i\in\omega_{1,3}$ and each $j\in\omega_{1,3}$:

$$\widehat{E}_{ij}(t,\underline{x})\doteqdot\widehat{E}_{ij}(t;\underline{x},\varepsilon)\sim\widehat{E}_{[\infty,2]ij}\left(t;\underline{x}_2,\varepsilon\right)\doteqdot\sum_{l=2}^{\infty}\varepsilon^l\widehat{e}_{(l)ij}\left(t,\underline{x}_2\right), \tag{3.72}$$

where

$$\widehat{e}_{(2)ij}\left(t,\underline{x}_2\right)=\widehat{e}'_{(2)ij}\left(t,\underline{x}_2\right)=\int_{-d}^{0}e_{(2)ij}(t,\underline{x})dz=\rho_9\int_{-d}^{0}v_{(1)i}(t,\underline{x})v_{(1)j}(t,\underline{x})dz \text{ for } l\doteqdot 2, \tag{3.73}$$

$$\widehat{e}_{(l)ij}\left(t,\underline{x}_2\right)=\widehat{e}'_{(l)ij}\left(t,\underline{x}_2\right)+\widehat{e}''_{(l)ij}\left(t,\underline{x}_2\right) \text{ for each } l\in\omega_3\,, \tag{3.74}$$

subject to

$$\widehat{e}'_{(l)ij}\left(t,\underline{x}_2\right)=\int_{-d}^{0}e_{(l)ij}(t,\underline{x})dz=\rho_9\int_{-d}^{0}v_{(m)i}(t,\underline{x})v_{(l-m)j}(t,\underline{x})dz \text{ for each } l\in\omega_2\,, \tag{3.75}$$

$$\widehat{e}''_{(l)ij}(t,\underline{x}_2)=\sum_{q=2}^{l-1}\frac{1}{(l-q)!}\sum_{n=2}^{q}e_{(n)ij}^{(l-q-1)}(t,\underline{x}_2)\zeta_{(l-n)}^{<l-q>}(t,\underline{x}_2) \text{ for each } l\in\omega_3. \tag{3.76}$$

Particularly, equation (3.76) becomes

$$\widehat{e}''_{(3)ij}(t,\underline{x}_2)=e_{(2)ij}^{(l0)}(t,\underline{x}_2)\zeta_{(1)}^{<1>}(t,\underline{x}_2)=\left[e_{(2)ij}(t,\underline{x}_2)\right]_{z=0}\zeta_{(1)}(t,\underline{x}_2) \text{ at } l_2 \doteqdot 3, \tag{3.76$_1$}$$

$$\begin{aligned}\widehat{e}''_{\mathrm{k}(4)}(t,\underline{x}_2)&=\sum_{q=2}^{3}\frac{1}{(4-q)!}\sum_{n=2}^{q}e_{(n)ij}^{(3-q)}(t,\underline{x}_2)\zeta_{(4-n)}^{<4-q>}(t,\underline{x}_2)\\&=\frac{1}{2!}e_{(2)ij}^{(1)}(t,\underline{x}_2)\zeta_{(2)}^{<2>}(t,\underline{x}_2)+\frac{1}{1!}\sum_{n=2}^{3}e_{(n)ij}^{(0)}(t,\underline{x}_2)\zeta_{(4-n)}^{<1>}(t,\underline{x}_2)\\&=\frac{1}{2}e_{(2)ij}^{(1)}(t,\underline{x}_2)\zeta_{(2)}^{<2>}(t,\underline{x}_2)+e_{(2)ij}^{(0)}(t,\underline{x}_2)\zeta_{(2)}^{<1>}(t,\underline{x}_2)+e_{(3)ij}^{(0)}(t,\underline{x}_2)\zeta_{(1)}^{<1>}(t,\underline{x}_2)\\&=\frac{1}{2}e_{(2)ij}^{(1)}(t,\underline{x}_2)\zeta_{(1)}^{2}(t,\underline{x}_2)+e_{(2)ij}^{(0)}(t,\underline{x}_2)\zeta_{(2)}(t,\underline{x}_2)+e_{(3)ij}^{(0)}(t,\underline{x}_2)\zeta_{(1)}(t,\underline{x}_2) \text{ at } l \doteqdot 4.\end{aligned} \tag{3.76$_2$}$$

(cf. (3.51$_1$) and (3.51$_2$)).•

Corollary 3.11. In accordance with (3.6), (3.37)–(3.41), (3.47)–(3.51), (3.56), and (3.72)–(3.76), for each $i\in\omega_{1,3}$, for each $j\in\omega_{1,3}$, for each $\langle t,\underline{x}_2\rangle\in R\times\underline{E}_2$, and for each $\varepsilon\in[0,1)$, the *3×3 momentary depth-integrated radiation* (or *wave-related*) *stress tensor* $\widehat{S}_{\mathrm{w}ij}(t,\underline{x}_2)$, which is defined by (2.48), is expanded into the following recursive asymptotic series in powers of the latent parameter ε:

$$\widehat{S}_{\mathrm{w}ij}(t,\underline{x}_2)\doteqdot\widehat{S}_{\mathrm{w}ij}(t;\underline{x}_2,\varepsilon)\sim\widehat{S}_{\mathrm{w}[\infty,1]ij}(t;\underline{x}_2,\varepsilon)\doteqdot\sum_{l=1}^{\infty}\varepsilon^l\widehat{s}_{\mathrm{w}(l)ij}(t,\underline{x}_2), \tag{3.77}$$

where

$$\widehat{s}_{\mathrm{w}(1)ij}(t,\underline{x}_2)\doteqdot\left[P_0\zeta_{(1)}(t,\underline{x}_2)-\rho_0\widehat{\dot{\phi}}'_{(1)}(t,\underline{x}_2)\right]\delta_{ij} \text{ for } l\doteqdot 1, \tag{3.77$_1$}$$

$$\begin{aligned}\widehat{s}_{\mathrm{w}(2)ij}(t,\underline{x}_2)\doteqdot&\left\{P_0\zeta_{(2)}(t,\underline{x}_2)-\rho_0\left[\widehat{\dot{\phi}}'_{(2)}(t,\underline{x}_2)+\widehat{\dot{\phi}}''_{(2)}(t,\underline{x}_2)\right]\right\}\delta_{ij}\\&-\frac{1}{2}\rho_0 g\zeta_{(2)}^{<2>}(t,\underline{x}_2)(\delta_{ij}-\delta_{i3}\delta_{j3})+\widehat{e}'_{(2)ij}(t,\underline{x}_2)-\widehat{e}'_{\mathrm{k}(2)}(t,\underline{x}_2)\delta_{ij} \text{ for } l\doteqdot 2,\end{aligned} \tag{3.77$_2$}$$

$$\begin{aligned}\widehat{s}_{\mathrm{w}(l)ij}(t,\underline{x}_2)\doteqdot&\left\{P_0\zeta_{(l)}(t,\underline{x}_2)-\rho_0\left[\widehat{\dot{\phi}}'_{(l)}(t,\underline{x}_2)+\widehat{\dot{\phi}}''_{(l)}(t,\underline{x}_2)\right]\right\}\delta_{ij}\\&-\frac{1}{2}\rho_0 g\zeta_{(l)}^{<2>}(t,\underline{x}_2)(\delta_{ij}-\delta_{i3}\delta_{j3})+\widehat{e}'_{(l)ij}(t,\underline{x}_2)+\widehat{e}''_{(l)ij}(t,\underline{x}_2)\\&-\left[\widehat{e}'_{\mathrm{k}(l)}(t,\underline{x}_2)+\widehat{e}''_{\mathrm{k}(l)}(t,\underline{x}_2)\right]\delta_{ij} \text{ for each } l\in\omega_3;\end{aligned} \tag{3.77$_3$}$$

'P_0' is a constant in accordance with Convention 2.1. By (3.40) at $l\doteqdot 1$, equation (3.77$_1$) becomes

$$\widehat{s}_{\mathrm{w}(1)ij}(t,\underline{x}_2)\doteqdot\left[P_0\zeta_{(1)}(t,\underline{x}_2)-\rho_0\int_{-d}^{0}\dot{\phi}_{(1)}(t,\underline{x})\,dz\right]\delta_{ij} \text{ for } l\doteqdot 1, \tag{3.78}$$

At the same time, by the first equation (3.17) and by equation (3.41_1), it follows that

$$\hat{\dot{\phi}}''_{(2)}(t,\underline{x}_2)+\frac{1}{2}g\zeta^{<2>}_{(2)}(t,\underline{x}_2)=-g\zeta^2_{(1)}(t,\underline{x}_2)+\frac{1}{2}g\zeta^2_{(1)}(t,\underline{x}_2)=-\frac{1}{2}g\zeta^2_{(1)}(t,\underline{x}_2). \qquad (3.77_{2'})$$

Hence, by equations (3.40) at $l=2$, (3.48), (3.73), and ($3.77_{2'}$), equation (3.77_2) can be written as

$$\begin{aligned}\hat{s}_{\mathrm{w}(2)ij}(t,\underline{x}_2) &\doteqdot \left[P_0\zeta_{(2)}(t,\underline{x}_2)-\rho_0\hat{\dot{\phi}}'_{(2)}(t,\underline{x}_2)-\hat{e}'_{\mathrm{k}(2)}(t,\underline{x}_2)\delta_{ij}\right]\delta_{ij}\\ &+\frac{1}{2}\rho_0 g\zeta^2_{(1)}(t,\underline{x}_2)(\delta_{ij}+\delta_{i3}\delta_{j3})+\hat{e}'_{(2)ij}(t,\underline{x}_2)\\ &\doteqdot\left[P_0\zeta_{(2)}(t,\underline{x}_2)-\rho_0\int_{-d}^{0}\dot{\phi}_{(2)}(t,\underline{x})\,dz-\frac{1}{2}\rho_9\int_{-d}^{0}\sum_{i=1}^{3}\left[v_{(1)i}(t,\underline{x})\right]^2dz\right]\delta_{ij}\\ &+\frac{1}{2}\rho_0 g\zeta^2_{(1)}(t,\underline{x}_2)(\delta_{ij}+\delta_{i3}\delta_{j3})+\rho_9\int_{-d}^{0}v_{(1)i}(t,\underline{x})v_{(1)j}(t,\underline{x})dz \text{ for } l\doteqdot 2.\end{aligned} \qquad (3.79)$$

The expression on the right-hand side of equation (3.77_3) can be particularized likewise by making use of (3.40), (3.41), (3.50), (3.51), (3.56), (3.75), and (3.76). However, I shall not bother to do so, because I do not intend to calculate any term of the series (3.77), of the order higher than 2 in terms of elementary functions. That is to say, I shall confine to the following approximate asymptotic approximation to $\hat{S}_{\mathrm{w}ij}(t;\underline{x}_2,ka)$ both in the case of a PPPMGWW and in the case of PSPMGWW:

$$\hat{S}_{\mathrm{w}ij}(t,\underline{x}_2)\doteqdot\hat{S}_{\mathrm{w}ij}(t;\underline{x}_2,ka)\approx\hat{S}_{\mathrm{w}[2,1]ij}(t;\underline{x}_2,ka)\doteqdot\hat{S}_{\mathrm{w}(1)ij}(t;\underline{x}_2,ka)+\hat{S}_{\mathrm{w}(2)ij}(t;\underline{x}_2,ka), \qquad (3.80)$$

where

$$\hat{S}_{\mathrm{w}(l)ij}(t;\underline{x}_2,ka)\doteqdot(ka)^l\hat{s}_{\mathrm{w}(l)ij}(t,\underline{x}_2) \text{ for each } l\in\{1,2\}, \qquad (3.81)$$

subject to (3.77_1) or (3.78) and subject to (3.79).•

Corollary 3.12. In accordance with the pertinent instances of the general place-holding definition (3.8), it follows from Corollary 3.11 that for each $i\in\omega_{1,3}$, for each $j\in\omega_{1,3}$, for each $\underline{x}_2\in\underline{E}_2$, and for each $\varepsilon\in[0,1)$, the *3×3 time averaged depth-integrated radiation* (or *wave-related*) *stress tensor* $\bar{\bar{S}}_{\mathrm{w}ij}(\underline{x}_2)$, which is defined by (2.49), is expanded into the following recursive asymptotic series in powers of the latent parameter ε:

$$\bar{\bar{S}}_{\mathrm{w}ij}(\underline{x}_2)\doteqdot\bar{\bar{S}}_{\mathrm{w}ij}(\underline{x}_2,\varepsilon)\doteqdot\overline{\hat{S}_{\mathrm{w}ij}(t;\underline{x}_2,\varepsilon)}^{t}\sim\bar{\bar{S}}_{\mathrm{w}[\infty,1]ij}(\underline{x}_2,\varepsilon)\doteqdot\overline{\hat{S}_{\mathrm{w}[\infty,1]ij}(t;\underline{x}_2,\varepsilon)}^{t}\doteqdot\sum_{l=1}^{\infty}\varepsilon^l\bar{\bar{s}}_{\mathrm{w}(l)ij}(\underline{x}_2), \qquad (3.82)$$

where

$$\bar{\bar{s}}_{\mathrm{w}(l)ij}(\underline{x}_2)\doteqdot\overline{\hat{s}_{\mathrm{w}(l)ij}(t,\underline{x}_2)}^{t}\doteqdot\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\hat{s}_{\mathrm{w}(l)ij}(t,\underline{x}_2)dt \text{ for each } l\in\omega_1. \qquad (3.83)$$

subject to (3.77_3), (3.78), and (3.79). Hence, both in the case of a PPPMGWW and in the case of PSPMGWW, the 3×3-tensor $\bar{\bar{\hat{S}}}_{\mathrm{w}ij}(\underline{x}_2, ka)$ in the first non-vanishing approximation in powers of *ka* can, in accordance with (3.80) and (3.81), be written as:

$$\bar{\bar{\hat{S}}}_{\mathrm{w}ij}\left(\underline{x}_2, ka\right) \approx \bar{\bar{\hat{S}}}_{\mathrm{w}[2,1]ij}\left(\underline{x}_2, ka\right) \doteqdot \bar{\bar{\hat{S}}}_{\mathrm{w}(1)ij}\left(\underline{x}_2, ka\right) + \bar{\bar{\hat{S}}}_{\mathrm{w}(2)ij}\left(\underline{x}_2, ka\right), \tag{3.84}$$

where

$$\bar{\bar{\hat{S}}}_{\mathrm{w}(l)ij}\left(\underline{x}_2, ka\right) \doteqdot (ka)^l \bar{\bar{\hat{s}}}_{\mathrm{w}(l)ij}\left(\underline{x}_2\right) \text{ for each } l \in \{1,2\}, \tag{3.85}$$

subject to (3.83) at $l \in \{1,2\}$. That is to say,

$$\begin{aligned} \bar{\bar{\hat{S}}}_{\mathrm{w}(1)ij}\left(\underline{x}_2, ka\right) &\doteqdot \overline{\hat{S}_{\mathrm{w}(1)ij}\left(t; \underline{x}_2, ka\right)}^{t} = ka\,\overline{\hat{s}_{\mathrm{w}(1)ij}\left(t, \underline{x}_2\right)}^{t} \\ &= ka\left[P_0 \overline{\zeta_{(1)}\left(t, \underline{x}_2\right)}^{t} - \rho_0 \overline{\dot{\hat{\phi}}'_{(1)}(t, \underline{x}_2)}^{t}\right]\delta_{ij} \text{ for } l \doteqdot 1, \end{aligned} \tag{3.86}$$

$$\begin{aligned} \bar{\bar{\hat{S}}}_{\mathrm{w}(2)ij}\left(\underline{x}_2, ka\right) &\doteqdot \overline{\hat{S}_{\mathrm{w}(2)ij}\left(t; \underline{x}_2, ka\right)}^{t} = (ka)^2\,\overline{\hat{s}_{\mathrm{w}(2)ij}\left(t, \underline{x}_2\right)}^{t} \\ &= (ka)^2\left\{\left[P_0 \overline{\zeta_{(2)}\left(t, \underline{x}_2\right)}^{t} - \rho_0 \overline{\dot{\hat{\phi}}'_{(2)}(t, \underline{x}_2)}^{t} - \overline{\hat{e}'_{\mathrm{k}(2)}\left(t, \underline{x}_2\right)}^{t}\right]\delta_{ij}\right. \\ &\left. + \frac{1}{2}\rho_0 g \overline{\zeta^2_{(1)}\left(t, \underline{x}_2\right)}^{t}\left(\delta_{ij} + \delta_{i3}\delta_{j3}\right) + \overline{\hat{e}'_{(2)ij}\left(t, \underline{x}_2\right)}^{t}\right\} \text{ for } l \doteqdot 2. \end{aligned} \tag{3.87}$$

•

Comment 3.5. Making use of the appropriate corollaries that have been made explicit above in this subsection, one can deduce an infinite sequence of asymptotic continuity equations for each depth-integrated continuity equation occurring in section 2. However, I shall not bother to do these trivial deductions here.•

4. The 3×3 time averaged depth-integrated radiation stress tensor of a PPPMGWW and that of a PSPMGWW in the first non-vanishing approximation with respect to *ka*

Convention 4.1. Henceforth, I shall use the definitions (I.7.38) and (I.8.30), i.e,

$$\Omega(k) \doteqdot \Omega(k,d) \doteqdot \sqrt{gk\tanh kd} > 0\,, \tag{4.1}$$

$$\tau_1 \doteqdot \sin,\ \ \tau_{-1} \doteqdot \cos\,, \tag{4.2}$$

respectively, without any additional comments.•

4.1. The case of a PPPMGWW

1. Given $\mu \in \{1,-1\}$, the following four trains of equations are tokens of slightly modified (enriched) trains (I.10.22)–(I.10.25) respectively:

$$\begin{aligned}\Phi_{(1)}(t,\underline{x}) &\doteqdot \Phi_{(1)}(t;\underline{x},ka) = ka\phi_{(1)}(t,\underline{x},)\\ &\doteqdot -\mu a\sqrt{\frac{g}{k\tanh kd}}\frac{\cosh k(z+d)}{\cosh kd}\tau_\mu(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2),\end{aligned} \tag{4.3}$$

$$\begin{aligned}\Phi_{(2)}(t,\underline{x}) &\doteqdot \Phi_{(2)}(t;\underline{x},ka) = (ka)^2\phi_{(2)}(t,\underline{x})\\ &= \frac{\mu a^2}{2\cosh kd\sinh 2kd}\sqrt{\frac{gk}{\tanh kd}}\cosh k(z+d)\\ &\cdot\{-2\sin 2[\Omega(k)t-\underline{k}_2\cdot\underline{x}_2]+\sin[\Omega(k)t+2\underline{k}_2\cdot\underline{x}_2]+3\sin[\Omega(k)t-2\underline{k}_2\cdot\underline{x}_2]\},\end{aligned} \tag{4.4}$$

$$\begin{aligned}\mathrm{Z}_{(1)}(t,\underline{x}_2) &\doteqdot \mathrm{Z}_{(1)}(t;\underline{x}_2,ka) = ka\zeta_{(1)}(t,\underline{x}_2) = -\frac{ka}{g}\dot{\phi}^{(0)}_{(1)}(t,\underline{x}_2) = -\frac{ka}{g}\left[\dot{\phi}_{(1)}(t,\underline{x})\right]_{z=0}\\ &= -\frac{1}{g}\left[\dot{\Phi}_{(1)}(t;\underline{x},ka)\right]_{z=0} \doteqdot -\frac{1}{g}\left[\dot{\Phi}_{(1)}(t,\underline{x})\right]_{z=0} = a\tau_{-\mu}(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2),\end{aligned} \tag{4.5}$$

$$\begin{aligned}\mathrm{Z}_{(2)}(t,\underline{x}_2) &\doteqdot \mathrm{Z}_{21)}(t;\underline{x}_2,ka) = (ka)^2\zeta_{(3)}(t,\underline{x}_2)\\ &= -\frac{a^2k}{2\sinh 2kd}\{1+\mu[-(2+\cosh 2kd)\cos 2(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2)\\ &+\cos(\Omega(k)t+2\underline{k}_2\cdot\underline{x}_2)+3\cos(\Omega(k)t-2\underline{k}_2\cdot\underline{x}_2)]\}.\end{aligned} \tag{4.6}$$

2. Under the pertinent instances of the general (place-holding) definition (I.10.28) or (2.8), it follows from (4.3) and (4.4) that

$$\overline{\hat{\dot{\Phi}}'_{(l)}(t;\underline{x}_2,ka)}^{t} = ka\overline{\hat{\dot{\phi}}'_{(l)}(t,\underline{x}_2)}^{t} = 0\ \text{ for each } l\in\{1,2\}, \tag{4.7}$$

because depth-integrating $\dot{\Phi}_{(l)}(t;\underline{x}_2,ka)$ or $\dot{\phi}_{(l)}(t,\underline{x}_2)$ between -*d* and 0 reduces to calculating the elementary integral:

$$\int_{-d}^{0}\cosh k(z+d)\,dz=\frac{1}{k}\left[\sinh k(z+d)\right]_{x=-d}^{z=0}=\frac{1}{k}\sinh kd\ , \tag{4.8}$$

which does not affect any trigonometric functional forms involved. At the same time, it follows from (4.5) and (4.6) that

$$\overline{Z_{(1)}(t;\underline{x}_2,ka)}^t=ka\overline{\zeta_{(1)}(t,\underline{x}_2)}^t=0\,, \tag{4.9}$$

$$\overline{Z_{(2)}(t;\underline{x}_2,ka)}^t=(ka)^2\overline{\zeta_{(2)}(t,\underline{x}_2)}^t=-\frac{a^2k}{2\sinh 2kd}\,, \tag{4.10}$$

$$\overline{\dot{Z}_{(1)}(t;\underline{x}_2,ka)}^t=ka\overline{\dot{\zeta}_{(1)}(t,\underline{x}_2)}^t=(ka)^2\overline{\dot{\zeta}_{(2)}(t,\underline{x}_2)}^t=\overline{\dot{Z}_{(2)}(t;\underline{x}_2,ka)}^t=0\,. \tag{4.11}$$

Equations (4.9) and (4.10) are tokens of (I.10.29) and (I.10.30) respectively. Also, equation (4.10) agrees with equation (4.12) in Longuet-Higgins and Stewart [1962], which was deduced there from intuitive considerations. Given $a\in(0,\infty)$, given $k\in(0,\infty)$, it follows from (4.10) that

$$\lim_{d\to\infty}\overline{Z_{(2)}(t;\underline{x}_2,ka)}^t=(ka)^2\lim_{d\to\infty}\overline{\zeta_{(2)}(t,\underline{x}_2)}^t=-\lim_{d\to\infty}\frac{a^2k}{2\sinh 2kd}=-0\,. \tag{4.10$_1$}$$

3. By (4.7) and (4.9), it follows from (3.86) and (3.85) at $l\doteqdot 1$, subject to (3.78) and (3.83) that

$$\overline{\hat{S}}_{\mathrm{w}(1)ij}(\underline{x}_2,ka)\doteqdot\overline{\hat{S}_{\mathrm{w}(1)ij}(t;\underline{x}_2,ka)}^t=ka\overline{\hat{s}_{\mathrm{w}(1)ij}(t,\underline{x}_2)}^t=0\,. \tag{4.12}$$

4. In accordance with (4.3) and (4.4), the operation of depth-integration of any pertinent bulk functional form will always apply to a constituent combination of *elementary hyperbolic* functional forms '$\cosh k(z+d)$' and '$\sinh k(z+d)$', whereas the next operation of time-averaging the resulting expression will always apply to constituent *elementary trigonometric* functional forms such ac'$\sin\left[m\Omega(k)t\mp \underline{nk}_2\cdot\underline{x}_2\right]$' and '$\cos\left[m\Omega(k)t\mp \underline{nk}_2\cdot\underline{x}_2\right]$', where *m*>0 and *n*>0 are strictly positive natural numbers Hence, the two operations are commutative, i.e. schematically

$$\overline{\hat{F}}=\hat{\overline{F}}\,. \tag{4.13}$$

This rule allows avoiding calculation of depth-integrating functional forms that will vanish after their subsequent time-averaging.

5. By (4.2), it follows that

$$\nabla_i\tau_\mu\left(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\right)=-\mu k_i\tau_{-\mu}\left(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\right)\ \text{for each}\ i\in\{1,2\}. \tag{4.14}$$

Making use of (4.14) and also of the equation $\mu^2 = 1$, and letting that

$$k_3 \doteqdot 0\,, \tag{4.15}$$

it follows by (4.3) that

$$\begin{gathered}v_{(1)i}(t,\underline{x}) = \nabla_i \phi_{(1)}(t,\underline{x}) = \frac{\partial \phi_{(1)}(t,\underline{x})}{\partial x_i} = \frac{1}{k\cosh kd}\sqrt{\frac{g}{k\tanh kd}} \\ \cdot\left[k_i \cosh k(z+d)\tau_{-\mu}\left(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2\right) - \mu k\delta_{i3}\sinh k(z+d)\tau_{\mu}\left(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2\right)\right] \\ \text{for each } i\in\omega_{1,3},\end{gathered} \tag{4.16}$$

which incorporates both equations (I.8.35a) and (I.8.35b) for the sake of convenience in the following reasoning. By (4.16), it follows that

$$\begin{gathered}e_{(2)ij}(t,\underline{x}) \doteqdot \rho_0 v_{(1)i}(t,\underline{x}) v_{(1)j}(t,\underline{x}) \\ = \frac{\rho_0 g}{k^3\cosh^2 kd\tanh kd}\left[k_i k_j \cosh^2 k(z+d)\tau_{-\mu}{}^2\left(\alpha(t,\underline{x}_2)\right)\right. \\ - \mu k\left(k_i\delta_{j3} + k_j\delta_{i3}\right)\cosh k(z+d)\sinh k(z+d)\tau_{-\mu}\left(\alpha(t,\underline{x}_2)\right)\tau_{\mu}\left(\alpha(t,\underline{x}_2)\right) \\ \left. + \mu^2 k^2\delta_{i3}\delta_{j3}\sinh^2 k(z+d)\tau_{\mu}{}^2\left(\alpha(t,\underline{x}_2)\right)\right] \text{for each } i\in\omega_{1,3} \text{ and each } j\in\omega_{1,3},\end{gathered} \tag{4.17}$$

where

$$\alpha(t,\underline{x}_2) \doteqdot \Omega(k)t - \underline{k}_2\cdot\underline{x}_2\,, \tag{4.18}$$

for the sake of brevity. In this case,

$$\tanh kd\cosh^2 kd = \frac{\sinh kd\cosh^2 kd}{\cosh kd} = \sinh kd\cosh kd = \frac{1}{2}\sinh 2kd\ , \tag{4.19}$$

– in agreement with (I.8.44)¸

$$\tau_{\pm\mu}{}^2(\alpha) = \frac{1}{2}\left(1 \mp \mu\cos 2\alpha\right)\ \text{(a)},\ \ \tau_{\mu}(\alpha)\tau_{-\mu}(\alpha) = \frac{1}{2}\sin 2\alpha\ \ \text{(b)},\ \ \text{for each } \mu\in\{1,-1\}, \tag{4.20}$$

– in agreement with (I.8.40), and also

$$\begin{gathered}\sinh\alpha\cosh\alpha = \frac{1}{2}\sinh 2\alpha\ \ \text{(a)},\ \ \cosh^2\alpha + \sinh^2\alpha = \cosh 2\alpha\ \ \text{(b)}, \\ \cosh^2\alpha - \sinh^2\alpha = 1\ \ \text{(c)},\ \ \text{for } \beta \doteqdot k(z+d),\end{gathered} \tag{4.21}$$

– in agreement with (I.8.42a). Hence, equation (4.17) for each $i\in\omega_{1,3}$ and each $j\in\omega_{1,3}$ reduces to

$$
\begin{aligned}
&e_{(2)ij}(t,\underline{x}) \doteq \rho_0 v_{(1)i}(t,\underline{x}) v_{(1)j}(t,\underline{x}) \\
&= \frac{\rho_0 g}{k^3 \sinh 2kd} \Big\{ k_i k_j \cosh^2 k(z+d)\big[1+\mu\cos\big(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\big)\big] \\
&\quad -\frac{1}{2}\mu k\big(k_i\delta_{j3}+k_j\delta_{i3}\big)\sinh 2k(z+d)\sin 2\big(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\big) \\
&\quad +k^2\delta_{i3}\delta_{j3}\sinh^2 k(z+d)\big[1-\mu\cos\big(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\big)\big]\Big\},
\end{aligned}
\tag{4.22}
$$

because $\mu^2 = 1$. From (4.22), it immediately follows that

$$
\begin{aligned}
&e_{\mathrm{k}(2)}(t,\underline{x}) \doteq \frac{1}{2}\rho_9 \sum_{i=3}^{3} e_{(2)ii}(t,\underline{x}) \doteq \frac{1}{2}\rho_9 \sum_{i=3}^{3}\big[v_{(1)i}(t,\underline{x})\big]^2 \\
&= \frac{\rho_9 g}{2k\sinh 2kd}\Big\{\cosh^2 k(z+d)\big[1+\mu\cos 2\big(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\big)\big] \\
&\quad +\sinh^2 k(z+d)\big[1-\mu\cos 2\big(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\big)\big]\Big\} \\
&= \frac{\rho_9 g}{2k\sinh 2kd}\big[\cosh 2k(z+d)+\mu\cos 2\big(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\big)\big],
\end{aligned}
\tag{4.23}
$$

– in agreement with (I.8.45).

6. For convenience in further computations, I shall make explicit the special expressions for two simple integrals, which will be most useful in the sequel. Namely, for each $k \in (0,\infty)$:

$$
\begin{aligned}
&\int_{-d}^{0}\sinh 2k(z+d)\,dz = \frac{1}{2k}\big[\cosh 2k(z+d)\big]_{x=-d}^{z=0} = \frac{1}{2k}\big(\cosh 2kd-1\big) \text{ (a)}, \\
&\int_{-d}^{0}\cosh 2k(z+d)\,dz = \frac{1}{2k}\big[\sinh 2k(z+d)\big]_{x=-d}^{z=0} = \frac{1}{2k}\sinh 2kd \text{ (b)},
\end{aligned}
\tag{4.24}
$$

$$
\begin{aligned}
&\int_{-d}^{0}\cosh^2 k(z+d)\,dz = \frac{1}{2}\int_{-d}^{0}\big[\cosh 2k(z+d)+1\big]dz = \frac{1}{4k}\big[\sinh 2k(z+d)\big]_{x=-d}^{z=0} + \frac{1}{2}d \\
&= \frac{1}{4k}\big(\sinh 2kd + 2kd\big) = \frac{1}{4k}\left(1+\frac{2kd}{\sinh 2kd}\right)\sinh 2kd \doteq \frac{1}{2k}m_1(2kd)\sinh 2kd,
\end{aligned}
\tag{4.25}
$$

$$
\begin{aligned}
&\int_{-d}^{0}\sinh^2 k(z+d)\,dz = \frac{1}{2}\int_{-d}^{0}\big[\cosh 2k(z+d)-1\big]dz = \frac{1}{4k}\big[\sinh 2k(z+d)\big]_{x=-d}^{z=0} - \frac{1}{2}d \\
&= \frac{1}{4k}\big(\sinh 2kd - 2kd\big) = \frac{1}{4k}\left(1-\frac{2kd}{\sinh 2kd}\right)\sinh 2kd \doteq \frac{1}{2k}m_{-1}(2kd)\sinh 2kd,
\end{aligned}
\tag{4.26}
$$

where

$$
m_{\pm 1}(2kd) = \frac{1}{2}\left(1 \pm \frac{2kd}{\sinh 2kd}\right),
\tag{4.27}
$$

so that

$$m_1(2kd)+m_{-1}(2kd)=1. \tag{4.28}$$

The physical sense of the quantities of $m_1(2kd)$ and $m_{-1}(2kd)$, defined by (4.27), is established by (I.9.1)–(I.9.5), (I.9.9)–(I.9.12), and (I.9.13)f, and it can be summarized as follows.

7. The group and phase speeds $c_{\mathrm{g}}(k,d)$ and $c_{\mathrm{p}}(k,d)$ of a PPPMGWW are defined in terms of its cyclic frequency $\Omega(k,d)$, given by (4.1), thus:

$$\begin{aligned}
c_{\mathrm{g}}(k,d) &\doteqdot \frac{\partial \Omega(k,d)}{\partial k}=\frac{\partial\sqrt{gk\tanh kd}}{\partial k}=\frac{\sqrt{g}}{2}\left(\sqrt{\frac{\tanh kd}{k}}+\frac{d}{\cosh^2 kd}\sqrt{\frac{k}{\tanh kd}}\right)\\
&=\frac{\sqrt{gk\tanh kd}}{2k}\left(1+\frac{kd}{\cosh^2 kd\tanh kd}\right)=\frac{\Omega(k,d)}{2k}\left(1+\frac{kd}{\cosh^2 kd\tanh kd}\right)\\
&=\frac{\Omega(k,d)}{2k}\left(1+\frac{2kd}{\sinh 2kd}\right)=\frac{\Omega(k,d)}{k}m_1(2kd)>0,
\end{aligned} \tag{4.29}$$

$$c_{\mathrm{p}}(k,d)\doteqdot\frac{\Omega(k,d)}{k}, \tag{4.30}$$

so that

$$m_1(2kd)=\frac{c_{\mathrm{g}}(k,d)}{c_{\mathrm{p}}(k,d)}>0, \tag{4.31}$$

and hence

$$m_{-1}(2kd)=1-\frac{c_{\mathrm{g}}(k,d)}{c_{\mathrm{p}}(k,d)}. \tag{4.32}$$

8. By (8.25) and (8.26) it follows from (8.22) and (8.23) that

$$\overline{\bar{e}'_{(2)ij}(t,\underline{x})}^{t}=\frac{\rho_0 g}{2k^2}\left[n_i n_j m_1(2kd)+\delta_{i3}\delta_{j3}m_{-1}(2kd),\right], \tag{4.33}$$

$$\overline{\bar{e}'_{\mathrm{k}(2)}(t,\underline{x})}^{t}=\frac{\rho_9 g}{4k^2}, \tag{4.34}$$

the understanding being that

$$n_i\doteqdot\frac{k_i}{k}\text{ for each } i\in\omega_{1,2},\, n_3\doteqdot 0. \tag{4.35}$$

9. By (4.20), it follows from (4.5) that

$$\begin{aligned}
\mathrm{Z}^2_{(1)}(t,\underline{x}_2)&\doteqdot \mathrm{Z}^2_{(1)}(t;\underline{x}_2,ka)=(ka)^2\zeta^2_{(1)}(t,\underline{x}_2)=a^2\tau_{-\mu}{}^2\left(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\right)\\
&=\frac{1}{2}a^2\left[1+\mu\cos 2\left(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\right)\right],
\end{aligned} \tag{4.36}$$

whence

$$\overline{Z^2_{(1)}(t;\underline{x}_2,ka)}^t = (ka)^2\overline{\zeta^2_{(1)}(t,\underline{x}_2)}^t = \frac{1}{2}a^2. \tag{4.37}$$

By (4.34) and (4.37), it follows that

$$\frac{1}{2}\rho_0 g\overline{\zeta^2_{(1)}(t,\underline{x}_2)}^t - \overline{\hat{e}'_{\mathrm{k}(2)}(t,\underline{x}_2)}^t = \frac{\rho_0 g}{4k^2} - \frac{\rho_0 g}{4k^2} = 0. \tag{4.38}$$

10. Thus, by (4.7) at $l \doteqdot 2$, (4.10)¸ (4.33), (4.37), and (4.38), equation (3.87) reduces to

$$\begin{aligned}
\bar{\hat{S}}^{\mathrm{p}}_{\mathrm{w}(2)ij}(\underline{x}_2,ka) &\doteqdot \overline{\hat{S}^{\mathrm{p}}_{\mathrm{w}(2)ij}(t;\underline{x}_2,ka)}^t = (ka)^2\overline{\hat{s}^{\mathrm{p}}_{\mathrm{w}(2)ij}(t,\underline{x}_2)}^t \\
&= P_0\overline{Z_{(2)}(t,\underline{x}_2)}^t\delta_{ij} + \frac{1}{2}\rho_0 g\overline{Z^2_{(1)}(t,\underline{x}_2)}^t\delta_{i3}\delta_{j3} + (ka)^2\overline{\hat{e}'_{(2)ij}(t,\underline{x}_2)}^t \\
&= -\frac{P_0 a^2 k}{2\sinh 2kd}\delta_{ij} + \frac{1}{4}\rho_0 g a^2\delta_{i3}\delta_{j3} + \frac{\rho_0 g a^2}{2}\left[n_i n_j m_1(2kd) + \delta_{i3}\delta_{j3}m_{-1}(2kd)\right] \\
&= \frac{\rho_0 g a^2}{2}\left[-\frac{P_0 k}{\rho_0 g\sinh 2kd}\delta_{ij} + n_i n_j m_1(2kd) + \left(m_{-1}(2kd) + \frac{1}{2}\right)\delta_{i3}\delta_{j3}\right],
\end{aligned} \tag{4.39}$$

where 'S' has, for more clarity, been furbished with the superscript '$^{\mathrm{p}}$', standing for "*progressive*". In separate components, (4.39) can be written as"

$$\begin{gathered}
\bar{\hat{S}}^{\mathrm{p}2\times2}_{\mathrm{w}(2)ij}(\underline{x}_2,ka) = \frac{\rho_0 g a^2}{2}\left[n_i n_j m_1(2kd) - \frac{P_0 k}{\rho_0 g\sinh 2kd}\delta_{ij}\right] \\
\text{for each } i \in \omega_{1,2} \text{ and each } j \in \omega_{1,2} \text{ (a)}, \\
\bar{\hat{S}}^{\mathrm{p}}_{\mathrm{w}(2)i3}(\underline{x}_2,ka) = \bar{\hat{S}}^{\mathrm{p}}_{\mathrm{w}(2)3i}(\underline{x}_2,ka) = 0 \text{ for each } i \in \omega_{1,2} \text{ (b)}, \\
\bar{\hat{S}}^{\mathrm{p}}_{\mathrm{w}(2)33}(\underline{x}_2,ka) = \frac{\rho_0 g a^2}{2}\left[-\frac{P_0 k}{\rho_0 g\sinh 2kd} + m_{-1}(2kd) + \frac{1}{2}\right] \\
= \frac{\rho_0 g a^2}{2}\left[1 - \frac{k}{\sinh 2kd}\left(\frac{P_0}{\rho_0 g} + d\right)\right] \text{ (c)}.
\end{gathered} \tag{4.40}$$

11. By (3.47)–(3.49), (3.57)–(3.58$_1$), and (4.36)–(4.38), it follows that

$$\bar{\hat{E}}_{\mathrm{k}(2)} = \bar{\hat{E}}_{\mathrm{pw}(2)} = \frac{1}{4}\rho_0 g a^2, \tag{4.39$_1$}$$

so that $\bar{\hat{E}}^{\mathrm{p}}_{(2)}$, defined as

$$\bar{\hat{E}}^{\mathrm{p}}_{(2)} \doteqdot \bar{\hat{E}}_{\mathrm{k}(2)} + \bar{\hat{E}}_{\mathrm{pw}(2)} = \frac{1}{2}\rho_0 g a^2, \tag{4.39$_2$}$$

is the *time-averaged depth-integrated total volumetric energy density of the* progressive (indicated by the superscript ‘p’) water wave in question. This is the pertinent interpretation of the factor ‘ $\rho_0 ga^2/2$ ’ occurring in the final expressions of the trains (4.39) and (4.40).

12. By the relations

$$\lim_{kd\to\infty} m_1(2kd) = \lim_{kd\to\infty} m_{-1}(2kd) = \frac{1}{2} \text{ (a)}, \ \lim_{kd\to\infty} \frac{1}{\sinh 2kd} = 0 \text{ (b)}, \tag{4.41}$$

the first two of which follow from (4.27) by the l’Hospitale rule and the last one is self-evident, or alternatively by the relations (4.29)–(4.32) subject to

$$\Omega_\infty(k) \doteqdot \lim_{kd\to\infty} \Omega(k,d) = \lim_{kd\to\infty} \sqrt{gk\tanh kd} = \sqrt{gk} > 0, \tag{4.42}$$

which is a token of (I.7.79) that follows from (4.1) (or (I.7.38)), it follows from (4.39) that for each $i \in \omega_{1,3}$ and each $j \in \omega_{1,3}$:

$$\lim_{kd\to\infty} \overline{\hat{S}^{\mathrm{p}}_{\mathrm{w}(2)ij}(t;\underline{x}_2,ka)}^t = (ka)^2 \lim_{kd\to\infty} \overline{\hat{s}^{\mathrm{p}}_{\mathrm{w}(2)ij}(t,\underline{x}_2)}^t = \frac{\rho_0 ga^2}{4}\left(n_i n_j + 2\delta_{i3}\delta_{j3}\right) \tag{4.43}\bullet$$

4.2. The case of a PSPMGWW

The tensor $\overline{\hat{S}}_{\mathrm{w}(2)ij}(\underline{x}_2,ka)$ of a PSPMGWW is calculated in accordance with the same scheme as that of a PPPMGWW, although details of the calculation are of course different.

1. Given $\mu \in \{1,-1\}$, given $\nu \in \{1,-1\}$, the following four trains of equations are tokens of slightly modified (enriched) trains (I.10.32)–(I.10.35) respectively:

$$\begin{aligned}\Phi_{(1)}(t,\underline{x}) &\doteqdot \Phi_{(1)}(t;\underline{x},ka) = ka\phi_{(1)}(t,\underline{x},) = \Phi_{\mu\nu}(t;\underline{x},\underline{k}_2,a) = ka\phi_{(1)\mu\nu}(t,\underline{x},)\\ &= -\mu a\sqrt{\frac{g}{k\tanh kd}}\,\frac{\cosh k(z+d)}{\cosh kd}\,\tau_\mu(\Omega(k)t)\tau_\nu(k_2\cdot\underline{x}_2),\end{aligned} \tag{4.44}$$

$$\begin{aligned}\Phi_{(2)}(t,\underline{x}) &\doteqdot \Phi_{(2)}(t,\underline{x},ka) = (ka)^2\phi_{(2)}(t,\underline{x},) = -\frac{\mu a^2(2\cosh 2kd-1)}{6\cosh kd\sinh 2kd}\sqrt{\frac{gk}{\tanh kd}}\\ &\cdot\cosh k(z+d)[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)][2\sin\Omega(k)t-\sin 2\Omega(k)t].\end{aligned} \tag{4.45}$$

$$\begin{aligned}\mathrm{Z}_{(1)}(t,\underline{x}_2) &\equiv \mathrm{Z}_{(1)}(t;\underline{x}_2,ka) = ka\zeta_{(1)}(t,\underline{x}_2) = -\frac{ka}{g}\dot{\phi}^{(0)}_{(1)}(t,\underline{x}_2) = -\frac{ka}{g}\left[\dot{\phi}_{(1)}(t,\underline{x})\right]_{z=0}\\ &= -\frac{1}{g}\left[\dot{\Phi}_{(1)}(t;\underline{x},ka)\right]_{z=0} \doteqdot -\frac{1}{g}\left[\dot{\Phi}_{(1)}(t,\underline{x})\right]_{z=0} = \frac{\mu a}{\sqrt{gk\tanh kd}}\frac{\partial\tau_\mu(\Omega(k)t)}{\partial t}\tau_\nu(k_2\cdot\underline{x}_2)\\ &= \frac{\mu^2 a\Omega(k)}{\sqrt{gk\tanh kd}}\tau_{-\mu}(\Omega(k)t)\tau_\nu(k_2\cdot\underline{x}_2) = a\,\tau_{-\mu}(\Omega(k)t)\tau_\nu(k_2\cdot\underline{x}_2),\end{aligned} \tag{4.46}$$

$$\begin{aligned}\mathrm{Z}_{(2)}(t,\underline{x}_2) &\equiv \mathrm{Z}_{(2)}(t;\underline{x}_2,ka) = (ka)^2\zeta_{(2)}(t,\underline{x}_2)\\ &= \frac{a^2k}{12\sinh 2kd}\{-3[1+\nu\cosh 2kd\cos 2(\underline{k}_2\cdot\underline{x}_2)]\\ &+4\mu(2\cosh 2kd-1)[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)]\cos\Omega(k)t\\ &+\mu[1-2\cosh 2kd+\nu(5\cosh 2kd+2)\cos 2(\underline{k}_2\cdot\underline{x}_2)]\cos 2\Omega(k)t.\end{aligned} \tag{4.47}$$

2. As before, depth-integrating $\dot{\Phi}_{(l)}(t;\underline{x}_2,ka)$ or $\dot{\phi}_{(l)}(t,\underline{x}_2)$ at $l\in\{1,2\}$, subject to (4.44) and (4.45),. between -*d* and 0 reduces to calculating the elementary integral (4.8), so that the pertinent homograph of equation (4.7) is semantically sound:

$$\overline{\hat{\dot{\Phi}}'_{(l)}(t;\underline{x}_2,ka)}^t = ka\overline{\hat{\dot{\phi}}'_{(l)}(t,\underline{x}_2)}^t = 0 \text{ for each } l\in\{1,2\}, \tag{4.48}$$

At the same time, it follows from (4.46) and (4.47) that

$$\overline{\mathrm{Z}}_{(1)}(\underline{x}_2) \doteqdot \overline{\mathrm{Z}_{(1)}(t,\underline{x}_2)}^t = a\overline{\tau_{-\mu}(\Omega(k)t)}^t\tau_\nu(k_2\cdot\underline{x}_2) = 0\,, \tag{4.49}$$

$$\overline{\mathrm{Z}}_{(2)}(\underline{x}_2) \doteqdot \overline{\mathrm{Z}_{(2)}(t,\underline{x}_2)}^t = -\frac{a^2k}{4\sinh 2kd}[1+\nu\cosh 2kd\cos 2(\underline{k}_2\cdot\underline{x}_2)], \tag{4.50}$$

$$\overline{\dot{\mathrm{Z}}_{(1)}(t;\underline{x}_2,ka)}^t = ka\overline{\dot{\zeta}_{(1)}(t,\underline{x}_2)}^t = (ka)^2\overline{\dot{\zeta}_{(2)}(t,\underline{x}_2)}^t = \overline{\dot{\mathrm{Z}}_{(2)}(t;\underline{x}_2,ka)}^t = 0\,. \tag{4.51}$$

Equations (4.49) and (4.51) are tokens of (4.9) and (4.11) and also those of (I.10.38) and (I.10.40) respectively. The interval of values of the functional form $\overline{\mathrm{Z}_{(2)}(t,\underline{x}_2)}^t$ is given by (I.10.39$_1$).

Under the general definition

$$\overline{F(\underline{x}_2)}^{\underline{x}_2} \doteqdot \lim_{X_1\to\infty}\lim_{X_2\to\infty}\frac{1}{X_1X_2}\int_{-X_1/2}^{X_1/2}\int_{-X_2/2}^{X_2/2}F(\underline{x}_2)dx_2dx_1 \tag{4.52}$$

(see (10.42)), it follows from (4.50) that

$$\overline{\overline{\mathrm{Z}_{(2)}(t,\underline{x}_2)}^t}^{\underline{x}_2} = -\frac{a^2k}{4\sinh 2kd} \tag{4.53}$$

(cf. (4.10)). Consequently, given $a\in(0,\infty)$, given $k\in(0,\infty)$, it follows from (4..53) that

$$\lim_{d\to\infty}\overline{\overline{\mathrm{Z}_{(2)}(t,\underline{x}_2)}^t}^{\underline{x}_2} = -\lim_{d\to\infty}\frac{a^2k}{4\sinh 2kd} = -0 \tag{4.53$_1$}$$

(cf. (4.10$_1$)).

3. By (4.47) and (4.49), it follows from (3.86) and (3.85) at $l\doteqdot 1$, subject to (3.78) and (3.83) that

$$\overline{\hat{S}}_{\mathrm{w}(1)ij}(\underline{x}_2,ka) \doteqdot \overline{\hat{S}_{\mathrm{w}(1)ij}(t;\underline{x}_2,ka)}^t = ka\overline{\hat{s}_{\mathrm{w}(1)ij}(t,\underline{x}_2)}^t = 0\,, \tag{4.54}$$

which is a homograph of (4.12).

4. The item 4 of the previous subsection applies to the case of a PSPMGWW.

5. By (4.2), it follows that

$$\nabla_i \tau_\nu\left(\underline{k}_2 \cdot \underline{x}_2\right) = \nu k_i \tau_{-\nu}\left(\underline{k}_2 \cdot \underline{x}_2\right) \text{ for each } i \in \{1,2\}. \tag{4.55}$$

Making use of (4.55) and (4.15) it follows by (4.3) that

$$\begin{gathered} v_{(1)i}\left(t,\underline{x}\right) = \nabla_i \phi_{(1)}\left(t,\underline{x}\right) = \frac{\partial \phi_{(1)}\left(t,\underline{x}\right)}{\partial x_i} = -\mu \frac{1}{k \cosh kd} \sqrt{\frac{g}{k \tanh kd}} \tau_\mu\left(\Omega(k)t\right) \\ \cdot \left[\nu k_i \cosh k(z+d) \tau_{-\nu}\left(\underline{k}_2 \cdot \underline{x}_2\right) + \delta_{i3} k \sinh k(z+d) \tau_\nu\left(\underline{k}_2 \cdot \underline{x}_2\right)\right] \\ \text{for each } i \in \omega_{1,3}, \end{gathered} \tag{4.56}$$

which incorporates both equations (I.8.57a) and (I.8.57b) for the sake of convenience in the following reasoning. By (4.56), it follows that

$$\begin{gathered} e_{(2)ij}\left(t,\underline{x}\right) \doteqdot \rho_0 v_{(1)i}\left(t,\underline{x}\right) v_{(1)j}\left(t,\underline{x}\right) \\ = \frac{\rho_0 g}{k^3 \cosh^2 kd \tanh kd} \tau_\mu{}^2\left(\Omega(k)t\right) \left[\nu^2 k_i k_j \cosh^2 k(z+d) \tau_{-\nu}{}^2\left(\underline{k}_2 \cdot \underline{x}_2\right) \right. \\ + \nu k\left(k_i \delta_{j3} + k_{ji} \delta_{i3}\right) \cosh k(z+d) \sinh k(z+d) \tau_{-\nu}\left(\underline{k}_2 \cdot \underline{x}_2\right) \tau_\nu\left(\underline{k}_2 \cdot \underline{x}_2\right) \\ \left. + k^2 \delta_{i3} \delta_{j3} \sinh^2 k(z+d) \tau_\nu{}^2\left(\underline{k}_2 \cdot \underline{x}_2\right)\right] \text{for each } i \in \omega_{1,3} \text{ and each } j \in \omega_{1,3}, \end{gathered} \tag{4.57}$$

whence, by (4.19),

$$\begin{gathered} e_{\mathrm{K}(2)}\left(t,\underline{x}\right) \doteqdot \frac{1}{2} \rho_0 \sum_{i=1}^{3} v_{(1)i}\left(t,\underline{x}\right) v_{(1)i}\left(t,\underline{x}\right) \\ = \frac{\rho_0 g}{k \sinh 2kd} \tau_\mu{}^2\left(\Omega(k)t\right) \left[\cosh^2 k(z+d) \tau_{-\nu}{}^2\left(\underline{k}_2 \cdot \underline{x}_2\right) + \sinh^2 k(z+d) \tau_\nu{}^2\left(\underline{k}_2 \cdot \underline{x}_2\right)\right], \end{gathered} \tag{4.58}$$

because $\nu^2 = 1$. By (4.19), (4.21,a), (4.24,a), (4.25), and (4.26), depth-integrating both sides of either equation (4.57) or (4.58) between –*d* and 0 yields:

$$\begin{gathered} \hat{e}'_{(2)ij}\left(t,\underline{x}\right) \doteqdot \frac{\rho_0 g}{k^4} \tau_\mu{}^2\left(\Omega(k)t\right) \left[k_i k_j m_1\left(2kd\right) \tau_{-\nu}{}^2\left(\underline{k}_2 \cdot \underline{x}_2\right) \right. \\ + \nu k\left(k_i \delta_{j3} + k_i \delta_{i3}\right) \frac{\cosh 2kd - 1}{2 \sinh 2kd} \tau_{-\nu}\left(\underline{k}_2 \cdot \underline{x}_2\right) \tau_\nu\left(\underline{k}_2 \cdot \underline{x}_2\right) \\ \left. + k^2 \delta_{i3} \delta_{j3} m_{-1}\left(2kd\right) \tau_\nu{}^2\left(\underline{k}_2 \cdot \underline{x}_2\right)\right] \text{for each } i \in \omega_{1,3} \text{ and each } j \in \omega_{1,3}, \end{gathered} \tag{4.59}$$

$$\hat{e}'_{\mathrm{K}(2)}\left(t,\underline{x}\right) \doteqdot \frac{\rho_0 g}{2k^2} \tau_\mu{}^2\left(\Omega(k)t\right) \left[m_1\left(2kd\right) \tau_{-\nu}{}^2\left(\underline{k}_2 \cdot \underline{x}_2\right) + m_{-1}\left(2kd\right) \tau_\nu{}^2\left(\underline{k}_2 \cdot \underline{x}_2\right)\right]. \tag{4.60}$$

6. By the pertinent instance of definition (2.8), it follows from the instance of the equation (4.20,a) with $\alpha \doteqdot \Omega(k)t$ that

$$\overline{\tau_{\pm\mu}{}^2(\Omega(k)t)}^t = \frac{1}{2}\left[1 \mp \mu c \overline{\cos 2(\Omega(k)t)}^t\right] = \frac{1}{2}. \qquad (4.61)$$

Hence, time-averaging equations (4.59) and (4.60) yields:

$$\begin{aligned}\overline{\bar{e}'_{(2)ij}(t,\underline{x})}^t \doteqdot{}& \frac{\rho_0 g}{2k^2}\Big[n_i n_j m_1(2kd)\tau_{-\nu}{}^2(\underline{k}_2\cdot\underline{x}_2) \\ &+ \nu(n_i\delta_{j3}+n_j\delta_{i3})\frac{\cosh 2kd - 1}{2\sinh 2kd}\tau_{-\nu}(\underline{k}_2\cdot\underline{x}_2)\tau_\nu(\underline{k}_2\cdot\underline{x}_2) \\ &+ \delta_{i3}\delta_{j3}m_{-1}(2kd)\tau_\nu{}^2(\underline{k}_2\cdot\underline{x}_2)\Big] \text{for each } i\in\omega_{1,3} \text{ and each } j\in\omega_{1,3},\end{aligned} \qquad (4.62)$$

$$\overline{\bar{e}'_{\mathrm{K}(2)}(t,\underline{x})}^t \doteqdot \frac{\rho_0 g}{4k^2}\left[m_1(2kd)\tau_{-\nu}{}^2(\underline{k}_2\cdot\underline{x}_2) + m_{-1}(2kd)\tau_\nu{}^2(\underline{k}_2\cdot\underline{x}_2)\right], \qquad (4.63)$$

subject to (4.35). By the following instances of equations (4.20),

$$\begin{gathered}\tau_{\pm\nu}{}^2(\underline{k}_2\cdot\underline{x}_2) = \frac{1}{2}[1\mp\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)] \text{ (a)}, \ \tau_\nu(\underline{k}_2\cdot\underline{x}_2)\tau_{-\nu}(\underline{k}_2\cdot\underline{x}_2) = \frac{1}{2}\sin 2(\underline{k}_2\cdot\underline{x}_2) \text{ (b)}, \\ \text{for each } \nu\in\{1,-1\},\end{gathered} \qquad (4.64)$$

equations (4.62) and(4.63) can be reduced further thus:

$$\begin{aligned}\overline{\bar{e}'_{(2)ij}(t,\underline{x})}^t \doteqdot{}& \frac{\rho_0 g}{4k^2}\Big\{n_i n_j m_1(2kd)[1+\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)] \\ &+ \nu(n_i\delta_{j3}+n_j\delta_{i3})\frac{\cosh 2kd - 1}{2\sinh 2kd}\sin 2(\underline{k}_2\cdot\underline{x}_2) \\ &+ \delta_{i3}\delta_{j3}m_{-1}(2kd)[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)]\Big\} \text{for each } i\in\omega_{1,3} \text{ and each } j\in\omega_{1,3},\end{aligned} \qquad (4.65)$$

$$\begin{aligned}\overline{\bar{e}'_{\mathrm{k}(2)}(t,\underline{x})}^t &\doteqdot \frac{\rho_0 g}{8k^2}\{m_1(2kd)+m_{-1}(2kd)+\nu[m_1(2kd)-m_{-1}(2kd)]\cos 2(\underline{k}_2\cdot\underline{x}_2)\} \\ &= \frac{\rho_0 g}{8k^2}\left[1+\nu\frac{2kd}{\sinh 2kd}\cos 2(\underline{k}_2\cdot\underline{x}_2)\right].\end{aligned} \qquad (4.66)$$

7. By (4.61) and by (4.64,a), it follows from (4.46) (or (I.8.48)) that

$$\overline{\zeta^2_{(1)}(t,\underline{x}_2)}^t = \frac{1}{k^2}\overline{\tau_{-\mu}{}^2(\Omega(k)t)}^t\tau_\nu{}^2(\underline{k}_2\cdot\underline{x}_2) = \frac{1}{4k^2}[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)]. \qquad (4.67)$$

Hence, by (4.66) and (4.67), it follows that

$$\begin{aligned}&\frac{1}{2}\rho_0 g\overline{\zeta^2_{(1)}(t,\underline{x}_2)}^t - \overline{\bar{e}'_{\mathrm{K}(2)}(t,\underline{x})}^t \\ &= \frac{\rho_0 g}{8k^2}[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)] - \frac{\rho_0 g}{8k^2}\left[1+\nu\frac{2kd}{\sinh 2kd}\cos 2(\underline{k}_2\cdot\underline{x}_2)\right] \\ &= -\frac{\nu\rho_0 g}{8k^2}\left(1-\frac{2kd}{\sinh 2kd}\right)\cos 2(\underline{k}_2\cdot\underline{x}_2) = -\frac{\nu\rho_0 g}{4k^2}m_{-1}(2kd)\cos 2(\underline{k}_2\cdot\underline{x}_2).\end{aligned} \qquad (4.68)$$

6. By (4.52), it follows that

$$\overline{\cos 2(\underline{k}_2\cdot\underline{x}_2)}^{x_2}=\overline{\sin 2(\underline{k}_2\cdot\underline{x}_2)}^{x_2}=0. \tag{4.69}$$

Hence, upon averaging with respect to $\underline{x}_2$, equations (4.65)–(4.68) become:

$$\overline{\overline{e'_{(2)ij}(t,\underline{x})}^{t}}^{x_2}=\frac{\rho_0 g}{4k^2}n_i n_j m_1(2kd)+\delta_{i3}\delta_{j3}m_{-1}(2kd) \text{ for each } i\in\omega_{1,3} \text{ and each } i\in\omega_{1,3}, \tag{4.70}$$

$$\overline{\overline{e'_{\mathrm{k}(2)}(t,\underline{x}_2)}^{t}}^{x_2}=\frac{\rho_0 g}{8k^2}, \tag{4.71}$$

$$\overline{\overline{\zeta^2_{(1)}(t,\underline{x}_2)}^{t}}^{x_2}=\frac{1}{4k^2}, \tag{4.72}$$

$$\frac{1}{2}\rho_0 g\overline{\overline{\zeta^2_{(1)}(t,\underline{x}_2)}^{t}}^{x_2}-\overline{\overline{e'_{\mathrm{K}(2)}(t,\underline{x})}^{t}}^{x_2}=0. \tag{4.73}$$

8. By (4.48) at $l \doteqdot 2$, (4.53), and (4.70)–(4.73), upon averaging it with respect to $\underline{x}_2$, equation (3.87) becomes

$$\begin{aligned}\overline{\overline{\widehat{S}}^{\mathrm{s}}_{\mathrm{w}(2)ij}(\underline{x}_2,ka)}^{x_2} &\doteqdot \overline{\overline{\widehat{S}^{\mathrm{s}}_{\mathrm{w}(2)ij}(t;\underline{x}_2,ka)}^{t}}^{x_2}=(ka)^2\overline{\overline{\widehat{s}^{\mathrm{s}}_{\mathrm{w}(2)ij}(t,\underline{x}_2)}^{t}}^{x_2}\\ &=P_0\overline{\overline{Z_{(2)}(t,\underline{x}_2)}^{t}}^{x_2}\delta_{ij}+\frac{1}{2}\rho_0 g\overline{\overline{Z^2_{(1)}(t,\underline{x}_2)}^{t}}^{x_2}\delta_{i3}\delta_{j3}+(ka)^2\overline{\overline{e'_{(2)ij}(t,\underline{x}_2)}^{t}}^{x_2}\\ &=-\frac{P_0a^2k}{4\sinh 2kd}\delta_{ij}+\frac{1}{8}\rho_0 ga^2\delta_{i3}\delta_{j3}+\frac{\rho_0 ga^2}{4}\left[n_i n_j m_1(2kd)+\delta_{i3}\delta_{j3}m_{-1}(2kd)\right]\\ &=\frac{\rho_0 ga^2}{4}\left[-\frac{P_0k}{\rho_0 g\sinh 2kd}\delta_{ij}+n_i n_j m_1(2kd)+\left(m_{-1}(2kd)+\frac{1}{2}\right)\delta_{i3}\delta_{j3}\right],\end{aligned} \tag{4.74}$$

where 'S' has, for more clarity, been furbished with the superscript '$^{\mathrm{s}}$', standing for "*standing*". Comparison of $\overline{\widehat{S}^{\mathrm{p}}_{\mathrm{w}(2)ij}(t;\underline{x}_2,ka)}^{t}$, defined by (4.39), and $\overline{\overline{\widehat{S}^{\mathrm{s}}_{\mathrm{w}(2)ij}(t;\underline{x}_2,ka)}^{t}}^{x_2}$, defined by (4.74), shows that the latter differs from the former by the factor $1/2$. It goes without saying that the limiting values of the two quantities as $kd\to\infty$ are interrelated likewise.

9. Namely, in analogy with item 11 of subsection 4,1, using (4.71) and (4.72) instead of (4.34) and (4.38), one finds that

$$\overline{\overline{\widehat{E}_{\mathrm{k}(2)}(t,\underline{x}_2)}^{t}}^{x_2}=\overline{\overline{\widehat{E}_{\mathrm{pw}(2)}(t,\underline{x}_2)}^{t}}^{x_2}=\frac{1}{8}\rho_0 ga^2, \tag{4.74$_1$}$$

so that $\overline{\overline{\widehat{E}^{\mathrm{s}}_{(2)}(t,\underline{x}_2)}^{t}}^{x_2}$, defined as

$$\overline{\overline{\widehat{E}^{\mathrm{s}}_{(2)}(t,\underline{x}_2)}^{t}}^{x_2}\doteqdot\overline{\overline{\widehat{E}_{\mathrm{k}(2)}(t,\underline{x}_2)}^{t}}^{x_2}+\overline{\overline{\widehat{E}_{\mathrm{pw}(2)}(t,\underline{x}_2)}^{t}}^{x_2}=\frac{1}{4}\rho_0 ga^2, \tag{4.74$_2$}$$

is the $\underline{x}_2$-*averaged time-averaged depth-integrated total volumetric energy density of the* s*tanding* (indicated by the superscript '$^{\mathrm{s}}$') water wave in question. This is the pertinent interpretation of the factor '$\rho_0 g a^2/4$' occurring in the final expression of the train (4.74).

10. By (4.41) and (4.42), it follows from (4.74) that for each $i \in \omega_{1,3}$ and each $j \in \omega_{1,3}$:

$$\lim_{kd\to\infty} \overline{\overline{\widehat{S}^{\mathrm{s}}_{\mathrm{w}(2)ij}(t;\underline{x}_2,ka)}^{t}}^{\underline{x}_2} = (ka)^2 \lim_{kd\to\infty} \overline{\overline{\widehat{s}^{\mathrm{s}}_{\mathrm{w}(2)ij}(t,\underline{x}_2)}^{t}}^{\underline{x}_2} = \frac{\rho_0 g a^2}{8}\left(n_i n_j + 2\delta_{i3}\delta_{j3}\right) \tag{4.75}$$

(cf. (4.43)).•s

5. Progressive and standing monochromatic gravity waves on a water layer on a liquid layer with a mildly varying bed

5.1. Formal empiric rules of generalization of basic characteristics of a PPPMGWW or PSPMGWW in the first non-vanishing approximation with respect to *ka* on a liquid layer of a constant depth to the respective characteristics of a priming progressive or standing quasi-plane monochromatic gravity water wave (PPQPMGWW or PSQPMGWW) on a water layer with a mildly varying bed

Definition 5.1. 4) By (4.29) and (4.31), given $d \in (0,+\infty)$, the values of the functional form '$\Omega(k,d)$', as defined by (4.1) (originally by (I.7.38), monotonically increase from $\Omega(0,d)=0$ to $\Omega(+\infty,d)=+\infty$ as k increases from 0 to +∞. Hence, given $d \in (0,+\infty)$, given $\omega \in (0,+\infty)$, the equation

$$\Omega(k,d)=\omega \tag{5.1}$$

has a unique solution with respect to 'k', which will be denoted by '$K(\omega,d)$', so that

$$\Omega(k,d)=\omega \text{ if and only if } k=K(\omega,d), \tag{5.2}$$

the understanding being that $K(\omega,d)>0$ (cf. Corollary I.9.2). Given $\omega \in (0,+\infty)$, let for each $\underline{x}_2 \in \underline{E}_2$:

$$\kappa(\omega,\underline{x}_2) \doteq \left|\underline{\kappa}_2(\omega,\underline{x}_2)\right| = K(\omega,h(\underline{x}_2)), \tag{5.3}$$

so that

$$\begin{aligned} \underline{\kappa}_2(\omega,h(\underline{x}_2)) &\doteq \underline{\nu}_2\kappa(\omega,h(\underline{x}_2)) = \langle \nu_1\kappa(\omega,h(\underline{x}_2)), \nu_2\kappa(\omega,h(\underline{x}_2))\rangle \\ &\doteq \langle \kappa_1(\omega,h(\underline{x}_2)), \kappa_2(\omega,h(\underline{x}_2))\rangle, \end{aligned} \tag{5.4}$$

where $\underline{\nu}_2$ is an arbitrary unit vector, defined as

$$\underline{\nu}_2 \doteq \langle \nu, \nu_2\rangle; \tag{5.5}$$

it goes without saying that

$$\kappa_3(\omega,\underline{x}_2) \doteq 0\,. \tag{5.6}$$

2) Let, also,

$$C_{\mathrm{g}}(\omega,\underline{x}_2) \doteq c_{\mathrm{g}}(\kappa(\omega,\underline{x}_2),h(\underline{x}_2)) = m_1(2\kappa(\omega,\underline{x}_2)h(\underline{x}_2))C_{\mathrm{p}}(\omega,\underline{x}_2), \tag{5.7}$$

$$C_{\mathrm{p}}\left(\omega,\underline{x}_2\right)\doteqdot c_{\mathrm{p}}\left(\kappa\left(\omega,\underline{x}_2\right),h\left(\underline{x}_2\right)\right)=\frac{\omega}{\kappa\left(\omega,\underline{x}_2\right)},\tag{5.8}$$

subject to (4.27)-(4.32). Thus, $C_{\mathrm{g}}\left(\omega,\underline{x}_2\right)$ and $C_{\mathrm{p}}\left(\omega,\underline{x}_2\right)$ are respectively the wave group speed and the wave phase speed of a certain *priming progressive quasi-plane monochromatic gravity water wave* (*PPQPMGWW*) *on a liquid layer with a mildly varying bed* $z=-h\left(\underline{x}_2\right)$ (to be specified).•

Definition 5.2: ***The empiric syntactic rule of the generalization.*** 1) In agreement with the pertinent previous notation, given $\nu\in\omega_1$, for each $l\in\omega_\nu$, $F_{(l)}\left(t;\underline{x},ka\right)$ and $F_{\mathrm{s}(l)}\left(t,\underline{x}_2,ka\right)$ are *scaled* asymptotic approximations of lth order with respect to ka to a given momentary bulk functional form $F\left(t;\underline{x},ka\right)$ and to a given momentary free-surface functional form $F_{\mathrm{s}}\left(t,\underline{x}_2,ka\right)$ respectively, so that

$$F\left(t;\underline{x},ka\right)\sim\sum_{l=\nu}^{\infty}F_{(l)}\left(t;\underline{x},ka\right)\doteqdot\sum_{l=\nu}^{\infty}(ka)^l\,f_{(l)}\left(t,\underline{x}\right),\tag{5.9}$$

$$F_{\mathrm{s}}\left(t;\underline{x}_2,ka\right)\sim\sum_{l=\nu}^{\infty}F_{\mathrm{s}(l)}\left(t;\underline{x}_2,ka\right)\doteqdot\sum_{l=\nu}^{\infty}(ka)^l\,f_{\mathrm{s}(l)}\left(t,\underline{x}_2\right),\tag{5.10}$$

where $f_{(l)}\left(t,\underline{x}_2\right)$ and $f_{\mathrm{s}(l)}\left(t,\underline{x}_2\right)$ are the respective *non-scaled* asymptotic approximations of lth order. In this case,

$$F_{\mathrm{s}}\left(t;\underline{x}_2,ka\right)\doteqdot \mathrm{Z}\left(t;\underline{x}_2,ka\right)\ \text{or}\ F_{\mathrm{s}}\left(t;\underline{x}_2,ka\right)\doteqdot\left[F\left(t;\underline{x},ka\right)\right]_{z=\mathrm{Z}\left(t;\underline{x}_2,ka\right)}\tag{5.11}$$

Relation (5.9) implies that

$$\hat{F}\left(t;\underline{x}_2,ka\right)\sim\sum_{l=\nu}^{\infty}\hat{F}_{(l)}\left(t;\underline{x}_2,ka\right)\doteqdot\sum_{l=\nu}^{\infty}(ka)^l\,\hat{f}_{(l)}\left(t,\underline{x}_2\right)\tag{5.12}$$

subject to (3.1)–(3.3), (3.7)–(3.9), (3.20), and (3.21). Also, by the pertinent instances of 3.8), it follows from (5.9), (5.10), and (5.12) that

$$\bar{F}\left(\underline{x},ka\right)\sim\sum_{l=\nu}^{\infty}\bar{F}_{(l)}\left(\underline{x},ka\right)\doteqdot\sum_{l=\nu}^{\infty}(ka)^l\,\bar{f}_{(l)}\left(\underline{x}\right),\tag{5.13}$$

$$\bar{F}_{\mathrm{s}}\left(\underline{x}_2,ka\right)\sim\sum_{l=\nu}^{\infty}\bar{F}_{\mathrm{s}(l)}\left(\underline{x}_2,ka\right)\doteqdot\sum_{l=\nu}^{\infty}(ka)^l\,\bar{f}_{\mathrm{s}(l)}\left(\underline{x}_2\right),\tag{5.14}$$

$$\bar{\bar{F}}\left(\underline{x}_2,ka\right)\sim\sum_{l=\nu}^{\infty}\bar{\bar{F}}_{(l)}\left(\underline{x}_2,ka\right)\doteqdot\sum_{l=\nu}^{\infty}(ka)^l\,\bar{\bar{f}}_{(l)}\left(\underline{x}_2\right).\tag{5.15}$$

2) Besides the independent variables occurring in the postpositive parentheses after each upper case or lover case functional variable occurring in the relations (5.9)–(5.15), the respective functional form is supposed to depend on the four parameters:

$$‘\varpi’,\ ‘k’,\ ‘d’,\ ‘\underline{k}_2’, \tag{5.16}$$

subject to (5.2), which should be added to the pertinent list of independent variables after a preceding semicolon. Thus, for instance,

$$\begin{gathered} F(t;\underline{x},ka) \doteqdot F(t;\underline{x},ka;\omega,k,d,k_2),\ F_{(l)}(t;\underline{x},ka) \doteqdot F_{(l)}(t;\underline{x},ka;\omega,k,d,k_2), \\ f_{(l)}(t,\underline{x}) \doteqdot f_{(l)}(t;\underline{x};\omega,k,d,k_2), \end{gathered} \tag{5.17}$$

$$\begin{gathered} \bar{\bar{F}}(\underline{x}_2,ka) \doteqdot F(\underline{x}_2,ka;\omega,k,d,k_2),\ \bar{\bar{F}}_{(l)}(\underline{x}_2,ka) \doteqdot \bar{\bar{F}}_{(l)}(\underline{x}_2,ka;\omega,k,d,k_2), \\ \bar{\bar{f}}_{(l)}(\underline{x}_2) \doteqdot \bar{\bar{f}}_{(l)}(\underline{x}_2;\omega,k,d,k_2), \end{gathered} \tag{5.18}$$

so that the definiendum of each one of the six of definitions (5.17) and (5.18) is the abbreviation of its definiens that is obtained by omission of the list (5.16).

3) The functional form that is obtained by furnishing the basic functional variable or constant of any given full or abbreviated functional form of those occurring or obviously understood in the previous item with a superscript tilde $^{\sim}$ is understood as one, in which the last three parameters of the list (5.16) are mentally replaced in accordance with the following definitions:

$$k \doteqdot \kappa(\omega,\underline{x}_2),\ d \doteqdot h_2(\underline{x}_2),\ \underline{k}_2 \doteqdot \underline{\kappa}_2(\omega,\underline{x}_2). \tag{5.19}$$

Once the substitutions (5.19) are actually (syntactically) executed, the superscript tilde should be omitted.

4) The above rule applies also to any other appropriate original functional form. For instance, in accordance with (5.7) and (5.8),

$$\begin{gathered} c_{\mathrm{g}}^{\sim}(k,d) \doteqdot c_{\mathrm{g}}(\kappa(\omega,\underline{x}_2),h(\underline{x}_2)),\ c_{\mathrm{p}}^{\sim}(k,d) \doteqdot c_{\mathrm{p}}(\kappa(\omega,\underline{x}_2),h(\underline{x}_2)), \\ m_{\pm 1}^{\sim}(2kd) \doteqdot m_{\pm 1}(2\kappa(\omega,\underline{x}_2)h(\underline{x}_2)). \end{gathered} \tag{5.20}$$

5) The *tilde-carrying variant of an initial uniform-bed-related functional form* is called the *mental non-uniform-bed-related interpretand* of the latter functional form. The *tilde-free variant of an initial uniform-bed-related functional form subject to the syntactic* (*actual*) *substitutions (5.19)* is called the *syntactic non-uniform-bed-related interpretand* of the latter functional form.•

Hypothesis 5.1: ***The empiric semantic rule of the generalization.*** Both the mental and the syntactic non-uniform-bed-related interpretand of the *first non-vanishing asymptotic approximation with respect to ka* to a characteristic of a PPPMGWW or PSPMGWW on a water layer of a constant depth is a *semantically sound* (*meaningful, having a denotatum*) characteristic of a *priming progressive or standing quasi-plane monochromatic gravity water wave* (*PPQPMGWW* or *PSQPMGWW*) *on a liquid layer with a mildly varying bed*, such that, e.g.,

$$\left|\underline{\nabla}_2 h(\underline{x}_2)\right| << 1 \text{ for each } \underline{x}_2 \in \underline{E}_2 . \tag{5.21}$$

In this case, all higher asymptotic approximations with respect to ka to the given characteristic of the given PPPMGWW or PSPMGWW are disregarded.•

Comment 5.1. In accordance with (4.3) and (4.16),

$$\begin{aligned}\Phi_{(1)}(t,\underline{x}) &\doteqdot \Phi_{(1)}(t;\underline{x},ka) = ka\phi_{(1)}(t,\underline{x},) \\ &\doteqdot -\mu a\sqrt{\frac{g}{k\tanh kd}}\frac{\cosh k(z+d)}{\cosh kd}\tau_{\mu}\left(\Omega(k,d)t-\underline{k}_2\cdot\underline{x}_2\right),\end{aligned} \tag{5.22}$$

$$\begin{aligned}V_{(1)i}(t,\underline{x}) &\doteqdot V_{(1)i}(t;\underline{x},ka) = kav_{(1)i}(t,\underline{x}) = ka\nabla_i\phi_{(1)}(t,\underline{x}) = ka\frac{\partial\phi_{(1)}(t,\underline{x})}{\partial x_i} \\ &\doteqdot \frac{1}{k\cosh kd}\sqrt{\frac{g}{k\tanh kd}} \\ &\cdot\left[k_i\cosh k(z+d)\tau_{-\mu}\left(\Omega(k,d)t-\underline{k}_2\cdot\underline{x}_2\right)-\mu k\delta_{i3}\sinh k(z+d)\tau_{\mu}\left(\Omega(k,d)t-\underline{k}_2\cdot\underline{x}_2\right)\right] \\ &\text{for each } i\in\omega_{1,3},\end{aligned} \tag{4.23}$$

Hence, for instance, $\Phi_{(1)}(t;\underline{x},ka)$, $\phi_{(1)}(t,\underline{x},)$, each scaled velocity component $V_{(1)i}(t;\underline{x},ka)$, and each non-scaled velocity component $v_{(1)i}(t,\underline{x})$ decrease with z decreasing from 0 to $-d$ either as $\frac{\cosh k(z+d)}{\cosh kd}$ or as $\frac{\sinh k(z+d)}{\cosh kd}$, i.e. mainly as $\exp(kz)$. Therefore, the *sufficient* criterion (5.21) of mildly varying can intuitively be replaced with this *sufficient* one:

$$\left|\underline{\nabla}_2 h(\underline{x}_2)\right|\exp\left(-\kappa(\omega,\underline{x}_2)h(\underline{x}_2)\right) << 1 \text{ for each } \underline{x}_2 \in \underline{E}_2 . \tag{5.24}$$

The bottom of a water layer, whose depth satisfies the criterion (5.24), is called an *effective mildly varying one*. Particularly, criterion (5.24) is satisfies if the minimum depth h_{m} satisfies the condition $h_{\mathrm{m}} \to \infty$.•

Example 5.1. In the result of substitutions (5.19),equations (4.39) and (4.74) become

$$\begin{aligned}\overline{\overline{\hat{S}}}_{\mathrm{w}(2)ij}\left(\underline{x}_2,\kappa(\omega,\underline{x}_2)a\right) &\doteqdot \overline{\hat{S}_{\mathrm{w}(2)ij}\left(t;\underline{x}_2,\kappa(\omega,\underline{x}_2)a\right)}^{t} = \left[\kappa(\omega,\underline{x}_2)a\right]^2\overline{\hat{s}_{\mathrm{w}(2)ij}(t,\underline{x}_2)}^{t} \\ &= \frac{\rho_0 g a^2}{2}\left\{-\frac{P_0\kappa(\underline{x}_2,\omega)}{\rho_0 g\sinh 2\kappa(\omega,\underline{x}_2)h(\underline{x}_2)}\delta_{ij}+\nu_i\nu_j m_1\left(2\kappa(\omega,\underline{x}_2)h(\underline{x}_2)\right)\right. \\ &\left.+\left[m_{-1}\left(2\kappa(\omega,\underline{x}_2)h(\underline{x}_2)\right)+\frac{1}{2}\right]\delta_{i3}\delta_{j3}\right\},\end{aligned} \tag{5.25}$$

$$\overline{\overline{\hat{S}}_{\mathrm{w}(2)ij}\left(\underline{x}_2,\kappa(\omega,\underline{x}_2)a\right)}^{\underline{x}_2} \doteq \overline{\overline{\hat{S}_{\mathrm{w}(2)ij}\left(t;\underline{x}_2,\kappa(\omega,\underline{x}_2)a\right)}^{t}}^{\underline{x}_2} = [\kappa(\omega,\underline{x}_2)a]^2\,\overline{\overline{\hat{s}_{\mathrm{w}(2)ij}\left(t,\underline{x}_2\right)}^{t}}^{\underline{x}_2}$$

$$= \frac{\rho_0 g a^2}{4}\left\{-\frac{P_0\kappa(\underline{x}_2,\omega)}{\rho_0 g \sinh 2\kappa(\omega,\underline{x}_2)h(\underline{x}_2)}\delta_{ij} + \nu_i\nu_j m_1\left(2\kappa(\omega,\underline{x}_2)h(\underline{x}_2)\right)\right. \tag{5.26}$$

$$\left. + \left[m_{-1}\left(2\kappa(\omega,\underline{x}_2)h(\underline{x}_2)\right)+\frac{1}{2}\right]\delta_{i3}\delta_{j3}\right\},$$

respectively.●

5.2. An empirical relation between the longshore sediment transport rate and the revised radiation stress tensor of the representative PPQPMGWW

Inman and Bagnold [1963], Komar and Inman [1970], and others have suggested that the potential transport rate $I(\omega)$ of the immersed weight of sand along the surf zone due to a real progressive quasi-plane wave of a cyclic frequency ω and of an amplitude a is given by the empirical formula

$$I(\omega) = K(\omega)L\left(\omega,\underline{x}_{2\mathrm{b}}\right), \tag{5.27}$$

where

$$L\left(\omega,\underline{x}_2\right) \equiv \overline{\hat{E}}_{\mathrm{w}(2)}\left(\omega,\underline{x}_2\right)C_{\mathrm{g}}\left(\omega,\underline{x}_2\right)\sin\alpha\left(\omega,\underline{x}_2\right)\cos\alpha\left(\omega,\underline{x}_2\right). \tag{5.28}$$

subject to (5.7). In this case, $\overline{\hat{E}}_{\mathrm{w}(2)}\left(\omega,\underline{x}_2\right)$, defined as

$$\overline{\hat{E}}_{\mathrm{w}(2)}\left(\omega,\underline{x}_2\right) \doteq \frac{\rho_0 g a^2}{2}, \tag{5.29}$$

is the time-averaged depth-integrated total volumetric energy of the PPQPMGWW, $\alpha\left(\omega,\underline{x}_2\right)$ is the angle between the local wave vector $\underline{\kappa}_2\left(\omega,\underline{x}_2\right)$ and the unit vector $\underline{N}_2\left(\underline{x}_2\right) \doteq \left\langle N_1\left(\underline{x}_2\right), N_2\left(\underline{x}_2\right)\right\rangle$ of the inland normal to the depth contour at the point $\underline{x}_2$, positive counterclockwise; both vectors are parallel to the XY-plane. The subscript ‘b’ to ‘$\underline{x}_2$’ in (5.27) indicates that L must be evaluated at the breaker depth contour. If $\overline{\hat{E}}_{\mathrm{w}(2)}\left(\omega,\underline{x}_2\right)$ is measured in *joule per meter squared*, $C_{\mathrm{g}}(\underline{x}_2,\omega)$ in *meter per second*, and $I(\omega)$ in *newton*, then $K(\omega)$ is an empirical dimensionless constant coefficient of the order 1. Thus, equation (5.27) subject to (5.28) and (5.29) is a scaling relation, which takes into account both the energy density flux into the serf zone and its orientation relative to the breaker depth contour. This relation is often called “*the CERC formula*”, “CERC” being an acronym of “Coastal Engineering Research Council”. All equivalent counterparts of equation (5.27) subject to

(5.28) and (5.29) that occur in the literature are as a rule written with the help of one or another notation. It is hoped that the reader will easily recognize (5.27) subject to (5.28) and (5.29) as an adequate version of any one of those in use. The interested reader will find a discussion of some physical mechanisms underlying the CERC formula, e.g., in Komar [1976, pp. 204–213].

If all depth contours are parallel to the shore line, it is convenient to choose a new coordinate system in such a way that its *X*-axis is directed shoreward, whereas its *Y*-axis is parallel to the shore line and oriented so as to form a right-handed coordinate system along with the above *X*-axis and along with the Z-axis oriented vertically downward. In this case, (5.28) can be written as

$$L(\omega, \underline{x}_2) \equiv \nu_1(\omega, \underline{x}_2)\nu_2(\omega, \underline{x}_2)\bar{\bar{E}}_{\mathrm{w}(2)}(\omega, \underline{x}_2)C_{\mathrm{g}}(\omega, \underline{x}_2). \tag{5.30}$$

By (4.31) and (5.25), equation (5.30) becomes

$$L(\omega, \underline{x}_2) = C_{\mathrm{p}}(\omega, \underline{x}_2)\bar{\hat{S}}_{(2)12}(\omega, \underline{x}_2). \tag{5.31}$$

Hence, (5.27) takes the form

$$I(\omega) = K(\omega)C_{\mathrm{p}}(\omega, \underline{x}_{2\mathrm{b}})\bar{\hat{S}}_{(2)12}(\omega, \underline{x}_{2\mathrm{b}}), \tag{5.32}$$

Relative to a coordinate system with arbitrary oriented axes *X* and *Y* in the horizontal plane, equation (5.32) can be rewritten in covariant form as

$$I(\omega) = K(\omega)C_{\mathrm{p}}(\omega, \underline{x}_{2\mathrm{b}})\sum_{i=1}^{2}\sum_{j=1}^{2} N_i(\underline{x}_{2\mathrm{b}})N_j(\underline{x}_{2\mathrm{b}})\bar{\hat{S}}_{(2)ij}(\omega, \underline{x}_{2\mathrm{b}}). \tag{5.33}$$

In this case, it is tacitly assumed that the transformation from the original coordinate system to the new one has been made, and that the meaning of the variables ‘ i ’, ‘ j ’, and ‘ $\underline{x}_2$ ’ has been changed accordingly. Thus, equation (5.33) is an empirical scaling relation between the longshore transport rate of the immersed weight of sand on the one hand, and the 2×2-TADIRST (time-averged depth-integrated radiation stress tensor) of the representative progressive quasi-plane mode on the other hand. Instead of the empirical coefficient ‘ $K(\omega)$ ’, one can introduce another coefficient ‘ $K'(\omega)$ ’ by the equation

$$K(\omega) \doteq K'(\omega)m_1(2\kappa(\omega, \underline{x}_{2\mathrm{b}})h(\underline{x}_{2\mathrm{b}})). \tag{5.34}$$

In this case, equation (5.33) becomes

$$I(\omega) = K'(\omega)C_{\mathrm{g}}(\omega, \underline{x}_{2\mathrm{b}})\sum_{i=1}^{2}\sum_{j=1}^{2} N_i(\underline{x}_{2\mathrm{b}})N_j(\underline{x}_{2\mathrm{b}})\bar{\hat{S}}_{(2)ij}(\omega, \underline{x}_{2\mathrm{b}}). \tag{5.35}$$

Since (5.27) subject to (5.28) is a scaling relation, therefore the fact that the right-hand side of (5.27) is expressed in terms of the radiation stress tensor components either by (5.32) or by (5.33) or else by (5.35) seems to be natural. At the same time, when written in any one of the above three forms, , the CERC formula acquires a new sense value.

6. Historico-philosophical remarks regarding radiation stress tensors of water waves

6.1. “Radiation stress tensor” paradoxes

I have already mention in Comment 2.7 that in the literature on water wave dynamics, there are several *different,* but *intuitively* as if the same, definitions of a 2×2 TADIRST in the XY-plane of the second order with respect to *ka*, which are regarded by their authors as unique adequate logographic interpretands of Longuet-Higgins and Stewart’s wordy definition of the generic name “*radiation stress tensor*”. This results in the phenomenon of paradoxical uses of the latter term, which I shall call “*radiation stress tensor*” *paradoxes* (“*RST*” *paradoxes*). In this case, in the different articles on the matter different notations are used. Therefore, in order to make explicit at least some of the “RST” paradoxes I shall, for the sake of being specific, consider the following three-stage definition of his ‘ S_{ij} ’ by Mei [1989].

> 1) «For the convenience of vertical integration, the vertical and horizontal directions are distinguished. Specifically, we denote the vertical velocity by *w* and the vertical coordinate by *z*, the horizontal velocity components by u_i ($i = 1,2$, $u_1 = u$, $u_2 = v$), and the horizontal coordinates by x_i ($i = 1,2$, $x_1 = x$, $x_2 = y$). We define the mean velocity U_i ($i = 1,2$) by integrating u_i over the instantaneous water depth and then over the time period *T*, tat is,
>
> $$U_i(x,y,z,t) = \frac{1}{\overline{\zeta} + h}\overline{\int_{-h}^{\zeta} u_i dz}, \qquad i = 1,2, \tag{2.1}$$
>
> where $\zeta(x,y,t)$ is the free-surface displacement, $\overline{\zeta}$ is its time mean, and $h(x,y)$ is the sill water depth. Physically, $u_i = \rho U_i(\overline{\zeta} + h)$ is the mean rate of mass flux across a vertical plane of unit width along x_i=const. The vector (U_1, U_2) may, therefore, be called the *mass flux velocity*, which depends only on the horizontal coordinates and the long time scale. Denoting the deviation from the mean by $\overline{u}_i$, we have
>
> $$u_i = U_i + \overline{u}_i(x,y,z,t). \tag{2.2}$$
>
> It follows from the definition that

$$\overline{\int_{-h}^{\zeta}\bar{u}_i dz} = 0\,. \tag{2.3)»}$$

(*ibid.* pp. 453–454))

2) «… the following definition has been introduced

$$S_{ij} = \overline{\int_{-h}^{\zeta}\left[P\delta_{ij}+\rho\bar{u}_i\bar{u}_j\right]dz} - \frac{\rho g}{2}\left(\bar{\zeta}+h\right)^2\delta_{ij}\,. \tag{2.25}$$

Physically, S_{ij} is the (i, j) component of the stress tensor, representing the *excess momentum fluxes.*† Since

$$\frac{\rho g}{2}\left(\bar{\zeta}+h\right)^2 = \rho g\int_{-h}^{\bar{\zeta}}\left(\bar{\zeta}-z\right)dz$$

is the total mean hydrostatic pressure over the mean depth, S_{ij} may be written

$$S_{ij} = \left[\overline{\int_{-h}^{\zeta}Pdz} - \int_{-h}^{\bar{\zeta}}\rho g\left(\bar{\zeta}-z\right)dz\right]\delta_{ij} + \overline{\int_{-h}^{\zeta}\rho\bar{u}_i\bar{u}_j dz}\,. \tag{2.26}$$

Thus, S_{ij} represents the sum of the *i*th component of the excess hydrostatic pressure on, and the net momentum flux across, a surface normal to the *j*th direction.

†The definition of S_{ij} is slightly different from Phillips [1977, Eq. (3.6.12).The difference is of the fourth order for the infinitesimal waves.» (*ibid.* pp. 457–458)

3) «It is straightforward to show that the radiation stresses are

$$\begin{aligned} S_{ij} &= \frac{\rho g A^2}{4}\left\{\frac{k_i k_j}{k^2}\left(1+\frac{2kh}{\sinh 2kh}\right)+\delta_{ij}\frac{2kh}{\sinh 2kh}\right\} \\ &= \frac{E}{2}\left\{\frac{k_i k_j}{k^2}\frac{2C_{\mathrm{g}}}{C}+\delta_{ij}\left(\frac{2C_{\mathrm{g}}}{C}-1\right)\right\} \end{aligned} \tag{3.10}$$

(Longuet-Higgins and Stewart, 1962, 1964).» (*ibid.* p. 465)

In referring to any one of the above cited numbered equation, I shall prefix its double numeral with the letter 'M'. Each one of the subscripts '*i*' and '*j*' occurring in the above quotations takes on values in the set $\{1,2\}$. Also, Mei's variable 'ζ' has the same range as *Longuet-Higgins and Stewart*'s (briefly *LH&S*'s) variable 'ζ' and the same range as my variable 'Z'. Hence, the *definiens* of Mei's definition (M2.25) of *his* radiation stress tensor S_{ij} subject to $i \in \omega_{1,2}$ and $j \in \omega_{1,2}$ contains the *wave-dependent* term

$$\frac{\rho g}{2}(\bar{\zeta}+h)^2\delta_{ij}\,, \tag{6.1}$$

i.e. $\frac{\rho_0 g}{2}(\bar{Z}+h)^2\delta_{ij}$ in my notation, which is subtracted from the term

$$\overline{\int_{-h}^{\zeta}\left[P\delta_{ij}+\rho\bar{u}_i\bar{u}_j\right]dz} \tag{6.2}$$

instead of the *wave-independent* term $\overline{\hat{S}_{eij}(t,\underline{x}_2)}^{t}$ (see (2.42e)) subject to $i\in\omega_{1,2}$ and $j\in\omega_{1,2}$. Accordingly, my equations (4.9) and (4.10), being tokens of equations (I.10.29) and (I.10.30) respectively, should be written in Mei's notation thus:

$$\bar{\zeta}_{(1)}=0\,, \tag{6.3}$$

$$\bar{\zeta}_{(2)}=-\frac{A^2k}{2\sinh 2kh}\,. \tag{6.4}$$

In this case, with an accuracy to the ambiguous usages both of '$\bar{\zeta}$' and of '=', equations (6.3) and (6.4), i.e. my equations (4.9) and (4.10), agree respectively with the statement that $\bar{\zeta}=0$ before equation (3.18) in LH&S [1960] and with equation (4.12) in LH&S [1962], which was deduced there from intuitive considerations. The former statement of LH&S means that their depths function h is the same as that of Mei. At the same time, in connection with the latter result of LH&S, it is noteworthy that, in accordance with the subject matter of sub-subsection I.10.5.2 (e.g.), the velocity potential $\Phi_{(2)}$ of the second order approximation with respect to ka is, like $\Phi_{(1)}$, bounded as a function of time, so that

$$\overline{\frac{\partial\Phi_{(2)}(t,\underline{x})}{\partial t}}^{t}=0\,. \tag{6.5}$$

Otherwise, the whole of the second-order approximation would be incorrect in principle. Therefore, for computing $\bar{Z}_{(2)}$, one should at first compute $\Phi_{(2)}$ in order to prove (6.5). Hence, $\bar{Z}$ cannot, in principle, be computed in the second-order approximation with respect ka in the framework of the linear theory, unless of course (6.5) is taken for granted. Accordingly, it is accidental that equation (4.12) in Longuet-Higgins and Stewrt [1962], along with (6.5), turns accidentally out to be true.

Occurrence of A^2 in (6.4) explicitly demonstrates that the term (6.1) is, not only wave-related, but that it is of the order of $(kA)^2$, i.e. of the order of the entire S_{ij}. Therefore, when used in its natural (default) sense as "*net wave-induced*", the qualifier '*radiation*" is not applicable to the tensor 'S_{ij}' defined by (M2.25) or (M2.76). That is to say, either one of Mei's last two definitions disagrees with LH&S's original wordy definiens of their 2×2 time-averaged depth-integrated radiation stress tensor S_{ij}^{LS} (see Comment 2.7) and is therefore *semantically incorrect* as a logographic interpretand of 'S_{ij}^{LS}'. At the same time, Mei explicitly states that, firstly, his S_{ij} represents "the *excess momentum fluxes*" (see the first line below (M2.25)) and that, secondly, the final expression for his 'S_{ij}', given by (M3.10), is due to LH&S. This is actually the main reason for confusion and for the creation of the pertinent radiation stress paradox.

Comparison of the separate components of S_{ij}, given by (M3.10), and the respective components of $\bar{\bar{S}}_{\mathrm{w}(2)ij}^{2\times 2}\left(\underline{x}_2, ka\right)$, given by (4.40) subject (4.31), (5.7), and (5.8) (see (4.12)), shows that, up to the different notations used in these two cases,

$$S_{xy} = S_{yx} = \bar{\bar{S}}_{\mathrm{w}(2)xy}^{2\times 2}\left(\underline{x}_2, ka\right) = \bar{\bar{S}}_{\mathrm{w}(2)yx}^{2\times 2}\left(\underline{x}_2, ka\right). \tag{6.6}$$

whereas

$$S_{xx} \neq \bar{\bar{S}}_{\mathrm{w}(2)xx}^{2\times 2}\left(\underline{x}_2, ka\right) \text{ and } S_{yy} \neq \bar{\bar{S}}_{\mathrm{w}(2)yy}^{2\times 2}\left(\underline{x}_2, ka\right). \tag{6.7}$$

Since Mei'sefinition of 'S_{ij}' differs from my definition of '$\bar{\bar{S}}_{\mathrm{w}(2)ij}^{2\times 2}\left(\underline{x}_2, ka\right)$' (cf. (2.49), (2.51), and (4.12)) and since the methods of calculation of the components of the two tensors are different, therefore the relations (6.7) are not surprising.

Surprising is Mei's statement that his definition (M2.25) or (M2.26) of 'S_{ij}' results in equation (M3.10), i.e. in the same values of 'S_{ij}' as those resulted from the completely different definition of 'S_{ij}' by LH&S. Mei does not make explicit any derivation of equation (M3.10) from his definition (M2.25) or (M2.26). Therefore, there are two options to explain his paradoxical statement. First, Mei correctly performed all calculations leading from (M2.25) or (M2.26) to (M3.10), while LH&S committed some errors in their calculations, which were not detected by their followers including Mei. In this case, the coincidence of the above two final results is accidental. Second, Mei did not performed all necessary calculations by himself, but rather he was confident both that his definition of 'S_{ij}' was an adequate logographic interepretand of 'S_{ij}^{LS}' (see Comment 2.7) and that the result of calculations of LH&S of their S_{ij}^{LS}, which were expected to be subjected by that time to

25-years interpersonal verifications, were correct. Therefore, Mei just quoted the supposedly correct result of LH&S for their S_{ij}^{LS} as the unavoidable result of his own definition of 'S_{ij}'. In this case, the possibility that LH&S had erred is not excluded, and this is what had unfortunately happened.

I have already indicated previously that intuitive considerations can be a useful heuristic tool in calculating linear characteristics of a non-linear phenomenon or in calculating bilinear characteristics, which are expressible in the framework of the pertinent linear approximation, – such bilinear characteristics, e.g., the volumetric kinetic energy density or the volumetric energy density flux of ideal fluid. If, however, calculating a bilinear characteristic of interest requires knowledge of the velocity potential of second order with respect to *ka* then the missing information is unavoidably compensated by making apparently plausible but often incorrect *ad hoc* assumptions. Accordingly, in such cases intuition turns often out to be, not only useless, but harmful. Therefore, one of the main objects of my developing the recursive asymptotic theory of nonlinear surface gravity water waves on a water layer with an even or infinitely deep bottom has been to avoid, as far as possible, making in the sequel any *ad hoc* non-systematic assumptions.

At the same time, rigorous (syntactic) systematic rules of deductive inference can result in some relations, which are not predictable intuitively, but which become intuitively comprehensible after they are made explicit formally. Ones of such results are, in my view, the expression (4.39) (or its equivalent (4.40)) for the 3×3 TADIRST $\overline{\hat{S}_{\mathrm{w}(2)ij}(t;\underline{x}_2,ka)}^{t}$ of a PPPMGWW in the second order, i.e. first non-vanishing, approximation with respect to *ka* (see equation (4.12)) and the expression (4.74) for the 3×3 *hotison(ally averaged* (*HA*) TADIRST $\overline{\overline{\hat{S}_{\mathrm{w}(2)ij}(t;\underline{x}_2,ka)}^{t}}^{\underline{x}_2}$ of a PSPMGWW in the same approximation (see equation (4.54) being s homograph of (4.12)). The diagonal elements of each one of the above two RST's involves the constant 'P_0', such that $P_0 = 0$ if the part of space above the free upper surface of the water layer is vacuous, whereas $P_0 = P_{\mathrm{a}}$, where P_{a} is the atmospheric pressure on the free water surface, if the above-mentioned part of space is occupied by atmospheric air. In this case, since P_{a} has a prepositive sign '–', opposite to the prepositive signs of all other terms contributing to any diagonal element of each given RST, one can thought that P_{a} produces surface tension of some kind, which tends to straighten (suppress, diminish) the existing surface water waves. In this connection, the following question of pure academic interest can e raised

Let in general $P_0 > 0$ be external pressure on the free (upper) surface of the water layer, and let, for the sake of being specific, the coordinate system be chosen so as

$$n_x = 1 \text{ and } n_y = 0 . \tag{6.8}$$

It follows from (4.43) and (4.75) that every component of $\overline{\hat{S}^{\mathrm{p}}_{\mathrm{w}(2)ij}\left(t; \underline{x}_2, ka\right)}^{t}$ or $\overline{\overline{\hat{S}^{\mathrm{s}}_{\mathrm{w}(2)ij}\left(t; \underline{x}_2, ka\right)}^{t}}^{\underline{x}_2}$ converges. Moreover, given

$$\delta \mathrel{\dot{\overset{\rightharpoonup}{=}}} kd > 0 , \tag{6.9}$$

given $d > 0$ ¸ by (4.27), it follows from (4.39) or (4.40) and from (4.74) independently of *a* that

$$\overline{\hat{S}^{\mathrm{p}}_{\mathrm{w}(2)xx}\left(t; \underline{x}_2, ka\right)}^{t} = \frac{1}{2} \overline{\overline{\hat{S}^{\mathrm{s}}_{\mathrm{w}(2)xx}\left(t; \underline{x}_2, ka\right)}^{t}}^{\underline{x}_2} = 0 \tag{6.10}$$

if and only if

$$P_0 = P_{0x}(d, \delta) \mathrel{\dot{\overset{\rightharpoonup}{=}}} \rho_0 g d u_x(\delta) \tag{6.11}$$

subject to

$$u_x(\delta) \mathrel{\dot{\overset{\rightharpoonup}{=}}} \frac{\sinh 2\delta m_1(2\delta)}{\delta} = \frac{\sinh 2\delta}{2\delta} + 1, \tag{6.12}$$

and that

$$\overline{\hat{S}^{\mathrm{p}}_{\mathrm{w}(2)zz}\left(t; \underline{x}_2, ka\right)}^{t} = \frac{1}{2} \overline{\overline{\hat{S}^{\mathrm{s}}_{\mathrm{w}(2)zz}\left(t; \underline{x}_2, ka\right)}^{t}}^{\underline{x}_2} = 0 \tag{6.13}$$

if and only if

$$P_0 = P_{0z}(d, \delta) \mathrel{\dot{\overset{\rightharpoonup}{=}}} \rho_0 g d u_z(\delta) \tag{6.14}$$

subject to

$$u_z(\delta) \mathrel{\dot{\overset{\rightharpoonup}{=}}} \frac{\sinh 2\delta}{\delta}\left(m_{-1}(2\delta) + \frac{1}{2}\right) = \frac{\sinh 2\delta}{\delta} - 1. \tag{6.15}$$

In this case, use of (4.27) has been made for developing the *universal* functional forms $u_x(\delta)$ and $u_z(\delta)$.

Given $\delta \mathrel{\dot{\overset{\rightharpoonup}{=}}} kd > 0$, it follows from (6.11) and (6.14) that

$$\lim_{d \to \infty} P_{0x} = \lim_{d \to \infty} P_{0z} = +\infty . \tag{6.16}$$

Assuming therefore that $d<\infty$, let us consider the equation

$$u_x(\delta) \mathrel{\dot{\overset{\rightharpoonup}{=}}} \frac{\sinh 2\delta}{2\delta} + 1 = \frac{\sinh 2\delta}{\delta} - 1 \mathrel{\dot{\overset{\leftharpoonup}{=}}} u_z(\delta). \tag{6.17}$$

This equation can be rewritten as

$$\frac{\sinh 2\delta}{2\delta} - 2 = 0\,. \tag{6.18}$$

It is clear that $\dfrac{\sinh 2\delta}{2\delta} - 2$ monotonically increases with δ monotonically increasing from 0 to ∞. Therefore, equation (6.18) has a unique solution $\delta \stackrel{\rightarrow}{=} \delta_*$. This solution has the property that

$$P_{0*}(d,\delta_*) \stackrel{\rightarrow}{=} P_{0x}(d,\delta_*) = P_{0z}(d,\delta_*), \tag{6.19}$$

the understanding being that at this p0ressure

$$\overline{\widehat{S}^{\mathrm{p}}_{\mathrm{w}(2)xx}(t;\underline{x}_2,ka)}^{t} = \overline{\widehat{S}^{\mathrm{p}}_{\mathrm{w}(2)zz}(t;\underline{x}_2,ka)}^{t} = \frac{1}{2}\overline{\overline{\widehat{S}^{\mathrm{s}}_{\mathrm{w}(2)xx}(t;\underline{x}_2,ka)}^{t}}^{\underline{x}_2} = \frac{1}{2}\overline{\overline{\widehat{S}^{\mathrm{s}}_{\mathrm{w}(2)zz}(t;\underline{x}_2,ka)}^{t}}^{\underline{x}_2} = 0\,. \tag{6.20}$$

6.2. “Amicus Plato, sed magis amica veritas”

1. The noble Latin dictum, which is taken as the heading of this subsection, is translated into English as: «*Dear is Plato, but dearer still is truth*» and also, less pretentiously, by any one of the following sentences: «*Plato is my friend, but truth is a better friend*», «*Plato is my friend, but truth is more my friend [than he is]*», or «*Plato I love, but I love truth more*». That Latin dictum is in turn a translation from Ancient Greek into Latin of the phrase credited to Aristotle, who as though said it inorder to express his uncompromising rejection of Plato’s teaching of Universals (for greater detail, see Iosilevskii [2016b, Essay 5, subsection 2.2, p. 112 f]). At the same time, according to the online «*Dictionary of Phrases and Fable*» of Wikipedia, the above Latin dictum is a free translation of a phrase of «*Nicomachean Ethics*» (1096a15) by Aristotle, which is literally translated as: «*Where both are friends, it is right to prefer truth*». In any case, my usage of the Latin dictum as the heading of this subsection ix an allegoric one, under which I shall make explicit some most conspicuous inconsistencies of the four articles by Longuet-Higgins and Stuart (briefly LH&S) [1960, 1961, 1962, 1964] that have resulted in their expressions for the TADIRST of a PPPMGWW and for the HATADIRST of a PSPMGWW, – the incorrect expressions that were canonized about 55 years ago. I shall also make a few remarks regarding the old article Tadjbaksh and Keller [1960], which concerns with some relevant aspects of PSPMGWW’s.

The above-mentioned objects of this subsection are not of course objects of the *recursive asymptotic theory of gravity surface water waves* (*RATOGSWW*, briefly *RAT*) that I have developed in I and in this article. I have developed this theory, firstly, for the sake of completeness as another peculiar branch of the perturbation WKB (LG) method and, secondly, as a reliable rigorous (syntactic) alternative of the presently common unreliable intuitive (semantic) method. Analyzing or revising any article published in the literature on water wave dynamics previously in the framework of the intuitive method is beyond the scope of the RAT, except the papers of LH&S for the following reason. On the one hand, the TADIRST of a PPPMGWW and the HATADIRST of a PSPMGWW are *minor* characteristics of surface water waves of gravity. These characteristics are distributed and are not therefore measurable immediately and straightforwardly. But on the other hand, the RST's of water waves are secondary characteristics of water waves, which are deduced from the basic equations of hydrodynamics and therefore they must be well-defined (unique).

One of the most general laws of philosophy is *the triad of motion of thought: thesis-antithesis-synthesis* due to the German philosopher Georg Wilhelm Friedrich Hegel (1770–1831). The above triad is interpreted as *the dialectic principle of unity*, or *identity*, *of opposites due* to another German philosopher Johann Gottlieb Fichte (1752–1814), a contemporary of Hegel. In a sense, Hegel's triads and its interpretation by Fichte were foreshadowed by the principle of *golden mean* of ancient Greek philosophers and also by the following two Latin dicta: «In medio stat veritas» – «The truth stands in the middle» and «In medio stat virtus» – «Virtue stands in the middle» or «Virtue is in the moderate». Aristotle represents the principle of *golden mean* and discusses its importance for ethics in his «*Nicomachean Ethics*» (cf. Iosilevskii [2016b, Essay 5, subsection 4.2, p. 168 f]). At the same time, golden mean is not applicable to the RST of a water wave: the RST either is correct or is not correct, but not both at the same time. And if it is incorrect then it is necessary to explain why, when possible. Needless to say that there is nothing personal in my criticism of inconsistencies of articles by LH&S. They are pioneer in the field but they have committed some errors. Somebody has to correct their errors. Unfortunately, it is me who happens to do this job. I may also err, particularly in my attempt to explicate errors of LH&S or mutual inconsistencies of some of their statements. If this happens, somebody else will correct errors of my own. And so on.

2. I distinguish between the qualifiers "true" and "valid" and hence between the respective substantives "validity" and "truth". In Latin, e.g., there are special words for "true" and "truth' on the one hand and there are some other special words for "valid" and "validity" on the other hand, – in accordance with the following vocabulary entries of the English-Latin part of the Latin-English dictionary by Simpson [1968]:

«**true,** (**1**) = in accordance with fact, *verus*, iverax(=truthful)' to be –. in parenthesis, as an admission, *quidem*; as –I live, supporting a statement, *ita vivam ut*; with indic.; as an answer, see of YES. (**2**) = genuine, real, *verus, sinserusm*, germanus. (**3**) = loyal, *fidus, fidelus.*

truth, as a quality, *veritas*; the – , in a particular case, = the fact (s), *verum*, *vera* (= n. sing., or plur., of *veus*) sometimes *veritas*; in accordance with –, *ex veritate* (Sall.); in –, *enimvero*; see also TRULLY.

valid, (**1**) of arguments, reasons, etc., *firmus, gravis, iustus, certus, ligitimus.* (**2**) of laws, *ratus*, to be –, *ratum esse, valěre*; to make –, *ratum facěre, ratum esse iuběre.*

validity, (of reasons, etc.) *gravitas*, *pondus* (-*ěris*, n.); otherwise rendered by adj.»

The interested reader will find a discussion of the difference between the senses of the words "valid" and "validity" on the one hand and the senses of the words "true" and "truth" on the one hand, in trial formal logic (TFL), in Iosilevskii [2016a]. For my purpose at hand, it is sufficient to notice that every *euautographic* (*genuinely non-interpreted syntactic*) *relation* (contrasted to "*euautographic term*") of that logic, having *academic or practical interest*, is classified in accordance with a certain built-in *algebraic* (and hence *analytical*, *not tabular*) *decision method* (*ADM*) either as a *valid* one or as an *antivalid* one, or else as a *vav-neutral* (*vav-indeterminate*) one, i.e. as one being *neither valid nor antivalid.* In this case, the *negation of a valid relation* is an antivalid relation and vice versa, whereas the *negation of a vav-neutral relation* is *another vav-neutral relation*. A euautographic relation is said to be: (i) *invalid* if it either is antivalid or vav-neutral, (ii) *non-antivalid* if it either is valid or vav-neutal, (iii) *vav-unneutral* (*vav-determinate*) if it either is valid or antivalid. A euautographic relation or term of TFL is just a semantically insignificant *chip* (*fish*) that is analogous to a chessman. In order to be interpreted semantically, some selected valid and vav-neutral euautographic relations and also the *master* (*decision*) *theorems* of the latter relations are replaced, in accordance with certain rules of substitution, by certain semantically significant relations as their *semantic interpretands*. In this case, the semantic interpretands of valid relations, both of *slave* ones and of *master* ones, are said to be *universally* (*tautologously*, *tautologically*) *true*, whereas the semantic interpretands of some vav-neutral relations, which are taken for granted to be *true*, are said to be *veracious*, i.e. *accidentally true.* Consequently, all *mathematical postulates*, *permanent* ones, called *mathematical axioms*, and *temporary* (*ad hoc*) ones, called *mathematical hypotheses,* are semantic interpretands of vav-neutral euautographic relations. Accordingly, all *mathematical theorems*, i.e. true mathematical relations that are proved from mathematical postulates or from other mathematical theorems, or from both, are also semantic interpretands of some vav-neutral euautographic relations of TFL. The recursive asymptotic theory of gravity water waves that is developed in I and in this article is a mathematical theory and therefore all its true relations (postulates and theorems) have the above character.

3. I have already indicated previously that Longuet-Higgins and Stewart [1960, 1961, 1962, 1964], to be refferred henceforth to as LH&S1–LH&S4 in that order, seem to have been the first writers to introduce the term '*radiation stress*' and to define its physical sense. Here follows a brief review of those aspects of the above papers, which are relevant to the radiation stress paradoxes of *progressive* water waves.

Equation (3.14) of LH&S1 is a definition of the quantity, which the authors denote by 'S_x' and which they commonly call 'a *radiation stress*'. Also, in accordance with that paper, S_x is the $\langle x,x \rangle$-component S_{xx} of a time-averaged 2×2 tensor, which the authors denote by '*S*', and which they call "*the stress tensor*" (*ibid*., p. 577), – instead of the presently common term "the radiation stress tensor". None of the three other components of the tensor is defined in the paper formally. However, equation (3.14) of LH&S1, which is actually the definition of 'S_{xx}', unambiguously indicates that any component S_{ij} of the tensor should be defined analogously. By equation (3.34) in LH&S1, the authors suggest the well-known expressions for all the four components S_{ij} with $i \in \{1,2\}$ and $j \in \{1,2\}$ in the case of a real priming progressive plane *mode* (*monochromatic wave*) of an amplitude a and of a wave number k, that travels in the direction of the X-axis. Since then these expressions are widely cited and used in the literature for more than 55 years. It has been shown in subsection 6.1 that the above expressions are incorrect. Still, LH&S1 do not make explicit any details of the calculations, in the result of which they arrive at their final expressions for the radiation stresses. One cannot therefore put his finger on a specific error, or errors, which the authors have committed.

No essentially new results relevant to radiation stresses are reported in LH&S2. In this paper, the expressions for the radiation stress components as computed for a real progressive plane mode in LH&S1 are just cited in connection with some applications. Also, in this paper, the radiation stress tensor of a real progressive plane mode has, for the first and for the last time, been mentioned as a 3×3 tensor, whose all $\langle i,x\rangle$ and $\langle x,i\rangle$components with $i \in \omega_{1,3}$ equal null (*ibid.*, equation (6.2)). Since then the radiation stress tensor has never been mentioned in the literature as a 3×3 tensor, but rather it has always been treated as a 2×2 tensor.

In their paper LH&S3, the authors consider some further applications of their radiation stress S_x due to a real progressive plane mode. The definition of 'S_x' as given by equation (3.31) in LH&S3 essentially differs from the definition of the same symbol as given previously by equation (3.14) in LH&S1. Nevertheless, in the case of a real progressive plane mode, the next formula (3.32) in LH&S3 assigns the same denotatum to 'S_x' as that obtained in LH&S1. The assignment of the denotatum to 'S_x' is made as a citation of the corresponding result of LH&S1 without any calculations. Thus, the formulae (3.31) and (3.32) in LH&S3, along with the reference to LH&S1 as the source of formula (3.32), evidence that the authors did not notice the substitution, which they had involuntarily and erroneously made for the original definition of 'S_x' occurring in LH&S1. At the same time, it is worthy of noticing the amazing fact that, in LH&S3, equation (3.32) subject to definition (3.31) turns out to be *syntactically correct*. This correctness of (3.32) is, however, *accidental* because it is a result of two errors, one of which has been made in LH&S1, and the other in LH&S3. Among some other unfortunate coincidences of circumstances, the apparent correctness of (3.32) subject to (3.31) might, probably, have contributed to the creation of the paradox in question.

Paper LH&S4 is the last one he series of original papers of these writers, which are concerned with the notion of radiation stresses. In this paper, the original results of LH&St relevant to their theory of radiation stresses and of some other nonlinear effects in water waves are formulated in the most complete form. In particular, in this paper, the writers give the explicit definitions of all the four components S_{xx}, S_{yy}, and $S_{xy} = S_{yx}$ of their time-averaged radiation stress tensor. These definitions are in agreement with the explicit definition of $S_x = S_{xx}$ in LH&S1. Also, in LH&S4, the authors make explicit some details of their calculations, so that their most essential errors can be revealed. In the sequel, the original results of LH&S will therefore be cited mainly from LH&S4.

Incidentally, Mei's definition (M2.25) of 'S_{xx}' coincides with the definition of 'S_x' as given by equation (3.31) in LH&S3. Still, Mei has likely not noticed that the definition of 'S_x' in LH&S3 essentially differs from the definition of 'S_x' in LH&S1 or from the definition of 'S_{xx}' in LH&S4. He has not, also, noticed that no calculations of any radiation stresses are reported in LH&S3. The value of the only radiation stress component S_x, which is mentioned in that paper, is just cited from LH&S1 in no connection with the modified definition of the quantity. Also, Mei has not noticed that his own definition of 'S_{xx}' disagrees with the definition of the same symbol in LH&S4 and that, hence, it disagrees with the definition of 'S_x' in LH&S1. This is why Mei accompanies his equation (m3.10) with a reference to two mutually contradictory papers LH&S3 and LH&S4 as a single whole, while LH&S are mentioned as discoverers of the formula. in any way, the fact that Mei's final expression for 'S_{ij}' coincided with the expression that was derived in LH&S1 and re-derived in LH&S4 can be regarded as accidental.

No velocity potentials are written down in LH&S4. Still, equations (3) and (4) on p. 531 of LH&S4 indicate that the PPPMGWW, dealt with in that paper, are presumabbly described by the velocity potential defined as

$$\Phi_{(1)}(t,\underline{x}) = -\frac{a\sigma}{k\sinh kh}\cosh k(z+h)\sin(kx-\sigma t)\,. \tag{6.21}$$

This definition can be regarded as a particular case of equation (4.3), subject to (4.1) and (4.2), which corresponds to

$$\mu = 1\,,\ h = d\,,\ \sigma = \Omega(k,d)\,,\ \underline{k}_2 = \langle k,0\rangle\,. \tag{6.22}$$

Indeed, by (4.1) and (6.22), it follows that

$$\frac{\sigma}{k\sinh kd} = \frac{\Omega(k,d)}{k\sinh kd} = \frac{\sqrt{gk\tanh kd}}{k\sinh kd} = \frac{\sqrt{gk\tanh kd}}{k\tanh kd\cosh kd} = \sqrt{\frac{g}{k\tanh kd}}\frac{1}{\cosh kd}\,, \tag{6.21$_1$}$$

which is in agreement with the pertinent factor occurring in (4.3)

According to equations (21) and (36) on pp. 535 and 536 of LH&S4, the radiation stress tensor components S_{ij} as computed supposedly with the help of (6.21) and of some other assumptions (see below) in the second-order asymptotic approximation with respect to 'ka' are given by

$$‘\ S = \frac{1}{2}\rho g a^2 \begin{pmatrix} \dfrac{2kh}{\sinh 2kh} + \dfrac{1}{2} & 0 \\ 0 & \dfrac{kh}{\sinh 2kh} \end{pmatrix}’. \tag{6.23}$$

According to the last equation (6.21_1), equation (6.23) is written in a coordinate system whose axis X is collinear with the vector $\underline{k}_2$. In a coordinate system, whose axis X is arbitrarily oriented in the equilibrious plane relative to the vector $\underline{k}_2$, equation (6.24) turns into equation (M3.10).

In deducing (6.23), LH&S employed the equation

$$‘\ \overline{p + \rho w^2} = -\rho g z = p_0\ ’, \tag{6.24}$$

which occurs as equation (12) on page 533 of LH&S4 and which the writers deduced from some unspecified intuitive considerations in the second-order cumulative asymptotic approximation with respect to ‘ ka ’. This equation is wrong because it contradicts the time-averaged unsteady Bernoulli equation in the same approximation. Indeed, in accordance with (I.5.51) subject to Convention 2.1, the time-averaged of the unsteady Bernoulli equation in the second-order asymptotic approximation with respect to ‘ ka ’ can straightforwardly be written as.

$$\begin{aligned} \overline{P(t,\underline{x})}^t \doteqdot \overline{P(t;\underline{x},ka)}^t &\approx P_0 - \rho_0 g z - ka\rho_0 \overline{\frac{\partial\phi_{(1)}(t,\underline{x})}{\partial t}}^t - (ka)^2\left[\rho_0 \overline{\frac{\partial\phi_{(2)}(t,\underline{x})}{\partial t}}^t + \overline{e_{\mathrm{k}(2)}(t,\underline{x})}^t\right] \\ &= P_0 - \rho_0 g z - (ka)^2 \overline{e_{\mathrm{k}(2)}(t,\underline{x})}^t = P_0 - \rho_0 g z - \frac{1}{2}\rho_0 a^2 k^2 \sum_{i=1}^{3} \overline{\left[v_{(1)\mathrm{i}}(t,\underline{x})\right]^2}^t \\ &= P_0 - \rho_0 g z - \frac{1}{2}\rho_0 \sum_{i=1}^{3} \overline{\left[V_{(1)\mathrm{i}}(t,\underline{x})\right]^2}^t = P_0 - \rho_0 g z - \overline{E_{\mathrm{k}(2)}(t,\underline{x})}^t, \end{aligned} \tag{6.25}$$

because

$$\overline{\frac{\partial\phi_{(1)}(t,\underline{x})}{\partial t}}^t = \overline{\frac{\partial\phi_{(12}(t,\underline{x})}{\partial t}}^t = 0, \tag{6.26}$$

– in accordance with section I.10. In a coordinate system, satisfying the last equation (6.22), equation (6.25) can be written as:

$$\overline{P(t,\underline{x})}^t \approx P_0 - \rho_0 g z - \frac{\rho_0}{2}\left\{\overline{\left[V_{(1)1}(t,\underline{x})\right]^2}^t + \overline{\left[V_{(1)3}(t,\underline{x})\right]^2}^t\right\}. \tag{6.27}$$

or, in notation of LH&S4, as:

$$\overline{p + \frac{1}{2}\rho\left(u^2 + w^2\right)} = P_0 - \rho g z\,, \tag{6.28}$$

instead of (6.24). It is clear that equations (6.24) and (6.28) are mutually inconsistent. Formally, assuming that equations (6.24) and (6.28) are both true and also assuming that $P_0 \equiv 0$, one finds that $\overline{u^2(t,\underline{x})} = \overline{w^2(t,\underline{x})}$ for each $\underline{x} \in \underline{E}$, which is an absurd. Hence, (6.24) is false (antitrue).

Incidentally, equations (6.25)–(6.27) and hence equation (6.28) apply both in the case of a PPPMGWW and in the case of a PSPMGWW. However, in the former case,

$$\overline{E_{\mathrm{k}(2)}(t,\underline{x})}^{t} = (ka)^2 \overline{e_{\mathrm{k}(2)}(t,\underline{x})}^{t} = \frac{\rho_9 g a^2 k \cosh 2k(z+d)}{2\sinh 2kd}, \tag{6.29}$$

by (6.23), and hence

$$\int_{-d}^{0} \overline{E_{\mathrm{k}(2)}(t,\underline{x})}^{t}\, dz = \frac{\rho_9 g a^2 k}{2\sinh 2kd} \int_{-d}^{0} \cosh 2k(z+d)\, dz = \frac{1}{4}\rho g a^2. \tag{6.30}$$

If equation (6.28) were used instead of equation (6.24) in equations (14) and (30) on pp. 534 and 535 of LH&S4 then the authors would likely had arrived at the same RST components as those suggested by my recursive asymptotic theory.

4. In LH&S4, besides the TADIRST of a PPPMGWW, an attempt was made to calculate the HATADIRST of a PSPMGWW, but again from intuitive consideration. Just as in the former case, the authors have failed to get reliable final results in the latter case. Unfortunately, the notation that is used in LH&S4 is ambiguous (not self-consistent) and confusing. Therefore, I feel that an attempt to explicate all inconsistencies of LH&S4 in terms of the completely different and partly homographic notation of my recursive asymptotic theory would have been bothering and counterproductive. In order to justify the above-said, here follows just one example.

By its definition, the variable 'ζ' of LH&S4 is sense-concurrent to, i.e. it has the same range as, the variable 'Z' of my recursive asymptotic theory. In the most general case of a PSPMGWW, $Z_{(2)}$, i.e. Z in the second-order asymptotic approximation with respect to *ka*, has rigorously been proved to be determined by equation (4.47), so that its time-averaged and its horizontal and time averaged are given by equations (4.50) and (4.53) respectively. In contrast to (4.50), the time-averaged variable '$\bar{\zeta}$' of LH&S4, being sense-concurrent to '$\overline{\mathrm{Z}}$' of this exposition, is asserted (from unknown intuitive considerations) to be given in the same approximation by equation (10), occurring on p. 538 of LH&S4 and having the form

$$\bar{\zeta} = a^2 k \coth 2kh \cos 2kx\,\text{'}, \tag{6.31}$$

subject to the very last definition (6.22). This equation is not a misprint, because the absence of any constant on its right-hand side is predetermined by the statement

“...the horizontal average of $\bar{\zeta}$ is identically zero...”, (6.31$_1$)

which occurs *ibidem* (on the same p. 538 of LH&S4). The origin of both (6.31) and (6.31$_1$) remains unknown, and their appearance in LH&S4 is surprising, especially taking into account that in the case of a PPPMGWW, the parallel equation for (4.12) of LH&S3:

$$\bar{\zeta} = -\frac{ka^2}{2\sinh 2kh}, \tag{6.32}$$

which is sense-concurrent to equation (4.10) of this exposition, is correct.

5. Previously, an attempt to compute ‘ $\mathrm{Z}(t, \underline{x}_2)$ ’ in the second and third-order approximations with respect to ka for a PSPMGWW on a water layer of a finite depth was made by Tadjbakhsh and Keller [1960]. The method used by the writers reminds the method of Bogolubov and Mitropolsky [1961] in the sense that it is a fitting procedure rather than a genuine asymptotic method. All calculations are made by the former writers in terms of some non-conventional dimensionless independent and functional variables. In this case, the time variable has a peculiar property that it is scale depending on an unknown frequency and hence it actually depends on the specific approximation with respect to ‘ ka ’, in which the frequency is calculated . Everybody is of course free to use his own notation as he pleases. However, interpretation of the final results of Tadjbakhsh and Keller [1960] in terms of the ordinary dimensional variables of hydrodynamics is not straightforward. Since no results of that paper have been canonized, therefore their review is beyond the scope of this exposition. For the reader, who will wish to learn the above paper¸ it is noteworthy that the term ‘$\left(\omega_0^2 - \omega_0^{-2}\right)$’ in equation (30) of Tadjbakhsh and Keller [1960] is misprinted. That term must have the form ‘$\left(\omega_0^2 + \omega_0^{-2}\right)$’, which immediately follows from comparison of equations (30) and (28) of that paper.

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