Gelfond constant

Edgar Valdebenito

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ABSTRACT

This note presents some formulas related with Gelfond constant : e^{π}

Introduction

Gelfond constant : e^{π}

1. $$e^{\pi} = (-1)^{-i} = i^{-2i} \ , \ i = \sqrt{-1}$$

2. $$e^{\pi} = \sum_{n=0}^{\infty} \frac{\pi^n}{n!}$$

3. $$e^{\pi} = \sum_{n=0}^{\infty} \frac{(4i)_n}{n!} \left(\frac{\sqrt{2} - 1 - i}{\sqrt{2}} \right)^n$$

4. $$e^{\pi} = \sum_{n=0}^{\infty} \frac{(6i)_n}{n!} \left(\frac{2 - \sqrt{3} - i}{2} \right)^n$$

5. $$e^{\pi} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(6i)_k (-3i)_{n-k} (-1)^n i^k}{k! \, (n-k)!} \left(\frac{1}{\sqrt{3}} \right)^{2n-k}$$

6. $$e^{\pi} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(8i)_k (-4i)_{n-k} (-1)^n i^k}{k! \, (n-k)!} \left(\sqrt{2} - 1 \right)^{2n-k}$$

7. $$e^{\pi} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(12i)_k (-6i)_{n-k} (-1)^n i^k}{k! \, (n-k)!} \left(2 - \sqrt{3} \right)^{2n-k}$$

8. $$e^{\pi}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(4\,i)_k\,(-2\,i)_{n-k}}{k!\,(n-k)!}\left(-\frac{7}{18}\right)^{n-k}\left(\frac{1-5\,i}{6}\right)^{k}$$

Notation : $\lim_{n\to\infty}x_n=x \Leftrightarrow x_n\to x$

Iterative formulas

9. $$x_{n+1}=x_n+23\sin(\ln(x_n))\ ,\ x_1=23\Rightarrow x_n\to e^{\pi}$$

10. $$x_{n+1}=x_n+46\cos\left(\frac{\ln(x_n)}{2}\right)\ ,\ x_1=23\Rightarrow x_n\to e^{\pi}$$

11. $$x_{n+1}=x_n-46\left(\tan\left(\frac{\ln(x_n)}{4}\right)-1\right)\ ,\ x_1=23\Rightarrow x_n\to e^{\pi}$$

Gelfond constant via Ramanujan − Göllnitz − Gordon continued fraction

RGG continued fraction :

12. $$H(q)=\cfrac{q^{1/2}}{1+q+\cfrac{q^2}{1+q^3+\cfrac{q^4}{1+q^5+\cfrac{q^6}{1+q^7+\ldots}}}}$$

13. $$x_{n+1}=\left(\sqrt{4+2\sqrt{2}}+\sqrt{3+2\sqrt{2}}\right)^2 x_n\,(H(x_n))^2\ ,\ x_1=23\Rightarrow x_n\to e^{\pi}$$

14. $$x_{n+1}=\left(\sqrt{4+2\sqrt{2}}+\sqrt{3+2\sqrt{2}}\right)^2 x_n\left(H\left(\frac{1}{x_n}\right)\right)^2\ ,\ x_1=23\Rightarrow x_n\to e^{\pi}$$

Gelfond constant via Ramanujan cubic continued fraction

Ramanujan cubic continued fraction :

15. $$V(q) = \cfrac{q^{1/3}}{1 + \cfrac{q + q^2}{1 + \cfrac{q^2 + q^4}{1 + \cfrac{q^3 + q^6}{1 + \ldots}}}}$$

16. $$x_{n+1} = -\left(10 + 6\sqrt{3}\right) x_n \left(V\left(-\frac{1}{x_n}\right)\right)^3 , x_1 = 23 \Rightarrow x_n \to e^{\pi}$$

Gelfond constant via Ramanujan φ Theta *function*

Ramanujan $\varphi(q)$ Theta *function* :

17. $$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2 \sum_{n=0}^{\infty} q^{n^2}$$

18. $$x_{n+1} = \frac{2 x_n}{2 - x_n \varphi\left(x_n^{-1}\right) + x_n \sqrt[4]{6\sqrt{3} - 9}\ \varphi\left(x_n^{-3}\right)} , x_1 = 23 \Rightarrow x_n \to e^{\pi}$$

Gelfond constant via Ramanujan ψ Theta *function*

Ramanujan $\psi(q)$ Theta *function* :

19. $$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}$$

20. $$x_{n+1} = 3\sqrt{3}\ \left(\psi\left(-x_n^{-3}\right)\right)^3 \left(\psi\left(-x_n^{-1/3}\right)\right)^3 , x_1 = 23 \Rightarrow x_n \to e^{\pi}$$

Gelfond constant via Euler products

Euler products :

21. $$E1(q) = \prod_{n=1}^{\infty} \left(1 + q^n\right)^{24} , |q| < 1$$

22. $$x_{n+1} = \frac{1}{2}\, x_n + 4\, E1\left(\frac{1}{x_n}\right) \;,\; x_1 = 23 \;\Rightarrow\; x_n \to e^{\pi}$$

23. $$E2(q) = \prod_{n=1}^{\infty} \left(1 + q^{4n}\right)^6 \;,\; |q| < 1$$

24. $$x_{n+1} = 4\, \sqrt[8]{32}\, \left(\sqrt{2} + 1\right)^{3/2} E2\left(\frac{1}{x_n}\right) \;,\; x_1 = 23 \;\Rightarrow\; x_n \to e^{\pi}$$

25. $$E3(q) = \prod_{n=1}^{\infty} \left(1 + q^{6n}\right)^4 \;,\; |q| < 1$$

26. $$x_{n+1} = \frac{2\, \sqrt[3]{2\left(\sqrt{3} - 1\right)}}{\sqrt{2} - \sqrt[4]{3}}\, E3\left(\frac{1}{x_n}\right) \;,\; x_1 = 23 \;\Rightarrow\; x_n \to e^{\pi}$$

final Remark

27. $$e^{\pi} = 23.140692632779262 \ldots$$

Gelfond's constant is a trancendental number

References

[1] C. Adiga , T. Kim , M.S. Mahadeva Naika , and H.S. Madhusudhan : On Ramanujan's cubic continued fraction and explicit evaluations of theta-functions. arXiv:math/0502323v1 [math.NT] 15 Feb 2005.

[2] H.H. Chan and S.-S. Huang : On the Ramanujan-Göllnitz-Gordon continued fraction, Ramanujan J. 1 (1997) , 75-90 .