# A recursive asymptotic theory of nonlinear gravity surface waves on a water layer with an even or infinitely deep bottom

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## Abstract

It is shown with complete logical and mathematical rigor that under the appropriate hypotheses of analytical extension and of asymptotic matching, which are stated in the article, the *nonlinear problem* of irrotational and incompressible gravity waves on an infinite water layer of a constant depth *d* reduces to an infinite *recursive sequence of linear two-plane boundary value problems for a harmonic velocity potential* with respect to powers of a dimensionless real-valued scaling parameter '*ka*', where $k>0$ is the wave number and $a>0$ the amplitude of a *priming* (*seeding*) *progressive*, or *standing*, *plane monochromatic gravity water wave* (briefly *PPPMGWW* or *PSPMGWW* respectively). The method, by which the given nonlinear water wave problem is treated in the exposition from scratch, can be regarded as a peculiar instance of the general *perturbation method*, which is known as the *Liouville-Green* (*LG*) *method* in mathematics and as the *Wentzel-Kramers-Brillouin* (*WKB*) *method* in physics. In the framework of the recursive theory developed, the velocity potential and any bulk or surface measurable characteristic of the wave motion is represented by an infinite asymptotic power series with respect to '*ka*', whose all coefficients are expressed *in quadratures* in accordance with a well-established an algorithm for their successive calculation. The theory developed applies particularly in the case where the depth *d* is taken to infinity. Besides the priming velocity potential of the first, linear asymptotic approximation in *ka*, the partial velocity potential and all relevant characteristics of wave motion of the second order with respect to *ka* are calculated in terms of elementary functions both in the case of a PPPMGWW and in the case of a PSPMGWW. Accordingly, the recursive theory incorporates the conventional Airy (linear) theory of water waves linear as its first non-vanishing approximation with the following proviso. In the Airy theory, the boundary condition at the *perturbed free* (*upper*) *surface* of a water layer is *paradoxically* stated at the *equilibrium plane* $z=0$, in spite of the fact that at any instant of

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time some part of the plane is necessarily located in air or in vacuum, and not in water. This and also a similar paradox arising in computing the time averages of bulk characteristics at spatial points close to the perturbed free surface are solved in the article.

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# 1. Basic nomenclature

To start with, I shall specify the main general nomenclature (logographic notation and wordy terminology) as introduced in the articles Iosilevskii [2015 and 2016a–2016c] to the cases to be dealt with in this article.

## 1.1. General nomenclature

1) A set is a class, but a class is not necessarily a set. I call a class "*a regular class*" if it is a set and "*an irregular class*" if it is not a set. In the contemporary literature on logic and mathematics, an irregular class is called a *proper class,* whereas a regular class, i.e. a set, is sometimes called a *small class* (see, e.g., Fraenkel et al [1973, p. 128, DEFINITION VII] for the former term or the article **class** in Wikipedia for both terms). The difference between an irregular class and a set (regular class) is discussed in detail in Iosilevskii [2016a, subsection I.9.3.2]). For instance, taxons (taxa, taxonomic classes) of any biological taxonomy of bionts (BTB) are *irregular*, or *proper*, *classes*, i.e. *classes* that are *not sets*. Particularly, *the species* (*specific class*) *of men*, that is formally called '*Homo sapiens*' and informally "*man*", exists as an irregular class but the set of all men does not exist in the sense that the expression "the set of all men" has no denotatum. By contrasts, in mathematics, a well-defined class of numbers as the class of natural (natural integer) numbers, the class of rational numbers, the class of real numbers, or the class of complex numbers is a regular class, i.e. a set. The *memberless class* is a *set* that is *denoted* [*logographically*] by '∅' and is *called* [*phonographically,* i.e. *wordily*] "the *empty set*"; i.e. semantically ∅ *is* the *empty set*. The class of a single object *x* is a *set* that is *denoted* by '{*x*}' and is *called* "the *singleton of x*"; i.e. semantically {*x*} *is* the *singleton of x*.

2) $\stackrel{\rightarrow}{=}$, $\stackrel{\leftarrow}{=}$, and $\stackrel{\leftrightarrow}{=}$ are *equality signs by definition*, a *rightward one*, a *leftward one*, and a *two-sided one* respectively, which are rigorously defined in Iosilevskii [2015, 2016a, and 2016b].

3) A symbol of the form '$\{\boldsymbol{x}|\boldsymbol{P}(\boldsymbol{x})\}$', called a *class-builder* (or particularly *set-builder*), which is designed to convert a given *relation* (*condition*) $\boldsymbol{P}(\boldsymbol{x})$ into a certain *constant* or *variable class-*

*valued* (or correspondingly) *term* (‘$\boldsymbol{P}$’ and ‘$\boldsymbol{x}$’ are atomic placeholders having the appropriate ranges).

4) ‘$\omega_0$’ *denotes*, i.e. $\omega_0$ *is*, the set of all *natural numbers* from 0 to infinity. Given $n \in \omega_0$, $\omega_n$ *is* the set of natural numbers from *n* to infinity; i.e.

$$\omega_n \mathrel{\overrightarrow{\equiv}} \{i | i \in \omega_0 \text{ and } i \geq n\}.$$

Given $m \in \omega_0$, given $n \in \omega_m$, $\omega_{m,n}$ *is* the set of natural numbers from the given number *m* to another given number *n* subject to $n \geq m$, i.e.

$$\omega_{m,n} \mathrel{\overrightarrow{\equiv}} \{i | i \in \omega_0 \text{ and } n \geq i \geq m\}.$$

It is understood that $\omega_{m,n} = \varnothing$ if $m>n$ and also that $\omega_{m,m} = \{m\}$ and $\omega_{m,\infty} = \omega_m$.

5) ‘$I_{-\infty,\infty}$’ *denotes*, i.e. $I_{-\infty,\infty}$ *is*, the set of all *natural integers* (*natural integral numbers*) – strictly positive, strictly negative, and null. Given $n \in I_{-\infty,\infty}$,

$$I_{n,\infty} = I_{\infty,n} \mathrel{\overrightarrow{\equiv}} \{i | i \in I_{-\infty,\infty} \text{ and } i \geq n\},$$

$$I_{-\infty,n} = I_{n,-\infty} \mathrel{\overrightarrow{\equiv}} \{i | i \in I_{-\infty,\infty} \text{ and } i \leq n\},$$

i.e. $I_{n,\infty}$ or $I_{\infty,n}$ is the set of all natural integers greater than or equal to *n*, and $I_{-\infty,n}$ or $I_{n,-\infty}$ is the set of all natural integers less than or equal to *n*. Given $m \in I_{-\infty,\infty}$, given $n \in I_{m,\infty}$,

$$I_{m,n} \mathrel{\overrightarrow{\equiv}} \{i | i \in I_{-\infty,\infty} \text{ and } n \geq i \geq m\},$$

i.e. $I_{m,n}$ is the set of all natural integers that are greater than or equal to *m* and are less than or equal to *n*.

6) The previous two items are *explicative definitions*. A theory of natural integers in particular, and a theory of any numbers (as rational, real, or complex ones) in general can consistently be deduced from the five Peano axioms, which are, in turn, theorems of an axiomatic set theory (see, e.g., Halmos [1960, pp. 46–53], Burrill [1967], Feferman [1964]).

7) The *unordered pair of two different* (*distinct*) *objects x and y* is the set $\{x, y\}$ of those objects, such that

$$\{x, y\} \mathrel{\overrightarrow{\equiv}} \{z | z = x \text{ or } z = y\}.$$

subject to $x \neq y$ (cf. Halmos [1960, p. 10]). If $x = y$ then the set $\{x\}$ such that $\{x\} = \{x, x\}$, having *x* as its only member, is called *the singleton of x* or less explicitly (more generally) *a singleton.*

6) The *ordered pair* $(x, y)$ *of two objects x and y, different or not*, – particularly that of two different or same elements $x$ and $y$ of two different or same sets (or in general classes) $X$ and $Y$ respectively. – is conventionally defined as:

$$(x, y) \stackrel{\rightarrow}{=} \{\{x\}, \{x, y\}\}$$

(see, e.g., Halmos [1960, pp. 22–25]). Therefore, by *Axiom of extension* (*ibid.* p. 2), for any four objects $x$, $y$, $x'$, and $y'$,

$$(x, y) = (x', y') \text{ if and only if } x=x' \text{ and } y=y'.$$

The set $X \times Y$, defined as:

$$X \times Y \stackrel{\rightarrow}{=} \{z | z = (x, y) \text{ for some } x \in X \text{ and for some } y \in Y\},$$

is called the *Cartesian*, or *direct*, *product of X* and $Y$ (*ibid.* p. 24). Here and throughout this exposition, $\stackrel{\rightarrow}{=}$ is the rightward sign of equality by definition, which, along with $\stackrel{\leftarrow}{=}$ and $\stackrel{\leftrightarrow}{=}$, is rigorously defined, e.g., in Iosilevskii [2015, 2016a, and 2016b].

7) Given $n \in \omega_2$, an *ordered n-tuple* of objects $x_1, x_2, ..., x_{n-1}, x_n$ is defined as a *repeated*, ($n$–1)-*fold ordered pair* thus:

$$\underline{x}_{,n} \stackrel{\rightarrow}{=} (x_1, x_2, ..., x_{n-1}, x_n) \stackrel{\rightarrow}{=} (\underline{x}_{n-1}, x_n) = \underbrace{((..((}_{n-1} x_1, x_2), x_3), ..., x_{n-1}), x_n) .$$

More specifically, an ordered $n$-tuple that is defined by the above formula is called *the left-associated repeated* (or *reiterative*) ($n$–1)-*fold* (or ($n$–1)-*ary*) *ordered pair of* $x_1$, $x_2$, …, $x_n$ *in that order*. Accordingly, for any 2$n$ objects $x_1$, $x_2$,…, $x_n$, $x'_1$, $x'_2$, …, $x'_n$,

$$(x_1, x_2, ..., x_n) = (x'_1, x'_2, ..., x'_n) \text{ if and only if } x_1 = x'_1, x_2 = x'_2, ..., x_n = x'_n.$$

8) An ordered $n$-tuple with any $n \in \omega_2$ is indiscriminately called an *ordered multiple*. It is worthy of recalling that, in contrast to an ordered multiple, an *ordered set* is a set that serves as a *domain of definition of the liner order relation* (*predicate*) ≤. An ordered irregular class does not exist.

9) If an ordered $n$-tuple is an $n$-fold ordered pair that is not systematically associated to the left then the association must be indicated either explicitly or by the appropriate notation. For instance,

$$(x_{,0}; x_1, x_2, ..., x_{n-1}, x_n) \stackrel{\rightarrow}{=} (x_{,0}; \underline{x}_{n-1}, x_n) \stackrel{\rightarrow}{=} (x_{,0}, ((..((x_1, x_2), x_3), ..., x_{n-1}), x_n)).$$

10) An *one-component univalent holor* is a conceptual object, which is denoted by '$\underline{x}_1$' or '$(x_1)$' and which can therefore be also called an *ordered one-tuple*, or *ordered single*, the understanding being that such an object *is distinct from a scalar* (*nilvalent holor*) and that it *can have a scalar as its only component*. Therefore, without loss of generality, $\underline{x}_1$ or $(x_1)$ can be identified with the singleton $\{x_1\}$ – the set having $x_1$ as its only member (element), so that

$$\underline{x}_1 \doteqdot (x_1) \doteqdot \{x_1\}.$$

At the same time, a set of *n* elements with $n \in \omega_2$ can alternatively be called an *unordered n-tuple*. Therefore, $(x_1)$ as defined above can be regarded as *an ordered one-tuple and as an unordered one-tuple simultaneously*. Thus, for any $n \in \omega_1 \doteqdot \{1,2,...\}$, an ordered *n*-tuple, i.e. an *n*-component univalent holor, is a nonempty set and is *not a nonempty individual*. A definition of the term "holor" can be found, e.g., in Moon and Spencer [1965, pp. 1, 14]), and also Iosilevskii [2016b, sub-subsection 2.3.1].

11) If *X* and *Y* are two classes (or particularly sets) then $X - Y$, called *the difference of X and Y*, is the set of all those elements of *X* which are not elements of *Y*.

12) Whenever confusion can result, the end of an article as a comment, preliminary remark, proof, etc will be marked by a heavy dot '•', – just as in I.

## 1.2. Specific interpretation of some logographic symbols by default

1) *R* is the set of real numbers and equivocally the field of real numbers.

2) Each of the letters '*x*', '*y*', and '*z*', alone or together with some labels on it (alphanumeric or not) is a real-valued variable, i.e. a variable whose range is a set (or field) *R* of real numbers. In the statements below, each of the bold-faced letters '***x***', '***y***', and '***z***' is, for the sake of brevity, a placeholder for any one of the light-faced letters '*x*', '*y*', and '*z*'.

3) $\underline{\boldsymbol{x}}_n$, i.e. $(\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_{n-1}, \boldsymbol{x}_n)$ or $\langle \boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_{n-1}, \boldsymbol{x}_n \rangle$, is an ordered *n*-tuple of real numbers.

4) $\underline{\boldsymbol{x}}_3$ will as a rule be abbreviated as $\underline{\boldsymbol{x}}$, i.e, $\underline{\boldsymbol{x}} \doteqdot \underline{\boldsymbol{x}}_3 \doteqdot \langle \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \rangle$. For instance, $\underline{x} \doteqdot \underline{x}_3 \doteqdot \langle x_1, x_2, x_3 \rangle$, $\underline{y} \doteqdot \underline{y}_3 \doteqdot \langle y_1, y_2, y_3 \rangle$, and $\underline{z} \doteqdot z_3 \doteqdot \langle z_1, z_2, z_3 \rangle$.

5) $\underline{E}_n$ is a *real n*-dimensional arithmetical vector space, i.e. an *n*-dimensional arithmetical vector space over the field *R*. Equivocally, $\underline{E}_n$ is the underlying set of vectors (elements) of that space, i.e. the set of ordered *n*-tuples of real numbers. Hence, given $n \in \omega_1$,

$$\underline{E}_n \doteqdot R^{n\times} \doteqdot \underbrace{R\times R\times \ldots \times R\times R}_{n \text{ times } R}$$
$$\doteqdot R^{(n-1)\times}\times R \doteqdot \underbrace{[[\ldots[}_{n-2}[R\times R]\times R]\times\ldots]\times R]\times R$$
$$\doteqdot \{(x_1, x_2, \ldots, x_{n-1}, x_n) | x_1 \in R, x_2 \in R, \ldots, x_{n-1} \in R, x_n \in R\},$$

i.e. $\underline{E}_n$ is *the left-associated repeated* (or *reiterative*) ($n$–1)-*fold* (or ($n$–1)-*ary*) *direct* (or *Cartesian*) *product of R by itself*, called *the left-associated* $n$th *direct* (or *Cartesian*) *power of R*, the understanding being that

$$\underline{E}_1 \doteqdot R^{1\times} \doteqdot \{(x_1\ ) | x_1 \in R\} = \{\{x_1\} | x_1 \in R\} \neq R\,.$$

6) ‘$t$’ is a real-valued time variable, whose every value $t\in R$ is interpreted as an instant of time assotiated with .the reading of a certain clock. In the actual fact, an instant of time is the singleton of $t$, i.e. $\underline{t} \doteqdot \underline{t}_1 \doteqdot \{t\} \in \underline{T}$, that is regarded as an *arithmetical vector of the space* $\underline{E}_1$, which is in turn interpreted as the *time continuum*. However, for the sake of simplicity, I shall employ ‘$t$’ as stated above, and not ‘$\{t\}$’.

7) In accordance with the item 9 of the previous sub-section,

$$(t; x, y, z) \doteqdot (t, (x, y, z)) \doteqdot (t, \underline{x}),\ (t; \underline{x}, \varepsilon) \doteqdot (t, (\underline{x}, \varepsilon)) \doteqdot (t, (x, y, z, \varepsilon)).$$

### 1.3. Special quotations versus ordinary quotations

In this article, besides ordinary quotations that may be used but occasionally, I widely use various so-called *special*, or *attitudinal*, *quotations* (*SQ’s*), which indicate the kind of a value, and hence the value itself, of the interior of a quotation, which is put forward as its accidental (circumstantial) *denotatum* (*denotation value*, pl. “*denotata*”). The entire system of SQ’s was developed in Iosilevskii [2016a, Preface, subsection 3.4], while in this article, I employ only some kinds of SQ’s, which are briefly described below for the reader’s convenience.

In order to state an *ordinary quotation* (as a repetition of the exact passage of another work or of the title of a book), I employ French double angle quotation marks, « », instead of *ordinary* English single or double quotations marks (as defined, e.g., under the vocabulary entry **quotation mark** in A Merriam-Webster® [1981], whereas the latter are freed of their ordinary functions and are used only as *special quotation marks* (*SQ marks*). In this case, the light-faced or bold-faced single or double, straight and curly or slant, English quotation marks are used differently. The pair of quotation marks that is used for making an ordinary or special quotation will be called *the exterior of* *the quotation*, whereas the *graphonym* (*graphic expression*) quoted will be called *the interior of the quotation*.

I *do not* follow Frege [1893–1903, vol. 1, p. 4] and his followers either in *admitting only a single kind of SQ's, each of which is the name of its interior,* and which I call *Fregean*, or *Frege's, quotations* (*FQ's*), or in *obstinately attempting to indicate autonymy with the help of the appropriate SQ marks in all cases* simply because such an attempt is *impracticable*. For forming FQ's, which I also call *proper,* or *strict*, *autonymous quotations* or *kyrioautonymous quotations* (*KAQ's*) and which I shall use quite rarely if at all, I shall employ *slant light-faced single quotation marks*, ʻ ʼ. Most often, I shall employ, – I have already started to, – *curly* (*decisive*) or *straight* (*indecisive*) *light-faced quotation marks*, *single ones*, ‘ ’ or ' ', which I shall call *homoloautographic*, or *photoautographic*, *quotation marks* (*HAQ marks*), and *double ones*, “ ” or " ", which I shall call *iconoautographic*, or *pictoautographic*, *quotation marks* (*IAQ marks*). Accordingly, an SQ will be called a *homoloautographic*, or *photoautographic*, *quotation* (*HAQ*) if it is formed by enclosing a graphonym between HAQ marks; an *iconoautographic*, or *pictoautographic*, *quotation* (*IAQ*) if it is formed by enclosing a graphonym between IAQ marks. HAQ's and IAQ's are indiscriminately called *common*, or *lax*, *autographic quotations* and also *cenautographic quotations* (*CAQ's*). KAQ's and CAQ's are indiscriminately called *special autographic quotations* (*SAQ's*), whereas all quotation marks that are used for forming SAQ's are called *SAQ marks*.

I employ the exterior of an HAQ or IAQ for indicating my *ad hoc* (*epistemologically relativistic*) mental attitude, according to which its interior denotes the class of distinct recurrent recognizably same graphonyms, which occur in the article and which are called *isotokens of the interior*. Accordingly, the interior of an HAQ or IAQ is alternatively called its *percept-class*. In this case, an HAQ denotes *the class of homolographic* (*photographic*), i.e. *proportional* or particularly *congruent*, *isotokens of its interior*, whereas an IAQ denotes *the class of iconographic* (*pictographic*), i.e. of both homolographic and *analographic* (*stylized*), *isotokens of its interior*.

The interior of an IAQ may contain some constituent *logographs* (*logographonyms*) or *iconographs* (*iconographonyms*), indiscriminateiy called *pasigraphs* (*pasigraphonyms*), which are known from a previous definition or definitions to be *homolographs*, i.e. a graphonyms that have only homolographic isotokens. In this case, the isotoken-class of the interior of the IAQ is supposed to preserve this property. By contrast, the *phonic* (*vocal*) *sounds* that are produced when the interior of an IAQ is read orally, provided that the interior does not contain any pasigraph, is called a *paratoken of the interior* and also most generally a *phononym*. An isotoken or paratoken (if exists) of the interior of an IAQ is indiscriminately called a *token* of the interior. Accordingly, if the interior of an IAQ is a phonograph then the IAQ may, depending on the mental attitude of its interpreter, denote either the *isotoken-class* or a *paratoken-class* of its interior or else the union of the two classes that is called the token-class of the interior.

The interior of an HAQ is either a pasigraph, i.e. a graphonym that is intelligible to a sapient subject independent of the *phonemic* language or languages, in which he has command, or a *phonographic* (*wordy*) *name*, which is conventionally set in a certain font, – such a name, e.g., as the *Linnaean binomial* (*binomen*) of a *species* in a biological taxonomy of bionts, which is conventionally set in *italic* (‘*Homo sapiens*’ for instance). The latter case is irrelevant to this exposition, so that the interior of any HAQ occurring in the exposition is a pasigraph or, more specifically, a logograph, i.e. a graphonym that has no phonic paratokens. Consequently, an isotoken of the interior of such an HAQ is alternatively called a *token* of the interior.

Incidentally, *the sense* (*sense value*) *of*, or *expressed by*, *a complex* (*combined*) *linguistic graphonym*, – provided that the latter has the sense thus defined, – is a *biune mental process* (psychical entity, brain symbol) of the maker or interpreter of the graphonym (as me), which includes (i) a *sense operation of coordination of the classes* that are *designated* by the relevant simple constituent parts of the graphonyms and that are called *the object classes of the sense*, and which also includes (ii) *the class that is resulted by the sense operation and that is designated by the graphonym*. The latter class is called *the designatum* (pl. “*designata*”) *of the graphonym* and alternatively *the subject class of the sense of the graphonym*. It is understood that if a graphonym is regarded as a simple one or is an idiom then its sense coincides with its desigsnatum. Consequently, one of two given senses (sense values) of a *glossonym* (*linguistic onym*, or *nym*) is said to be broader, or narrower, than the other one if the subject class of the former is broader, or correspondingly narrower, then the latter. In Iosilevskii [2016a], in order to refer to the sense of a graphonyn, I enclose the latter in *light-faced virgule-like quotation marks*, \ /, which I shall call *enneoxenographic*, or *semantic*, or *sense*, *quotation marks* (*EXQ marks*). No *enneoxenographic* (*semantic*, *sense*) *quotations* (*EXQ’s*) thus formed are used in this exposition.

The bold-faced quotation marks **′  ′**, **‘ ’** or **' '**, and **“ ”** or **" "**, **/**, and quotations that they form will be qualified as *quasi-kyrioautographic* (*QKA*), *quasi-homoloautographic* (*QHA*), and *quasi-iconoautographic* (*QIA*) *quotation marks* (*Q marks*) and *quotations* (*Q’s*), respectively. The interior of a QKAQ, QHAQ, QIAQ, or QPAQ is either entirely a *placeholder* (*place-holding variable*) or it contains some placeholders, upon replacing all of which with appropriate concrete graphonyms the bold-faced quotation marks should be replaced with the corresponding light-faced ones. That is to say, QKAQ's, QHAQ's, and QIAQ's are placeholders for KAQ's, HAQ's, and IAQ's respectively, whereas the latter are constants. QHAQ's and QIAQ's are indiscriminately called *common*, or *lax*, *quasi-autographic quotations* and also *quasi-cenautographic quotations* (*QCAQ's*). QKAQ's and QCAQ's are indiscriminately called *special*, or *attitudinal*, *quasi-autographic quotations* (*SQAQ's*), whereas all quotation marks that are used for forming SQAQ's are called *SQAQ marks*.

Uses of the HAQ marks ' ' and of the QHAQ's **' '**, e.g., can be illustrated as follows. Any one of the *concrete logographs* (a) '$\sin x$', '$\sinh x$', and '$e^x$', and also any one of these (b) '$x+y$' '$x \cdot y$', '$\sin(x+y)$', '$\sinh(x \cdot y)$', etc are by definition *functional forms*, whereas 'sin', e.g., is the *associated function of* '$\sin x$' and '+', e.g., is the *associated function of* '$x+y$'. At the same time, the *abstract logographs* **'** $f_1(x)$**'**, **'** $f_2(x,y)$**'**, etc are *placeholders* for functional forms containing the respective independent variables. Any pair of SAQ marks can be replaced with the appropriate *prepositive added words*. For instance, instead of **'** $f_1(x)$**'**, I may use the phrase "the functional form placeholder $f_1(x)$", but I may not use the phrase "the functional form $f_1(x)$", because **'** $f_1(x)$**'** is any isotoken of the placeholder therein depicted between light-faced single quotation marks, and not a functional form. In this case, I may say that **'** $f_1(x)$**'** is *a* singulary functional form, the understanding being that, once the interior of the above QHAQ is replaced with a *concrete* singulary functional form such as '$\sin x$', '$\sinh x$', and '$e^x$', the bold-faced single quotation marks should be replaced with light-faced one.

The procedure of using SQ's (special quotations), which has been described above, is called the *Special Quotation Method* (*SQM*) or *Special Quotation Device* (*SQD*).

Thus, the reader should remember that quotations marks of the different forms and shapes, which he encounters in the treatise, are not selected spontaneously and that therefore they are not interchangeable. At the same time, as I have already pointed out previously, no attempt will made to indicate autonymy with the help of SAQ's (special autographic quotations) in *all* cases because such an attempt is doomed to failure. I resort to the SQD only where confusion between *autonymous* and *xenonymous* uses of *xenographs* might otherwise be harmful. In some cases, such confusion is harmless, while in many other cases, where a graphonym is used *polysemantically*, it is productive and indispensable. For instance, in stating verbal definitions, I shall often use the defining predicate "*is called*", which should in principle be followed by *the IAQ of the pertinent xenographic definiendum*. In many cases, however, it is, not only harmless, but useful to employ *unquoted xenographic definienda* after that predicate.

## 2. Mathematical physics (applied mathematics) versus pure mathematics

### 2.1. Molecular hypothesis versus continuum hypothesis in mathematical physics

One of the most fundamental epistemological axioms of modern natural philosophy is the *molecular hypothesis* – the presently common concept, according to which every substance consists of molecules. Therefore, merely the fact of applying the qualifier "continuum", or "continuous", to a physical theory (as in the titles of books Landau *et al* [1991] and Truesdell [1991]) signifies that, in constructing the theory, an implicit *assumption* is made that, *under certain restrictions*, the substance can be treated as continuum. The words "*hypothesis*" and "*assumption*" are English synonyms. Hence, the above mentioned implicit assumption is a hypothesis that is often called "*the continuum hypothesis*".

From the standpoint of differential and integral calculus, the treatment of the same substance as one consisting of molecules on the one hand, and as continuous medium on the other, is a *paradox* (*contradiction*). This paradox is solved physically and not mathematically, i.e. qualitatively and not quantitatively, by separating the underlying continuum *mathematical theory* and its *physical interpretations* by *real-valued functions.* In this case, the word "*interpretation*" can be understood in the sense of the technical term as defined in a theory of logistic systems and formalized languages (cf. Church [1956, §07 and footnote 199] or Fraenkel *et al* [1973, chapter V, §3ff]). For instance, the conventional wave equation can be interpreted as describing any given kind of waves such as electromagnetic or acoustic ones or such as water waves.

To be specific, the appropriate physical analysis based on the molecular hypothesis shows that all macroscopic (continuum) physical characteristics of the substance (as its mass density, elastic and viscosity coefficients, dielectric permeability, etc.) are physically meaningful only within a domain whose minimal linear size is much larger either than the maximal intermolecular distance, if the substance is condensed matter (liquid or solid), or is much larger than the mean-free path of the substance molecules, if the substance is a gas. Likewise, the minimal linear size of a macroscopic field source (such as an electric charge, an electric current, a dislocation, a vortex, etc.), and also the minimal thickness of a transition region in condensed matter which is treated mathematically as an infinitesimally thin interface between two continuous media or, in particular, as the infinitesimally thin boundary surface of a continuous condensed medium (the interface between condensed mater and vacuum), must satisfy the above mentioned conditions. Incidentally, in condensed matter, the mean intermolecular distance can be, and will be, treated as the mean-free path of the substance molecules.

Particularly, according to the continuum hypothesis, a fluid (liquid or gas) is treated mathematically as continuous medium. From the standpoint of mathematical analysis, this means that all macroscopic physical characteristics of the fluid, as its mass density or as components of its momentum flux density vector, are described by differentiable, and hence, continuous functions of appropriate independent variables. The functions are treated with the help of all available tools of mathematics including differential and integral calculus, and also including partial differential equations. Still, from the standpoint of physical analysis, a fluid consists of individual molecules. Therefore, a volume element $dV$ which is mathematically regarded as infinitesimal should physically be *interpreted* as being small macroscopically, but not microscopically. In other words, an infinitesimal volume element $dV$ of fluid mechanics is one that contains a very large number of molecules whereas $(dV)^{1/3} > 0$ is small as compared to some macroscopic linear characteristics of the fluid motion (as the characteristic wavelength $\lambda$ if appropriate). The presently common terms of fluid mechanics such as "a *fluid particle*", "a *material particle*", "a *point of the fluid*", and "a *point in the fluid*" are just connotative synonyms (class-synonyms) both of the verbal term "a *macroscopically small volume element of the fluid*" and of the logographic term '$dV$' (cf. Landau and Lifshitz [1987, p. 1]). Likewise, all mathematical points, curves, and surfaces of discontinuity of the field should be interpreted physically as three-dimensional manifolds, whose thickness (minimal linear size) is much larger than the mean-free path of the molecules constituting the medium in question.

In addition to the above-said, the continuum hypothesis has the following two important implications.

First, any macroscopic (phenomenological) theory of a specified physical system should be deducible from the relevant microscopic (molecular) theory of that same system. For instance, classical thermodynamics follows from statistical physics, macroscopic electrodynamics follows from microscopic electrodynamics (classical or quantum), the theory of elasticity and plasticity follows from the microscopic theory of crystal lattices, gas dynamics follows from the Boltzmann kinetic equation, etc. Macroscopic theories, as fluid mechanics, are as a rule well-established ones, whereas some of the corresponding microscopic theories, as any one of the existing microscopic theories of classical liquids, are not. Still, this fact does not cancel the restrictions which are imposed on any macroscopic theory by the molecular hypothesis through the continuum hypothesis. Thus, as contrasted to *mathematical foundations* of continuum mechanics (as various integral principles), a *microscopic* theory (either an actual one or a would-be one) can be regarded as the *physical foundation* of the corresponding macroscopic (continuum) theory.

The second implication of the continuum hypothesis is that every macroscopic differential equation of mathematical physics either is or can be regarded as a result of the appropriate averaging of a certain microscopic equation. As a consequence, all equations of mathematical physics in general, and those of continuum mechanics in particular, are interpreted in theoretical physics as conventional partial differential equations, all unknowns of which are conventional (non-generalized) functions having all necessary conventional (non-generalized) derivatives. Two other irrefutable arguments in favor of the requirement that all fields of continuum fluid mechanics must be differentiable real-valued functions follow from the *classical measurability and determinacy principles*.

Thus, when regarded as a part of theoretical physics, and not as a part of pure mathematics, continuum fluid mechanics is a classical phenomenological macroscopic physical theory, which is based on the *continuum hypothesis* (see, e.g., Batchelor [1967, pp. 4-6], Lamb [1932, p. 1], and Landau an Lifshitz [1987, p. 1]). In addition, being a classical theory, and not a quantum-mechanical one, continuum fluid mechanics is also based on two other fundamental principles of classical physics, which can be called the *classical measurability principle* and the *classical determinacy principle*, as contrasted to the *quantum-mechanical indeterminacy,* or *uncertainty, principle.* The principles themselves and also their implications in classical macroscopic (continuous) theories are discussed below.

### 2.2. The classical measurability principle

According to the *classical principle of measurability*, or briefly the *classical measurability principle*, any *simple* (*one-component*) *classical* (*non-quantum*) characteristic of a discrete or continuous physical system is *physically measurable* in the sense that there exists an *imaginary experiment* that allows to assign a certain *real* number to that characteristic. This means that there exists an *imaginary physical instrument* that allows performing the above measurements. Some, but not all, imaginary experiments or imaginary instruments are abstractions of the corresponding real ones. Any measuring instrument, imaginary or real, has the following two fundamental imaginary (conceptual) or real properties, respectively:

(i) The instrument is a *macroscopic classical* physical system.

(ii) The instrument is *small enough* in order not to disturb the measured characteristic noticeably.

Typical examples of applications of the classical principle of measurability can be found both in the theory of a classical electromagnetic field in vacuum (special theory of relativity) and in the theory of a classical gravitational field (general theory of relativity). In these theories, abstract mathematical computations and proofs are often supported by the pertinent discourses of imaginary measurements of time and length intervals with the help of imaginary clocks and imaginary rulers (see, e.g., Landau an Lifshitz [1989, §§ 1-4, 97]). Some other imaginary measurement procedures as described in classical electrodynamics are based on the notion of an imaginary *test body*, although this notion is not, always, made explicit. Such a body either in the form of a small charged particle or in the form of a small conducting contour is used in classical electrodynamics for defining the electric or magnetic field intensity, respectively (cf. Landau an Lifshitz [1989, §17)]). In fluid mechanics, a test body in the form of an imaginary neutral classical material particle or in the form of an imaginary small manometer (cf. an imaginary ruler or an imaginary clock in either theory of relativity) can be used in the obvious way for conceptually measuring the momentary local fluid velocity or pressure, respectively.The classical principle of measurability is the foundation for various *physical interpretations* of an underlying mathematical theory by *real*-valued functions.

By specification, it immediately follows from the classical measurability principle that any simple physically measurable field characteristic of fluid, as the momentary local fluid pressure or as any given component of the momentary local fluid velocity, or acceleration, is necessarily *real*-valued (*not complex*-valued and, in general, *not abstract*-valued). It also follows from that principle that the fluid velocity (e.g.) must be a continuously differentiable real-valued function of Eulerian variables. Otherwise, the fluid particle that has a given location at a given instant of time would have no physically measurable acceleration, and therefore it could not be an object of the Newtonian (classical, non-quantum) mechanics.

## 2.3. The classical determinacy principle versus the quantum-mechanical indeterminacy principle

The *state* of a classical mechanical system at any given instant of time can be defined as the ordered set of the positions and momenta (or velocities). which the particles (as molecules or, in general, as any point material bodies) constituting the system have at that instant. In this case, according to the Newtonian equation of motion, the state of the system at any given instant of time is uniquely determined by the state of the system at any other given instant of time. The above statement is the *classical principle of determinacy* or briefly the *classical determinacy principle*. This principle is also equivalent to the statement that every classical particle moves along its *continuous trajectory*. As a consequence, a physical system that consists of *n* classical molecules (*n* can be as large as one pleases) moves along its *n-dimensional trajectory* in the 3*n-dimensional phase (Liouville) space* which is, by definition, the space of geometrical coordinates and momenta components of all particles (see, e.g., Landau and Lifshitz [1988, pp. 146, 147]). As contrasted to the classical (Newtonian) mechanics, as based either on the Galileo principle or on the Einstein principle of relativity, any quantum theory leads to the *indeterminacy*, or *uncertainty, principle*. According to this principle, a microscopic particle (as an electron or a photon) cannot, at the same time, be at a specified point and move with a specified velocity. As a consequence, a quantum (non-classical) particle *does not move along any trajectory*. Likewise, a many-body quantum system does not move along any multi-dimensional trajectory in any phase space.

The most essential difference between a classical and a quantum theory is that a state of a classical system is described by *real*-valued functions, while a state of a quantum system is described by *complex*-valued functions. As a result, a classical system is *deterministic*, whereas a quantum system is *not*. At the outlet of creation of the quantum mechanics, there were heated arguments among the creators of the theory, until the agreement about the presently common probabilistic interpretation of the results of measurements of the conventional classical attributes of a particle (as its coordinates and its momentum, or velocity, components) has been reached as the best. Incidentally, Einstein who received his Nobel Prize, not for his two theories of relativity as many people think, but for his *quantum* theory of the photo-electric effect, never accepted the probabilistic interpretation of measurements on quantum systems.

Thus, the passage from classical states of physical systems, which are described by differentiable real-valued functions, to quantum states, – or, in particular, to quantum-mechanical states, – which are described by still differentiable but complex-valued functions, has resulted in the passage from the deterministic interpretation of the results of a physical experiment to the indeterministic (probabilistic) interpretation. This has caused the entire revolution in the Weltenshauung of physicists. It is therefore clear that if one passes from the conventional states of classical mechanics, which are described by differentiable real-valued functions to some *abstract* states described by some non-differentiable and even non-continuous *abstract*-valued functions, then merely the fact of asserting that the weak theory so obtained is a part of physics and not of pure mathematics would have meant a radical revolution in the entire natural philosophy. Indeed, no classical physical system that has the above abstract states can move along any continuous *path* (*trajectory*) in any phase space. Moreover, in this case, *the very notion of phase space cannot be introduced at all*.

### 2.4. Postscript on mathematical physics (applied mathematics) and pure mathematics

The whole of the above discussion in this section can be summarized as follows. The term "*mathematical physics*" is actually a synonym of the term "*applied mathematics*". In this case, applied mathematics is not just non-rigorous (sloppy) mathematics, but rather it is a certain part of pure (rigorous) mathematics, which is provided with the appropriate physical interpretation. In this connection, the following general remarks about formal mathematical approach to physical problems can be made.

From the standpoint of logical analysis, it is desirable that an underlying mathematical theory should be as weak as possible, because the weaker is a theory, the less is the danger that the theory will, after all, turn out to be contradictory (paradoxical). However, from the standpoint of a physical analysis, any weak mathematical theory which treats the unknowns of classical physics equations as generalized (abstract-valued) functions belongs, by definition, to pure mathematics, and not to physics. As a consequence, such a theory does not belong to natural philosophy either. In this case, it does not matter whether the equations in their weak form are obtained from an integral principle (as the principle of least action or the principle of virtual work) or whether they are obtained straightforwardly by the corresponding generalization of some conventional differential equations of mathematical physics. On the other hand, if all corollaries that follow from a certain integral principle are properly interpreted on the base of the fundamental principles of classical physics then the formal theory so obtained must either be equivalent to or must exactly coincide with the pertinent naive theory, which is obtained from the corresponding intuitive considerations. For instance, both the Lagrangian mechanics and the Hamilton-Jacobi mechanics are equivalent to the original naive Newtonian mechanics (cf. Landau and Lifshitz [1988, chapters I, VII ff]).

In formulating or solving some problems of classical physics in general, and of continuum fluid mechanics in particular, one may of course use various abstract mathematical objects as complex numbers, matrices, generalized (abstract-valued) functions, etc. However, all simple (one-component) physical characteristics of any physical system must, after all, be interpreted as physically measurable. This particularly means that any simple classical field must be the field of values of a certain *real*-valued function, which has all necessary *real*-valued partial derivatives. In this case, all mathematical points, curves, and surfaces of discontinuity of the field should be interpreted physically as three-dimensional manifolds, whose thickness (minimal linear size) is much larger than the mean-free path of the molecules constituting the medium in question. Otherwise, the field would have been inconsistent with the basic principles of classical physics, which comprises the Newtonian mechanics and the Maxwell electrodynamics. In particular, in the absence of the above physical interpretation, a fluid particle has neither physically measurable velocity nor physically measurable acceleration, and it is not an object of the Newtonian mechanics. It is not, obviously, an object of quantum mechanics either. Hence, it is not an object of physics at all.

Thus, in every discipline of natural philosophy, physical principles must dominate mathematical ones. In particular, any weak mathematical theory, which is not provided with the physical interpretation on the base of the continuum hypothesis and on the base of the classical principles of measurability and determinacy, cannot be regarded as a conceptual model of any physical system. Such a theory is a branch of pure mathematics, rather than to be a branch of natural philosophy, until it is provided with the appropriate physical interpretation. Here follows two examples that illustrate this point.

i) Any given quantum-mechanical state (Shrödinger’s psi-function) is an abstract mathematical object, namely, a vector in a certain Hilbert space. This vector becomes an object of physics, only if it is provided with the appropriate quantum-mechanical, i.e. physical, interpretation. Accordingly, non-relativistic quantum mechanics is a part of physics, and hence, a part of natural philosophy, whereas a theory of Hilbert spaces is a part of pure mathematics.

ii) A theory of automorphisms (as a theory of automorphisms of Euclidean or pseudo-Euclidean vector spaces) is a branch of pure mathematics. However, a theory of automorphisms of the Minkowski space (the four-dimensional pseudo-Euclidean space of index 1), which is provided with the appropriate physical interpretation, is Einstein's special theory of relativity. The later is a part of physics.

In connection with the above, it should be recalled that besides solutions, which have all necessary continuous partial derivatives and which are said to be *strong* (or *strict* or *classical*), a partial differential equation of mathematical physics in 3×1 space-time continuum (e.g.) can have so-called *weak* (or *lax* or *generalized*) solutions (see, e.g. Garabedian [1965, pp. 284, 299, 445-447, 506] or Zauderer [1983, pp. 288-294]). Every strong solution is, by definition, a weak one, but not vice versa. In the general case, a weak solution is not required to be continuously differentiable, but rather it should be just integrable, and therefore it can be discontinuous at some surfaces. A shock wave with infinitesimally thin front is a solution of equations of gas dynamics of this kind (see. e.g., Landau and Lifshitz [1987, pp. 146, 147]). Still, from the standpoint of a physical analysis based on the molecular hypothesis, any singular mathematical surface (singular two-dimensional manifold) of a macroscopic field should be interpreted as a three-dimensional region whose thickness is mach larger than the mean-free path of the molecules constituting the medium. This is in accordance with the relevant general remarks as made at the beginning of the Introduction. At the same time, on both sides of the singular surface, the given macroscopic field is classical, i.e. it has all necessary continuous partial derivatives.

## 3. A general mathematical model of a perturbed liquid layer

### 3.1. The geometrical form of the model

Given a liquid layer with non-uniform bed in a vertical uniform field of gravity, suppose that the coordinate XY-plane ($z$=0) of a right-handed rectangular rectilinear laboratory coordinate system coincides with the *free* (*flexible, elastic, resilient*) *upper* boundary surface of the layer, whereas the positive Z-axis is opposite to the direction of gravity. Relative to that coordinate system, the acceleration due to gravity is characterized by the ordered triple $\underline{g} = \langle 0,0,-g \rangle$ subject to $g > 0$; the equation

$$s_{\mathrm{b}}(\underline{x}) \stackrel{=}{=} h(\underline{x}_2) + z = 0\,, \tag{3.1}$$

where ‘$h(\underline{x}_2)$’ is a *known functional form*, describes the *rigid* (*firm*, *inflexible*, *solid*) *bottom* boundary surface $S_{\mathrm{b}}$ of the layer, and the equation

$$s_{\mathrm{t}}(t,\underline{x}) \stackrel{=}{=} \mathrm{Z}(t,\underline{x}_2) - z = 0\,, \tag{3.2}$$

where ‘$\mathrm{Z}(t,\underline{x}_2)$’ is an *unknown functional form*, which describes its upper boundary surface $S_{\mathrm{t}}$ of the disturbed layer. It is assumed that $h$ and $\mathrm{Z}$ are *real*-valued functions, which, along with their all first-order partial derivatives, are defined, continuous, and bounded on $\underline{E}_2$ and on $R \times \underline{E}_2$, respectively. In this case, the equation $h(\underline{x}_2) \stackrel{=}{=} h(x,y) = 0$ implicitly defines a *closed contour* $\Gamma_0$ in the plane $z$=0 that serves the boundary perimeter of the unperturbed liquid layer.

**Hypothesis 3.1:** ***A hypothesis of an infinite liquid layer.*** 1) In the sequel, I shall assume that the liquid layer is *infinite* in all longitudinal directions and that $S_{\mathrm{b}}$ and $S_{\mathrm{t}}$ do not intersect. Accordingly, I shall assume that for each $t \in R$ and each $\underline{x}_2 \in \underline{E}_2$:

$$0 < h_{\mathrm{m}} \le h(\underline{x}_2) \le h_{\mathrm{M}} < +\infty\,, \tag{3.3}$$

$$-\infty < -h_{\mathrm{m}} < \mathrm{Z}_{\mathrm{m}} \le \zeta_{\mathrm{m}}(\underline{x}_2) \le \mathrm{Z}(t,\underline{x}_2) \le \zeta_{\mathrm{M}}(\underline{x}_2) \le \mathrm{Z}_{\mathrm{M}} \le h_{\mathrm{m}} < \infty\,, \tag{3.4}$$

$$\mathrm{Z}_{\mathrm{m}} \le z_{\mathrm{m}}(t) \le \mathrm{Z}(t,\underline{x}_2) \le z_{\mathrm{M}}(t) \le \mathrm{Z}_{\mathrm{M}}\,, \tag{3.5}$$

$$\zeta_{\mathrm{m}}(\underline{x}_2) \le 0 \le \zeta_{\mathrm{M}}(\underline{x}_2)\,,\; z_{\mathrm{m}}(t) \le 0 \le z_{\mathrm{M}}(t)\,, \tag{3.6}$$

where $h_{\mathrm{m}}$ and $h_{\mathrm{M}}$ are the infimum and supremum of all values of ‘$h(\underline{x}_2)$’, $\mathrm{Z}_{\mathrm{m}}$ and $\mathrm{Z}_{\mathrm{M}}$ are those of all values of ‘$\mathrm{Z}(t,\underline{x}_2)$’, $\zeta_{\mathrm{m}}(\underline{x}_2)$ and $\zeta_{\mathrm{M}}(\underline{x}_2)$ are the infimum and supremum of the values of ‘$\mathrm{Z}(t,\underline{x}_2)$’ at $\underline{x}_2$ held constant, $z_{\mathrm{m}}(t)$ and $z_{\mathrm{M}}(t)$ are those at $t$ held constant. The inequality $h_{\mathrm{M}} < +\infty$ expresses the assumption that $h$ is bounded. This inequality, along with $h_{\mathrm{m}} < +\infty$, which follows from it, has been included in (3.4) for more clarity. After a given problem is solved, one may, when desired, pass either to the limit $h_{\mathrm{M}} \to +\infty$ or to the limit $h_{\mathrm{m}} \to +\infty$ (which implies that $h_{\mathrm{M}} \to +\infty$ as well) in all relevant final formulae.•

**Comment 3.1.** In accordance with the choice of the coordinate plane $z = 0$, the equation

$$\mathrm{Z}(t,\underline{x}_2) = z_{\mathrm{m}}(t) = z_{\mathrm{M}}(t) = \mathrm{Z}_{\mathrm{m}} = \mathrm{Z}_{\mathrm{M}} = 0$$

holds for each $t \in R$ and each $\underline{x}_2 \in E_2$ only in the absence of perturbations. At the same time, the equation

$$\mathrm{Z}(t, \underline{x}_2) = \zeta_{\mathrm{m}}(\underline{x}_2) = \zeta_{\mathrm{M}}(\underline{x}_2) = 0 \tag{3.7}$$

may hold for each for each $t \in R$ and some $\underline{x}_2 \in \underline{E}_2$ not only in the absence of perturbations. If the function $Z$ describes a *standing wave* then equation (3.7) determines the nodes of the wave.

2) At the same time, in accordance with Hypothesis 3.1, it is hereafter assumed that

$$\mathrm{Z}(t, \underline{x}_2) \neq 0 \text{ for some } \langle t, \underline{x}_2 \rangle \in R \times \underline{E}_2, \tag{3.8}$$

i.e. $\mathrm{Z} \neq C_0$, where $C_0$ is a *constant function*, every value of each equals 0. Hence,

$$\zeta_{\mathrm{m}}(\underline{x}_2) < 0 < \zeta_{\mathrm{M}}(\underline{x}_2) \text{ for some } \underline{x}_2 \in \underline{E}_2. \tag{3.9}$$

A liquid layer, for which (3.8), or (3.9), holds is called "*perturbed*".•

**Definition 3.1.** The subscript 'f', or 'n', in a functional constant (or variable) signifies that that constant (or that variable) stands for a *function of a function*, or for a *function of a real number*, respectively. In particular, with '$D$' being an ellipsis, $D_{\mathrm{f}}$ is a function of a function, whereas $D_{\mathrm{n}}$ is a function of a real number. Similarly, $D_{\mathrm{ff}}$ is a function of two arguments, both of which are functions; $D_{\mathrm{fn}}$ is a function of two arguments, of which the first is a function, and the second is a real number; $D_{\mathrm{nf}}$ is a function of two arguments, of which the first is a real number, and the second is a function; $D_{\mathrm{nn}}$ is a function of two real numbers.•

**Definition 3.2.**

$$D_{\mathrm{ff}}^{\mathrm{oo}}(-h, \mathrm{Z}(t,\,)) \doteqdot \{\langle \underline{x}_2, z\rangle | \underline{x}_2 \in \underline{E}_2 \text{ and } z \in (-h(\underline{x}_2), Z(t, \underline{x}_2))\} \text{ for each } t \in R, \tag{3.10}$$

$$\mathbf{D}_{\mathrm{ff}}^{\mathrm{oo}}(-h, \mathrm{Z}) \doteqdot \bigcup_{tt' \in R} [\{t'\} \times D_{\mathrm{ff}}^{\mathrm{oo}}(-h, \mathrm{Z}(t',\,))], \tag{3.11}$$

and similarly with each one of the following three pairs of strings: ⟨'oc', '$z \in (\;,\;]$'⟩, ⟨'co', '$z \in [\;,\;)$'⟩, and ⟨'cc', '$z \in [\;,\;]$' ⟩ in place of ⟨'oo', '$z \in (\;,\;)$'⟩ respectively. A double-letter superscript 'oo', 'oc', 'co', or 'cc' to '$D$' is descriptive of the fact that the entire variable bearing that superscript stands, respectively, for the *open-open* (*open*), *open-closed*, *closed-open*, or *closed-closed* (*closed*) liquid layer, which is enclosed between the surfaces described by the equations $z = -h(\underline{x}_2)$ and $z = Z(t, \underline{x}_2)$. Thus, particularly, $D_{\mathrm{ff}}^{\mathrm{oo}}(-h, C_9)$ or $D_{\mathrm{fn}}^{\mathrm{oo}}(-h, 0)$ is the liquid layer in equilibrium, the understanding being that $C_9$ is, as before, the constant function, every value of each equals 0. The set $\mathbf{D}_{\mathrm{ff}}^{\mathrm{oo}}(-h, Z)$ as defined by (3.11) is said to be *folded with respect to all values of the variable* '$t$' or, briefly, *time-folded*, and similarly with 'oc', 'co', or 'cc' in place of 'oo'. Either of the above two synonymous terms, and also the sets, which they denote, are suggestions of my own, – they are not in common usage. •

## 3.2. Eulerian variables of a fluid flow versus Lagrangian ones

### 3.2.1. Eulerian variables

No matter how a mass of fluid is geometrically configured, under the continuum hypothesis, two different ways of specification (description) of a fluid flow are possible. These are known as the *Eulerian specification* and the *Lagrangian specification*, although both are in reality due to Euler (see, e.g., Lamb [1932, pp. 2–15], Batchelor [1967, pp. 71–73], Landau. and Lifshitz [1987, pp. 1–5]).

In the framework of the Eulerian specification, all characteristics of a fluid flow are described by functional forms (extensional functional variables), which depend on two independent variables, namely, a time variable, as '$t$', and a spatial vector variable, as '$\underline{x}$' subject to $\underline{x} \doteqdot \langle x_1, x_2, x_3 \rangle = \langle x, y, z \rangle$. Thus for instance, $\underline{V}(t, \underline{x})$, defined as

$$\underline{V}(t, \underline{x}) \doteqdot \langle V_1(t, \underline{x}), V_2(t, \underline{x}), V_3(t, \underline{x}) \rangle,$$

is the fluid velocity, while $\rho(t, \underline{x})$ and $P(t, \underline{x})$ are the fluid mass density and fluid pressure, respectively – all at a temporo-spatial point $\langle t, \underline{x} \rangle$, which is given relative to a certain clock and also relative to a certain rectangular rectilinear right-handed coordinate system. The real-valued variables '$x$', '$y$', '$z$' and the corresponding vector-valued variable '$\underline{x}$' are said to be *Eulerian independent variables*, whereas a functional form depending of '$t$' and '$\underline{x}$' is said to be an *Eulerian functional form*.

**Definition 3.3.** 1) Unless stated otherwise. '$F(t, \underline{x})$' is hereafter a placeholder for any Eulerian functional form (as '$\underline{V}(t, \underline{x})$', '$V_i(t, \underline{x})$', '$\rho(t, \underline{x})$', or '$P(t, \underline{x})$').

2) In this case, both the definiendum and the definiens of the definition:

$$\frac{DF(t, \underline{x})}{Dt} \doteqdot \frac{\partial F(t, \underline{x})}{\partial t} + \left( \sum_{i=1}^{3} V_i(t, \underline{x}) \nabla_i \right) F(t, \underline{x}) \tag{3.12}$$

subject to

$$\nabla_i \doteqdot \frac{\partial}{\partial x_i} \text{ for each } i \in (1,2,3), \tag{3.13}$$

is called the *convectional*, or *path, derivative of the function F at the temporo-spatial point* $\langle t, \underline{x} \rangle$. Accordingly, the operator '$\frac{D}{Dt}$', defined as:

$$\frac{D}{Dt} \doteqdot \frac{\partial}{\partial t} + \underline{V}(t,\underline{x}) \cdot \underline{\nabla} \doteqdot \frac{\partial}{\partial t} + \sum_{k=1}^{3} V_k(t,\underline{x}) \nabla_k\,, \tag{3.14}$$

is called the *convectional*, or *path, differential operator at the above point.*•

**Definition 3.4.** 1) *Given t∈R,* if a functional form **'**$F(t,\underline{x})$**'** is defined for each $\underline{x} \in D_{\mathrm{ff}}^{\mathrm{oo}}(-h, \mathrm{Z}(t,\ ))$ then the set $D_{\mathrm{ff}}^{\mathrm{oo}}(-h, \mathrm{Z}(t,\ ))$ is said to be the *spatial domain of definition of that form* and also the *spatial domain of definition of the momentary associated function* $F(t,\ )$ of the form **'**$F(t,\underline{x})$**'**. If **'**$F(t,\underline{x})$**'** is defined *for each t∈R and each* $\underline{x} \in D_{\mathrm{ff}}^{\mathrm{oo}}(-h, \mathrm{Z}(t,\ ))$, i.e. *for each* $(t,\underline{x}) \in \mathbf{D}_{\mathrm{ff}}^{\mathrm{oo}}(-h, \mathrm{Z}\ )$, then the set $\mathbf{D}_{\mathrm{ff}}^{\mathrm{oo}}(-h, \mathrm{Z}\ )$ is said to be the *total* (*temporo-spatial*) *domain of definition of that form* and also that of the *total* (*complete*) *associated function F of the functional form* **'**$F(t,\underline{x})$**'**, whereas $\mathbf{D}_{\mathrm{ff}}^{\mathrm{oo}}(-h, \mathrm{Z}\ )$ is the domain of definition of the function $F$. The domains $D_{\mathrm{ff}}^{\mathrm{oo}}(-h, \mathrm{Z}(t,\ ))$ and $\mathbf{D}_{\mathrm{ff}}^{\mathrm{oo}}(-h, \mathrm{Z}\ )$ can be extended to $D_{\mathrm{ff}}^{\mathrm{cc}}(-h, \mathrm{Z}(t,\ ))$ and to $\mathbf{D}_{\mathrm{ff}}^{\mathrm{cc}}(-h, \mathrm{Z}\ )$ respectively with the help of the definitions:

$$[F(t,\underline{x})]_{z=-h(\underline{x}_2)} \doteqdot \lim_{\varepsilon \to +0} [F(t,\underline{x})]_{z=-h(\underline{x}_2)+\varepsilon},\ [F(t,\underline{x})]_{z=\mathrm{Z}(t,\underline{x}_2)} \doteqdot \lim_{\varepsilon \to +0} [F(t,\underline{x})]_{z=\mathrm{Z}(t,\underline{x}_2)-\varepsilon}, \tag{3.15}$$

provided of course that the limits are supposed to exist. In accordance with (3.15), it will hereafter be assumed that $F(t,\ )$ is defined on $D_{\mathrm{ff}}^{\mathrm{cc}}(-h, \mathrm{Z}(t,\ ))$ and that hence $F$ is defined on $\mathbf{D}_{\mathrm{ff}}^{\mathrm{cc}}(-h, \mathrm{Z}\ )$.

2) As contrasted to the *local bulk* characteristics of the fluid flow, which are denoted by '$F(t,\ \underline{x})$', the characteristics as defined by (3.15), and also $h(\underline{x}_2)$ and $\mathrm{Z}(t,\underline{x}_2)$ themselves, will be called "*local surface* characteristics of the fluid flow". Accordingly, local bulk and local surface characteristics of the fluid flow will collectively be called "*local* characteristics".•

**Comment 3.2.** The fact that $D_{\mathrm{ff}}^{\mathrm{oo}}(-h, \mathrm{Z}(t,\ ))$ or $D_{\mathrm{ff}}^{\mathrm{cc}}(-h, \mathrm{Z}(t,\ ))$ depends on $t$ is the very reason why all functional forms (extensional functional variables) occurring in this exposition are, in contrast to the common practice, formed in such a way that the time variable '$t$' is always mentioned (listed) before the spatial vector variable as '$\underline{x}$' or '$\underline{x}_2$', – just as in theory of relativity.•

**Definition 3.5.** Given a real or complex number $a$, $C_a$ is the *constant function*, with any given domain of definition, whose every value equals $a$. Particularly, $C_0$ is the *null-valued function*, i.e. the constant function whose every value equals *null*. For avoidance of notation conflicts, the null-valued function $C_0$ defined on a concrete domain may in the sequel be denoted differently.•

**Convention 3.1.** 1) In accordance with the common practice, in making statements about local characteristics of a fluid flow and particularly in stating equations for a fluid flow, I shall, for the sake of brevity, omit the strings '$(t,\underline{x}_2)$', '$(\underline{x}_2)$', '$(t,\underline{x})$', and '$(\underline{x})$' from some (strictly some or all) pertinent functional forms every time when this seems to be safe so as not to lead to the confusion between a function and its value at given arguments, and also be safe so as not to mislead with regard to which independent variables are actually involved in the abbreviated functional form. Accordingly, it is assumed that every abbreviated bulk relation is preceded either by the appropriate quantifiers such as 'for each $(t,\underline{x}_2)\in R\times\underline{E}_2$ and each $z\in[-h(\underline{x}_2),Z(t,\underline{x}_2)]$:' in that order or equivalentlyby these: 'for each $t\in R$ and each $\underline{x}\in D_{\mathrm{ff}}^{\mathrm{cc}}(-h,\mathrm{Z}(t,\,)$:' in that order, or else by this single quantifier: 'for each $(t,\underline{x})\in\mathbf{D}_{\mathrm{ff}}^{\mathrm{cc}}(-h,\mathrm{Z}\,)$:', – unless of course the quantifiers are written down or unless it is stated otherwise.

2) In order to indicate explicitly that an intensional functional variable as '$F$' is used equivocally both as a name of the associated function $F$ of a functional form as $F(t,\underline{x})$ and as an abbreviation of that form, I shall often make a definition either of the form '$F \doteqdot F(t,\underline{x})$' or of the form '$F(t,\underline{x}) \doteqdot F$'. If '$F$' is contextually regarded as an abbreviation of $F(t,\underline{x})$ then the equation '$F(t,\underline{x})=0$' will briefly be written as '$F(t,\underline{x})=0$'. If, however, '$F$' is contextually regarded as a name of the associated function $F$ of '$F(t,\underline{x})$' then the equation '$F(t,\underline{x})=0$' will briefly be written as '$F=C_0$'.

3) Unless stated otherwise, the subscript variable '$i$' which occurs in a statement as an apparent free variable is assumed to be bound by the quantifier 'for each $i\in\omega_{1,3}$:'; and similarly. with any other equivalent variable from '$j$' to '$n$' in place of '$i$'.•

### 3.2.2. Lagrangian variables

N In the framework of the Lagrangian specification, the ordered triple $\underline{x}$ of coordinates of a given fluid particle at any given instant of time $t$ is determined by the ordered triple

$\underline{a} \stackrel{=}{\to} \langle a_1, a_2, a_3 \rangle = \langle a, b, c \rangle$ of coordinates of that particle at some initial instant $t_0$, which are called *Lagrangian variables*. Thus, values of '$\underline{x}$' are actually values of a functional form '$\underline{\dot{x}}(t,\underline{a})$', which depends on the independent variables '$t$' and '$\underline{a}$', so that $\underline{\dot{x}}(t_0,\underline{a}) = \underline{a}$. Accordingly, the fluid velocity $\underline{\dot{v}}(t,\underline{a})$, mass density $\dot{\rho}(t,\underline{a})$, and pressure $\dot{P}(t,\underline{a})$ of the fluid particle $\underline{a}$ at the instant $t$ are values of the corresponding functional forms depending on '$\underline{\dot{x}}(t,\underline{a})$':

$$\underline{\dot{v}}(t,\underline{a}) \stackrel{=}{\to} \frac{\partial \underline{\dot{x}}(t,\underline{a})}{\partial t}, \ \dot{\rho}(t,\underline{a}) \stackrel{=}{\to} \tilde{\rho}\left(\underline{\dot{x}}(t,\underline{a})\right), \ \dot{P}(t,\underline{a}) \stackrel{=}{\to} \tilde{P}(\underline{x}(t,\underline{a})).$$

In this case, the condition of conservation of mass during the motion of a fluid element (the equation of continuity) is accordingly written as:

$$\tilde{\rho}\left(\underline{\dot{x}}(t,\underline{a})\right) \frac{\partial\left(\dot{x}(t,\underline{a}), \dot{y}(t,\underline{a}), \dot{z}(t,\underline{a})\right)}{\partial(a,b,c)} = \tilde{\rho}\left(\underline{a}\right),$$

where $\tilde{\rho}\left(\underline{a}\right)$ is the initial mass density at the temporo-spatial point $\langle t_0, \underline{a} \rangle$: $\tilde{\rho}\left(\underline{\dot{x}}(t_0,\underline{a})\right) = \tilde{\rho}\left(\underline{a}\right)$. If the fluid is *incompressible* fluid then $\tilde{\rho}\left(\underline{\dot{x}}(t_0,\underline{a})\right) = \tilde{\rho}\left(\underline{a}\right) = \rho_0$, where '$\rho_0$' is a constant. Consequently,

$$\frac{\partial\left(\dot{x}(t,\underline{a}), \dot{y}(t,\underline{a}), \dot{z}(t,\underline{a})\right)}{\partial(a,b,c)} = 1.$$

A functional form depending of '$t$' and '$\underline{a}$' is said to be a *Lagrangian functional form*. A Lagrangian functional form describes the whole dynamical history of the associated physical characteristic of each fluid particle and is therefore more fundamental than the corresponding Eulerian functional form. At the same time, the Lagrangian specification of a fluid flow leads, as a rule, to a very cumbersome analysis, largely because it does not allow determining directly any *spatial partial derivatives* of physical characteristics of the fluid flow as its velocity, mass density, or pressure (cf. Batchelor [1967, p. 71]). Therefore the Eulerian specification of fluid flows is taken for granted practically in all studies on fluid mechanics and particularly in this exposition.

Still, in all boundary value problems of fluid mechanics, in which boundary conditions are given at varying surfaces, especially in the cases where the surfaces are not known, the Eulerian formalism leads to some grave paradoxes (contradictions), which make questionable, not only the validity of solutions of some problems of this kind, but also the validity of the formulations of the problems as such. In this connection, it is to be remarked that all problems of fluid mechanics in general, and boundary value problems in particular, are nonlinear ones, which cannot be solved analytically. Therefore, various approximations (as the linear one), and also various analytical methods (as an asymptotic one or as the method of averaging an equation and its unknowns with respect to the time argument or with respect to some spatial arguments), are often used to allow solving the problem. The paradoxes, which arise in the result of the approximations, made or in the result of the analytical methods used, sometimes remain unnoticed by the writers or they are tacitly ignored by the writers as some freak properties of the approximations or methods, which should disappear once the problem is rigorously solved by somebody else. Still, as a rule, rigorous solutions of the problems are not and will not ever be available. Therefore, once a paradox is detected in an axiomatic theory, such as the water wave dynamics based on the Eulerian specification of fluid flows, it is important to show that this paradox can, not only be eliminated, but be eliminated in such a way that its elimination creates no other paradoxes. It is clear that any given paradox can be eliminated without eliminating the analytical methods, which engender it, only by explicitly formulating and granting certain additional implicit hypotheses (assumptions) underlying the methods. In this case, the additional hypotheses can result in some other (secondary) paradoxes (contradictions). If this happens then the entire strong theory, which includes both its basic principles (axioms) and the additional hypotheses, is contradictory. In order to get a consistent theory, some or all additional hypotheses, along with the corresponding analytical methods, should be rejected. In this connection, it should be recalled that according to the most general rule of constructing axiomatic theories, if an axiomatic theory turns out to be contradictory then the number of its axioms must be decreased.

# 4. Basic equations for a fluid flow

## 4.1. The continuity equation of the fluid mass density

The differential equation of conservation of mass during a fluid flow is called the *mass continuity equation* or simply the *continuity equation* whenever there is no danger of misunderstanding. In accordance, e.g., with Landau and Lifshitz [1987, pp. 1–2], the continuity equation for the fluid mass density flow relative to a given rectangular rectilinear laboratory coordinate system, can, in the given version of Eulerian specification, be written as:

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \left( \rho \underline{V} \right) = 0\,, \tag{4.1}$$

where $\rho$ is the mass density and $\underline{V} \doteqdot \langle V_1, V_2, V_3 \rangle$ is the fluid velocity, at a temporo-spatial point $\langle t, \underline{x} \rangle \doteqdot \langle t, x_1, x_2, x_3 \rangle = \langle t, x, y, z \rangle$. The vector

$$\underline{J} \doteqdot \rho \underline{V} \tag{4.2}$$

is called the *mass flux density*,

**Hypothesis 4.1.** It is hereafter assumed that the liquid is *incompressible* in the sense that its mass density $\rho(t,\underline{x})$ is constant throughout the layer; i.e. $\rho(t,\underline{x}) = \rho_0$, while '$\rho_0$,' is a constant. The continuity equation (4.1) of the liquid flow in the layer has the form

$$\underline{\text{div}}\,\underline{V} \doteqdot \underline{\nabla} \cdot \underline{V} \doteqdot \sum_{j=1}^{3} \nabla_j V_j = 0\,, \tag{4.3}$$

because

$$\frac{D\rho}{Dt} \doteqdot \frac{\partial \rho}{\partial t} + \underline{V} \cdot \underline{\nabla}\rho = 0\,. \tag{4.4}\bullet$$

## 4.2. The continuity equation of the fluid momentum flux density

In accordance, e.g., with Landau and Lifshitz [1987, pp. 44–45], the continuity equation for the fluid momentum flux density can be written as:

$$\frac{\partial(\rho V_i)}{\partial t} + \sum_{j=1}^{3} \frac{\partial \Pi_{ij}}{\partial x_j} = 0 \text{ for each } i \in \omega_{1,3}\,, \tag{4.5}$$

where $\Pi_{ij}$ is the *momentum flux density tensor*. The latter is defined thus:

$$\Pi_{ij} \doteqdot P\delta_{ij} + \rho V_i V_j - \tau_{ij} \doteqdot -\sigma_{ij} + \rho V_i V_j\,, \tag{4.6}$$

subject to:

$$\sigma_{ij} \doteqdot -P\delta_{ij} + \tau_{ij}\,, \tag{4.7}$$

$$\begin{aligned} \tau_{ij} &\doteqdot \eta\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_l} - \frac{2}{3}\delta_{ij}\sum_{k=1}^{3}\frac{\partial V_k}{\partial x_k}\right) + \zeta\delta_{ij}\sum_{k=1}^{3}\frac{\partial V_k}{\partial x_k} \\ &= \eta\left(\nabla_j V_i + \nabla_i V_j\right) + \left(\zeta - \frac{2}{3}\eta\right)\delta_{ij}\sum_{k=1}^{3}\nabla_k V_k. \end{aligned} \tag{4.8}$$

with coefficients $\eta$ and $\zeta$ independent of the velocity. The tensor $\sigma_{ij}$ is called the *total stress tensor*, $-P\delta_{ij}$ is called the *inviscid stress tensor*, and $\tau_{ij}$ is called the *viscous stress tensor*. The above form of the invicid stress tensor, in which the *scalar* coefficient *P* is *pressure*, is predetermined by the fact that that liquid as such (in the absence of the field of gravity, e.g.) is *isotropic*. The coefficients $\eta$ and $\zeta$ are *strictly positive* ($\eta$>0 and $\zeta$>0) and they are called the *coefficients of viscosity*; the *first* one and the *second* one respectively. In the general case, $\eta$ and $\zeta$ depend on pressure *P* and temperature *T*, the latter and hence the former not being *constant* throughout the fluid. Therefore, upon substitution '$\Pi_{ij}$', subject to (4.6)–(4.8), into (4.5), '$\eta$' and '$\zeta$' cannot be taken outside the differential operator '$\frac{\partial}{\partial x_j}$'.

In most cases of liquid flows, however, '$\eta$' and '$\zeta$' can be regarded as constant, so that equation (4.5) subject to (4.1) and (4.6)–(4.8) becomes:

$$\rho\left[\frac{\partial V_i}{\partial t} + \left(\sum_{j=1}^{3} V_j \nabla_j\right) V_i\right] = -\nabla_i P + \eta \Delta V_i + (\zeta + \frac{1}{3}\eta)\nabla_i \sum_{k=1}^{3} \nabla_k V_k , \tag{4.9}$$

where '$\Delta$' is the Laplacian operator, defined as:

$$\Delta \doteq \underline{\nabla} \cdot \underline{\nabla} = \sum_{k=1}^{3} \nabla_k{}^2 = \sum_{k=1}^{3} \frac{\partial^2}{\partial x_k{}^2} . \tag{4.10}$$

In developing (4.9) from (4.5), the expression on the left-hand side of equation (4.5) can be developed with the help of (4.1) thus:

$$\begin{aligned} \frac{\partial(\rho V_i)}{\partial t} &= \rho \frac{\partial V_i}{\partial t} + V_i \frac{\partial \rho}{\partial t} = \rho \frac{\partial V_i}{\partial t} - V_i \sum_{j=1}^{3} \nabla_j (\rho V_j) \\ &= \rho \frac{\partial V_i}{\partial t} - V_i \sum_{j=1}^{3} V_j \nabla_j \rho - \rho V_i \sum_{j=1}^{3} \nabla_j V_j . \end{aligned} \tag{4.9$_1$}$$

At the same time, the item '$\rho V_i V_j$' of (4.6) contributes into the expression on the right-hand side of equation (4.5) the following expression:

$$-\sum_{j=1}^{3} \frac{\partial \rho V_i V_j}{\partial x_j} = -\rho\left(\sum_{j=1}^{3} V_j \nabla_j\right) V_i - V_i \sum_{j=1}^{3} V_j \nabla_j \rho - \rho V_i \sum_{j=1}^{3} \nabla_j V_j . \tag{4.9$_2$}$$

The first two terms on the final expression of (4.9$_1$) are congruent tokens of the last two terms in (4.9$_2$), so that all these are cancelled in the final expression for (4.5).

The equation (4.9) is a conventional equation of motion of *compressible viscous fluid*, which called the *Navier-Stokes equation.* The *scalar* mass continuity equation (4.1) and the *3-vector* momentum flux density continuity equation (4.9) form the set of *four homogeneous partial differential equations* for *four unknown functional variables* $V_1$, $V_2$, $V_3$, and *P*, the understanding being that these equations should be supplemented by the appropriated boundary conditions at the upper (free) and bottom (rigid) *boundary surfaces*.

Equation (4.9) becomes considerably simpler if the fluid may be regarded as *incompressible*, so that equation (4.3) holds, and the last term on the right-hand side of (4.9) vanishes. In discussing viscous fluids, the latter are almost always regarded as incompressible, so that equation (4.9) takes the form:

$$\frac{\partial V_i}{\partial t} + \left( \sum_{j=1}^{3} V_j \nabla_j \right) V_i = -\frac{1}{\rho} \nabla_i P + \nu \Delta V_i \,, \tag{4.11}$$

where the coefficient '$\nu$', defined as:

$$\nu \doteqdot \eta / \rho \,. \tag{4.12}$$

is called the *kinematic viscosity*, while the coefficient '$\eta$' itself is called the *dynamic viscosity*. Consequently, the stress tensor in an incompressible fluid becomes:

$$\sigma_{ij} \doteqdot -P\delta_{ij} + \eta \left( \frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_l} \right). \tag{4.13}$$

If processes of dissipation of energy are unimportant in motion of a fluid then one may set

$$\eta = \zeta = 0 \tag{4.14}$$

and call this fluid *inviscid* or *ideal.* In this case, equation (4.9) turns into:

$$\frac{\partial V_i}{\partial t} + \left( \sum_{j=1}^{3} V_j \nabla_j \right) V_i = -\frac{1}{\rho} \nabla_i P \tag{4.15}$$

which is called the *Euler equation.*

In the motion of an ideal fluid, there is no heat exchange among different parts of the fluid and also between the fluid and the bodies adjoining it. This means that that motion must be *adiabatic* or *isentropic,* Denoting the entropy per unit fluid mass at a temporo-spatial point $(t, \underline{x})$ by '$s(t, \underline{x})$', the condition for its adiabatic motion:

$$s(t, \underline{x}) = s_0 \,, \tag{4.16}$$

where '$s_0$' is a constant, implies that

$$\frac{Ds}{Dt} \doteqdot \frac{\partial s}{\partial t} + \underline{V} \cdot \underline{\nabla} s = 0 \tag{4.17}$$

(cf. (4.4)). Owing to (4.16), the familiar thermodynamic equation:

$$dw = Tds + vdP \,, \tag{4.18}$$

where w is the *enthalpy* (*heat function*) per unit mass of fluid, $v = 1/\rho$ is the specific volume, and $T$ is the temperature, turns into

$$dw = vdP = dP/\rho\,. \tag{4.19}$$

Hence, (4.15) can be rewritten as

$$\frac{\partial V_i}{\partial t} + \left(\sum_{j=1}^{3} V_j \nabla_j\right) V_i = -\nabla_i w\,. \tag{4.20}$$

If the *ideal* fluid in question is *incompressible*, so that for every pertinent temporo-spatial point $(t, \underline{x})$, $\rho \stackrel{\rightarrow}{\equiv} \rho(t, \underline{x}) = \rho_0$, where '$\rho_0$,' is a constant, then either equation (4.15) or (4.20) can be rewritten as:

$$\frac{\partial V_i}{\partial t} + \left(\sum_{j=1}^{3} V_j \nabla_j\right) V_i = -\nabla_i \left(\frac{P}{\rho_0}\right). \tag{4.21}$$

In this case, either equation (4.15) or (4.21) implies that a necessary condition for the fluid in question to be at rest (in mechanical equilibrium) that corresponds to $\underline{V}(t, \underline{x}) = \text{constant}$ is that $P(t, \underline{x}) = P_0(t)$, where $P_0$ is a function of $t$ only. Nether equation (4.15) nor (4.21) has a solution that describes the *hydrostatic equilibrium of* the liquid layer in a homogeneous field of gravity.

If a fluid is in a homogeneous gravitational field then an additional mass force $\rho\underline{g}$, where $\underline{g} = \langle 0,0,-g\rangle$ subject to $g > 0$ is the acceleration due to gravity, acts on any unit volume. Therefore, the *i*th component of this force, $\rho g_i$ subject to

$$g_i = -g\delta_{i3} \text{ for each } i \in \omega_{1,3} \ (g > 0), \tag{4.22}$$

must be added to the *i*th component of pressure force, $-\nabla_i P$, on the right-hand side of equation (4.21). Hence, in the presence of the above homogeneous field of gravity, equation (4.21) is replaced with

$$\frac{\partial V_i}{\partial t} + \left(\sum_{j=1}^{3} V_j \nabla_j\right) V_i = -\frac{1}{\rho} \nabla_i P + g_i \tag{4.23}$$

subject to (4.22). If the fluid is *incompressible* one of a constant mass density $\rho_0$ then equation (4.23) can be written as:

$$\frac{\partial V_i}{\partial t} + \left( \sum_{j=1}^{3} V_j \nabla_j \right) V_i = -\nabla_i \left( \frac{P}{\rho_0} \right) + g_i \,, \tag{4.24}$$

which comes instead of (4.21).

**Comment 4.1.** If $\underline{V} = (V_1, V_2, V_3) = (0,0,0)$, which means that the liquid layer is *in equilibrium* (*at rest*), and if $P$ is denoted in this case by ‘$P_{\mathrm{e}}$’ then equation (4.24), subject to (4.22), reduces to the following set of three partial differential equations

$$\frac{\partial P_{\mathrm{e}}}{\partial x} = 0 \quad \text{(a)}, \ \frac{\partial P_{\mathrm{e}}}{\partial y} = 0 \quad \text{(b)}, \ \frac{\partial P_{\mathrm{e}}}{\partial z} = -\rho_0 g \quad \text{(c)}. \tag{4.25}$$

In accordance with equations (4.25,a) and (4.25,b), the function ‘$P_{\mathrm{e}}$’ is independent of ‘*x*’ and ‘*y*’, so that it can depend only on ‘*t*’ and ‘*z*’, i.e. $P_{\mathrm{e}} \doteqdot P_{\mathrm{e}}(t, z)$, while integration of both sides of equation (4.25,c) between 0 and a given real number *z* yields

$$P_{\mathrm{e}}(t, z) - P_{\mathrm{e}}(t, 0) = -\rho_0 g \int_0^z dz' = -\rho_0 g z \,, \tag{4.26}$$

which is valid for

$$\text{each } t \in R \text{, each } (x, y) \in \underline{E}_2 \text{, and each } \ z \in [-h(x, y), 0], \tag{4.27}$$

i.e for

$$\text{each } (t, \underline{x}) \in \mathbf{D}_{\mathrm{ff}}^{\mathrm{cc}}(-h, \mathrm{Z}) \,. \tag{4.27'}$$

In this case, the boundary functional form ‘$P_{\mathrm{e}}(t,0)$’ and hence equation (4.26) can be specified in the following two ways.

i) If the part of space above the upper boundary surface z=0 of the liquid layer is *vacuous* then $P_{\mathrm{e}}(t,0) = 0$ and hence equation (4.26) becomes

$$P_{\mathrm{e}}(t, z) = P_{\mathrm{hs}}(z) \doteqdot -\rho_0 g z \,, \tag{4.28}$$

so that $P_{\mathrm{hs}}(z)$ thus defined is the *net hydrostatic pressure at the depth* –*z*>0.

ii) If the above part of space is occupied with air then $P_{\mathrm{e}}(t,0) = P_{\mathrm{a}}(t)$, where $P_{\mathrm{a}}(t)$ is a given *atmospheric pressure at* z=0, and hence equation (4.26) becomes

$$P_{\mathrm{e}}(t, z) = P_{\mathrm{te}}(t, z) \doteqdot P_{\mathrm{e}}(t) + P_{\mathrm{hs}}(z) = P_{\mathrm{a}}(t) - \rho_0 g z \,, \tag{4.29}$$

so that $P_{\mathrm{te}}(t, z)$ thus defined is the *total equilibrium pressure at a depth* –*z*>0.•

## 4.3. A potential, or irrotational, fluid flow

The cross product of any two arithmetical vectors $\underline{a}$ and $\underline{b}$ in $E_3$ can be written as:

$$(\underline{a} \wedge \underline{b})_i \doteqdot \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} a_j b_k \text{ for each } i \in \omega_{1,3}, \tag{4.30}$$

where '$\varepsilon_{ijk}$' is the *completely antisymmetrical unit pseudo-tensor of Levi-Civita*. Equation (4.30) with '$\underline{\nabla}$' as '$\underline{a}$' and '$\underline{V}$' as '$\underline{b}$' becomes

$$(\underline{\text{curl}}\,\underline{V})_i \doteqdot (\underline{\nabla} \wedge \underline{V})_i \doteqdot \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \nabla_j V_k \text{ for each } i \in \omega_{1,3}, \tag{4.31}$$

A fluid flow is said to be a *potential*, or *irrotational*, one if $\underline{\text{curl}}\,\underline{V} = \underline{0}$ throughout the flow and a *rotational* one if $\underline{\text{curl}}\,\underline{V} \neq \underline{0}$ in some part of the flow; $\underline{0}$ defined as $\underline{0} \doteqdot (0,0,0)$ is the arithmetical null 3-vector. The velocity $\underline{V}$ in potential flow can be expressed as the gradient of some scalar, which is called the *velocity potential* and which will be denoted by 'Φ', so that

$$\underline{V} \doteqdot \underline{\text{grad}}\,\Phi, \text{ i.e. } V_i = \nabla_i \Phi \text{ for each } i \in \omega_{1,3}. \tag{4.32}$$

In this case, (4.31) becomes

$$(\underline{\text{curl}}\,\underline{\nabla}\Phi)_i = (\underline{\nabla} \wedge \underline{\nabla}\Phi)_i \doteqdot \left(\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \nabla_j \nabla_k\right)\Phi = 0 \text{ for each } i \in \omega_{1,3}. \tag{4.33}$$

By (4.32), equation (4.3) becomes

$$\Delta\Phi = 0 \tag{4.34}$$

subject to (4.10). Applying the operator $\nabla_i$ to both sides of (4.34) and making use of (4.32) once again yield

$$\Delta V_i = 0 \text{ for each } i \in \omega_{1,3}. \tag{4.35}$$

Hence, equation (4.11) turns into (4.15).

## 4.4. An unsteady Bernoulli equation: the first integral of the Euler equation

With '$\underline{V}$' as '$\underline{a}$' and '$\underline{\text{curl}}\,\underline{V}$' as '$\underline{b}$', equation (4.30) can be written as:

$$\begin{aligned}[\underline{V} \wedge (\underline{\text{curl}}\,\underline{V})]_i &= [\underline{V} \wedge (\underline{\nabla} \wedge \underline{V})]_i = \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \varepsilon_{ijk} V_j \varepsilon_{klm} \nabla_l V_m \\ &= \sum_{j=1}^{3} (V_j \nabla_i V_j - V_j \nabla_j V_i) = \frac{1}{2}\nabla_i \underline{V}^2 - \left(\sum_{j=1}^{3} V_j \nabla_j\right) V_i = 0,\end{aligned} \tag{4.36}$$

where use of the equation

$$\sum_{k=1}^{3} \varepsilon_{ijk}\,\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{4.37}$$

has been made. By (4.32) and (4.36), for each $(t,\underline{x}) \in \mathbf{D}_{\text{ff}}^{\text{cc}}(-h,\mathrm{Z})$ Euler's equation (4.24) becomes

$$\nabla_i \Psi(t,\underline{x}) = 0 \;\text{ for each }\; i \in \omega_{1,3} \tag{4.38}$$

subject to

$$\begin{aligned} \Psi(t,\underline{x}) &\stackrel{\rightarrow}{=} \rho_0 \frac{\partial \Phi(t,\underline{x})}{\partial t} + \frac{1}{2}\rho_0 [\underline{\nabla}\Phi(t,\underline{x})]^2 + P(t,\underline{x}) + \rho_0 g z \\ &= \rho_0 \frac{\partial \Phi(t,\underline{x})}{\partial t} + E_{\text{k}}(t,\underline{x}) + P(t,\underline{x}) + E_{\text{p}}(z) = \rho_0 \frac{\partial \Phi(t,\underline{x})}{\partial t} + E(t,\underline{x}) + P(t,\underline{x}), \end{aligned} \tag{4.39}$$

where

$$E_{\text{k}}(t,\underline{x}) \stackrel{\rightarrow}{=} \frac{1}{2}\rho_0 [\underline{\nabla}\Phi(t,\underline{x})]^2 = \frac{1}{2}\rho_0 [\underline{V}(t,\underline{x})]^2 , \tag{4.40}$$

$$E_{\text{p}}(z) \stackrel{\rightarrow}{=} -P_{\text{hs}}(z) = \rho_0 g z , \tag{4.41}$$

$$E(t,\underline{x}) \stackrel{\rightarrow}{=} E_{\text{k}}(t,\underline{x}) + E_{\text{p}}(z), \tag{4.42}$$

by (4.28) and (4.32). In this case, the former domain of definition of the functional form '$P_{\text{hs}}(z)$', which is defined by the relation

$$z \in [-h(x,y), 0], \tag{4.43}$$

occurring in (4.27), is supposed to be automatically extended so as to satisfy the relation

$$z \in [-h(x,y), \mathrm{Z}(t,x,y)]. \tag{4.44}$$

It is understood that $E_{\text{k}}(t,\underline{x})$, $E_{\text{p}}(z)$, and $E(t,\underline{x})$, defined by (4.40)–(4.42), are respectively the *volumetric kinetic*, *potential*, and *total energy densities* of the liquid at the temporo-spatial point $(t,\underline{x})$.

Given $t \in R$, let $\underline{x'} = \langle x', y', z' \rangle$ and $\underline{x''} = \langle x'', y'', z'' \rangle$ be two arbitrary different points located in the fluid at the instant $t$. Let, also, $L(\underline{x'},\underline{x''})$ be an arbitrary *Jordan arc* (see, e.g., Apostol [1963, p. 170]) joining $\underline{x'}$ and $\underline{x''}$ and lying entirely in the fluid at that instant, i.e. $L(\underline{x'},\underline{x''}) \subset D_{\text{ff}}^{\text{cc}}(-h,\mathrm{Z}(t,))$. Taking the line integrals of both sides of equation (4.38) along $L(\underline{x'},\underline{x''})$ from $\underline{x'}$ to $\underline{x''}$, one gets

$$\Psi(t,\underline{x'}) = \Psi(t,\underline{x''}) \tag{4.45}$$

independent of the path of integration. Since the points $\underline{x'}$ and $\underline{x''}$ are different, therefore equation (4.45) holds if and only if

$$\Psi(t,\underline{x}) = P_0(t) \text{ for each } t \in R \text{ and each } \underline{x} \in D_{\mathrm{ff}}^{\mathrm{cc}}(-h, Z(t,)\,, \tag{4.46}$$

where '$P_0(t)$' is an arbitrary real-valued functional form independent of '$\underline{x}$' and hence possibly depending only on '$t$'. Thus, equation (4.46) subject to (4.39) is the *first integral of equation (4.38)*, i.e. the *first integral of Euler's equation (4.24)*, – the one, which will be called the *unsteady Bernoulli equation for an ideal incompressible fluid flow in an infinite layer*.

Without loss of generality, '$P_0(t)$' can be specified thus:

$$P_0(t) \doteqdot [P(t,\underline{x})]_{z=Z(t,\underline{x}_2)}, \tag{4.47}$$

the understanding being that (a) if the part of space above the upper boundary surface $z = Z(t,\underline{x}_2)$ of the liquid layer is *vacuous* then

$$[P(t,\underline{x})]_{z=Z(t,\underline{x}_2)} \doteqdot 0 \tag{4.48}$$

and that (b) if the above part of space is occupied with air then

$$[P(t,\underline{x})]_{z=Z(t,\underline{x}_2)} \doteqdot P_{\mathrm{a}}(t), \tag{4.49}$$

where $P_{\mathrm{a}}(t)$ is a given *atmospheric pressure at* $z = Z(t,\underline{x}_2)$. In the latter case, I have tacitly assumed that $P_{\mathrm{a}}(t)$ is the same at least for $z \in [Z_{\mathrm{m}}, Z_{\mathrm{M}}]$ and also I have neglected the surface tension of the liquid. Consequently, in both above cases, the Bernoulli equation (4.46) subject to (4.39)–(4.42) can be written as:

$$\begin{aligned} P(t,\underline{x}) - [P(t,\underline{x})]_{z=Z(t,\underline{x}_2)} &= -\rho_0 \frac{\partial \Phi(t,\underline{x})}{\partial t} - E(t,\underline{x}) \\ = -\rho_0 \frac{\partial \Phi(t,\underline{x})}{\partial t} - E_{\mathrm{k}}(t,\underline{x}) - E_{\mathrm{p}}(z) &= -\rho_0 \frac{\partial \Phi(t,\underline{x})}{\partial t} - \frac{1}{2}\rho_0 [\underline{\nabla}\Phi(t,\underline{x})]^2 - \rho_0 g z \end{aligned} \tag{4.50}$$

and also as:

$$P(t,\underline{x}) - [P(t,\underline{x})]_{z=Z(t,\underline{x}_2)} = P_{\mathrm{d}}(t,\underline{x}) + P_{\mathrm{hs}}(z), \tag{4.51}$$

where $P_{\mathrm{d}}(t,\underline{x})$, defined as:

$$P_{\mathrm{d}}(t,\underline{x}) \doteqdot -\rho_0 \frac{\partial \Phi(t,\underline{x})}{\partial t} - E_{\mathrm{k}}(t,\underline{x}) = -\rho_0 \frac{\partial \Phi(t,\underline{x})}{\partial t} - \frac{1}{2}\rho_0 [\underline{\nabla}\Phi(t,\underline{x})]^2, \tag{4.52}$$

is the *dynamic pressure* at the temporo-spatial point $(t,\underline{x}) \in \mathbf{D}_{\mathrm{ff}}^{\mathrm{cc}}(-h, Z)$ and $P_{\mathrm{hs}}(z)$, defined by (4.41), is the *hydrostatic pressure* at each temporo-spatial point $(t,\underline{x}) \in \mathbf{D}_{\mathrm{ff}}^{\mathrm{cc}}(-h, Z)$ of the horizontal plane with the applicate $z$.

**Comment 4.2.** Equation (4.20) can be written as a variant of equation (4.38), namely

$$\nabla_i \Psi'(t,\underline{x}) = 0 \text{ for each } i \in \omega_{1,3} \tag{4.38'}$$

subject to

$$\Psi'(t,\underline{x}) \doteqdot \frac{\partial\Phi(t,\underline{x})}{\partial t} + \frac{1}{2}\underline{V}^2(t,\underline{x}) + w(t,\underline{x}). \tag{4.39'}$$

Therefore, in analogy with (4.46), the *first integral* of (4.20) can be written as:

$$\Psi'(t,\underline{x}) = w_0(t), \tag{4.46'}$$

where '$w_0(t)$' is as before an arbitrary real-valued functional form depending only on '$t$'. Let in (4.32)

$$\Phi(t,\underline{x}) = \Phi'(t,\underline{x}) - \Phi_0(t) \text{ subject to } \Phi'(t,\underline{x}) \doteqdot \Phi(t,\underline{x}) + \Phi_0(t), \tag{4.32$_1$}$$

where '$\Phi_0(t)$' is another arbitrary real-valued functional form depending only on '$t$'. Hence,

$$\underline{V}(t,\underline{x}) = \underline{\nabla}\Phi(t,\underline{x}) = \underline{\nabla}\Phi'(t,\underline{x}). \tag{4.32$_2$}$$

At the same time, given $t_0 \in R$, one may particularly set

$$\Phi_0(t) \doteqdot -\int_{t_0}^{t} w_0(t')\,dt', \tag{4.32'}$$

whence, by the Leibnitz rule of differentiation of an integral with variable limits,

$$\frac{\partial\Phi_0(t)}{\partial t} = -w_0(t). \tag{4.32″}$$

Consequently, by (4.32$_1$), (4.32$_2$), and (4.32″), equation (4.46') subject to (4.39') becomes

$$\frac{\partial\Phi'(t,\underline{x})}{\partial t} + \frac{1}{2}\underline{V}^2(t,\underline{x}) + w(t) = 0. \tag{4.46″}$$

Owing to (4.32$_2$)), '$\Phi'$' can be freed of its initial denotatum, defined by (4.32$_1$), and denote $\Phi$, while (4.46″) means that one can, without loss of generality, put $w_0(t) \doteqdot 0$ in (4.46').•

## 4.5. The momentum flux density tensor

When regarded as a pertinent instance of (4.5), equation (4.24) can be written as:

$$\rho_0 \frac{\partial V_i}{\partial t} + \sum_{j=1}^{3} \frac{\partial S_{ij}}{\partial x_j} = 0 \text{ for each } i \in \omega_{1,3} \tag{4.53}$$

subject to the pertinent *total momentum flux density tensor* $S_{ij}$, defined for each $i \in \omega_{1,3}$ and each $j \in \omega_{1,3}$ as

$$S_{ij} \doteqdot P\delta_{ij} + \rho_0 V_i V_j + \rho_0 gz\delta_{i3}\delta_{j3}\,. \tag{4.54}$$

Accordingly, in the presence of the field of gravity, the liquid becomes as if *anisotropic*.

Strictly speaking, however, the pertinent instance of (4.5) is the *homogeneous* equation:

$$\rho_0 \frac{\partial V_i}{\partial t} + \sum_{j=1}^{3} \frac{\partial S_{ij}^{-}}{\partial x_j} = 0 \text{ for each } i \in \omega_{1,3} \tag{4.53$_1$}$$

subject to the pertinent *abridged momentum flux density tensor* $S_{ij}^{-}$, defined as

$$S_{ij}^{-} \doteqdot P\delta_{ij} + \rho_0 V_i V_j\,, \tag{4.54$_1$}$$

whereas equation (4.24) subject to (4.22) can be written as the *inhomogeneous* equation:

$$\rho_0 \frac{\partial V_i}{\partial t} + \sum_{j=1}^{3} \frac{\partial S_{ij}^{-}}{\partial x_j} = -g\delta_{i3} \text{ for each } i \in \omega_{1,3} \tag{4.53$_2$}$$

In this case, passage from the homogeneous equation (4.53$_1$) to the inhomogeneous equation (4.53$_2$) changes the meaning of the unknown functional variables involved, particularly of '$P$'. The following simplest example illustrates the above said. The homogeneous Newtonian equation $\frac{d^2 x(t)}{dt^2} = 0$ implicitly defines a functional form '$x(t)$', descriptive of one-dimensional *steady* motion of a material particle with a constant velocity. At the same time, the inhomogeneous Newtonian equation $\frac{d^2 x(t)}{dt^2} = f(t)$ implicitly defines the *homographic* functional form '$x(t)$', descriptive of one-dimensional *unsteady* (*accelerated*) motion of a material particle with a varying velocity.

The unsteady Bernoulli equation (4.50) or (4.51) subject to (4.52) is the first integral of the Euler momentum flux density continuity equation (4.24), i.e. (4.53) subject to (4.54). Therefore, the latter equation subject to the former one is a *tautology,* which will be demonstrated before long in what follows. At the same time, the total momentum flux density tensor $S_{ij}$ itself subject to the unsteady Bernoulli equation, which is made explicit below, is a useful characteristic of wave motion.

Substitution of $P$, defined by (4.50), into (4.54) yields

$$S_{ij} \doteqdot \left[ P_0(t) - \rho_0 \frac{\partial \Phi}{\partial t} \right] \delta_{ij} - \rho_0 gz\left(\delta_{ij} - \delta_{i3}\delta_{j3}\right) + \left(E_{ij} - E_{\mathrm{k}}\delta_{ij}\right), \tag{4.55}$$

subject to (4.40) and (4.47)–(4.49) and also subject to

$$E_{ij} \doteqdot \rho_0 V_i V_j = \rho_0 \left(\nabla_i \Phi\right)\left(\nabla_j \Phi\right) = \rho_0 \left(\frac{\partial \Phi}{\partial x_i}\right)\left(\frac{\partial \Phi}{\partial x_j}\right), \tag{4.56}$$

by (4.32). The 3×3-tensor $S_{ij}$, defined by (4.55), satisfies the equation:

$$\sum_{j=1}^{3}\nabla_j S_{ij} \equiv \sum_{j=1}^{3}\frac{\partial S_{ij}}{\partial x_j} = \rho_0\frac{\partial V_i}{\partial t}, \tag{4.57}$$

because

$$\sum_{j=1}^{3}\nabla_j\left(\frac{\partial\Phi}{\partial t}\delta_{ij}\right) = \frac{\partial\nabla_i\Phi}{\partial t} = \frac{\partial V_i}{\partial t}, \tag{4.57$_1$}$$

$$\sum_{j=1}^{3}\nabla_j\left[P_0(t)\delta_{ij}\right] = \nabla_i P_0(t) = 0, \tag{4.57$_2$}$$

$$\sum_{j=1}^{3}\nabla_j\left[\rho_0 gz(\delta_{ij}-\delta_{i3}\delta_{j3})\right] = \rho_0 g(\nabla_i z - \delta_{i3}\nabla_3 z) = \rho_0 g\delta_{i3}(1-1) = 0, \tag{4.57$_3$}$$

$$\sum_{j=1}^{3}\nabla_j\left(E_{ij} - E_{\text{k}}\delta_{ij}\right) = \sum_{j=1}^{3}\nabla_j E_{ij} - \nabla_i E_{\text{k}} = 0. \tag{4.57$_4$}$$

The first three of equations (4.57$_1$)–(4.57$_4$) are self-evident, whereas the last one follows from (4.56) by (4.3), (4.36), and (4.40) thus:

$$\begin{aligned}\sum_{j=1}^{3}\nabla_j E_{ij} &= \rho_0\sum_{j=1}^{3}\nabla_j\left(V_i V_j\right) = \rho_0\left(\sum_{j=1}^{3}V_j\nabla_j\right)V_i + \rho_0 V_i\sum_{j=1}^{3}\nabla_j V_j\\ &= \rho_0\left(\sum_{j=1}^{3}V_j\nabla_j\right)V = \frac{1}{2}\rho_0\nabla_i\underline{V}^2 = \nabla_i E_{\text{k}}.\end{aligned} \tag{4.58}$$

Equation (4.57), being formally the same as (4.53), is *tautological* as expected.

### 4.6. The energy continuity equation

Besides (4.38) subject to (4.39), equation (4.24) subject to (4.22),can be written as:

$$\rho_0\frac{\partial V_i}{\partial t} + \rho_0\left(\sum_{j=1}^{3}V_j\nabla_j\right)V_i + \nabla_i(P + \rho_0 gz) = 0. \tag{4.59}$$

Multiplying both sides of equation (4.61) by $V_i$ and then summing up the result with respect to $i$ from 1 through 3 yields

$$\frac{\partial E_{\text{k}}}{\partial t} + \sum_{i=1}^{3}\nabla_i Q_i = 0, \tag{4.60}$$

subject to (4.40) and also subject to the equation

$$Q_i \equiv Q_i(t,\underline{x}) \equiv V_i(P + E_{\text{k}} + \rho_0 gz) = V_i(P + E_{\text{k}} + E_{\text{p}}) = V_i(P + E) \text{ for each } i \in \omega_{1,3}. \tag{4.61}$$

In this case, $V_i = \nabla_i \Phi$ by (4.32), while $E_{\mathrm{k}}$, $E_{\mathrm{p}}$, and $E$ are defined by (4.40)–(4.42) respectively. In developing (4.60) from (4.59), use has been made of the self-evident equation:

$$\sum_{i=1}^{3} V_i \frac{\partial V_i}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\sum_{i=1}^{3} V_i^2 = E_{\mathrm{k}} \qquad (4.60_1)$$

and also, by (4.3), of the equations:

$$\rho_0 \sum_{i=1}^{3}\sum_{j=1}^{3} V_i V_j \nabla_j V_i = \frac{\rho_0}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} V_j \nabla_j V_i V_i = \frac{\rho_0}{2}\sum_{j=1}^{3} V_j \nabla_j \underline{V}^2$$
$$= \frac{\rho_0}{2}\sum_{ij=1}^{3} \nabla_i \left(V_i \underline{V}^2\right) = \sum_{i=1}^{3} \nabla_i \left(V_i E_{\mathrm{k}}\right), \qquad (4.61_1)$$

$$\sum_{i=1}^{3} V_i \nabla_i \left(P + \rho_0 g z\right) = \sum_{i=1}^{3} \nabla_i \left[V_i \left(P + \rho_0 g z\right)\right]. \qquad (4.61_2)$$

By (4.42), it follows that

$$\frac{\partial E_{\mathrm{p}}(t,\underline{x})}{\partial t} = \frac{\partial \rho_0 g z}{\partial t} = 0\,, \qquad (4.62)$$

because '$\rho_0$' and '$g$' are constants, and also because the variables '$t$' and '$z$' are independent. At the same time, by (4.47)–(4.49), it follows from (4.50) that

$$P(t,\underline{x}) + E(t,\underline{x}) = \left[P(t,\underline{x})\right]_{z=Z(t,\underline{x}_2)} - \rho_0 \frac{\partial \Phi(t,\underline{x})}{\partial t} = P_0(t) - \rho_0 \frac{\partial \Phi(t,\underline{x})}{\partial t}\,. \qquad (4.63)$$

In this case, by (4.3), it follows from (4.61) and (4.63) that

$$\sum_{i=1}^{3} \nabla_i Q_i = \sum_{i=1}^{3} \nabla_i \left[V_i (P + E)\right] = \sum_{i=1}^{3} \nabla_i \left[V_i \left(P_0(t) - \rho_0 \frac{\partial \Phi}{\partial t}\right)\right]$$
$$= -\rho_0 \sum_{i=1}^{3} \nabla_i \left(V_i \frac{\partial \Phi}{\partial t}\right) \doteqdot \sum_{i=1}^{3} \nabla_i Q_{*i}\,, \qquad (4.64)$$

where

$$Q_{*i} \doteqdot Q_{*i}(t,\underline{x}) \doteqdot -\rho_0 V_i \frac{\partial \Phi}{\partial t} \text{ for each } i \in \omega_{1,3}\,. \qquad (4.65)$$

At the same time, equation (4.61) subject to (4.63) and (4.65) reduces to

$$Q_i \doteqdot Q_i(t,\underline{x}) \doteqdot V_i \left(P_0(t) - \rho_0 \frac{\partial \Phi}{\partial t}\right) = Q_i'(t,\underline{x}) + P_0(t) V_i \text{ for each } i \in \omega_{1,3}\,. \qquad (4.61a)$$

By (4.62) and (4.64), equation (4.60) is equivalent to this;

$$\frac{\partial E}{\partial t} = \frac{\partial E_{\mathrm{k}}}{\partial t} = -\sum_{i=1}^{3} \nabla_i Q_i = -\sum_{i=1}^{3} \nabla_i Q_{*i} \qquad (4.66)$$

subject to (4.40), (4.42), (4.61), and (4.65). The 3-vector $\underline{Q}(t,\underline{x})$, whose components are defined by (4.61), is called the *flux density vector across the unit area*, or *Poynting vector, of the liquid at the temporo-spatial point* $(t,\underline{x})$. The 3-vector $\underline{Q}_*(t,\underline{x})$, whose components are defined by (4.65), is called the *effective* flux density vector across the unit area, or *effective* Poynting vector, of the liquid at the temporo-spatial point $(t,\underline{x})$.

### 4.7. The dynamic boundary condition at the upper (free) surface of the liquid layer

For each for each $(t,\underline{x}_2)\in R\times\underline{E}_2$, equation (4.50) or (4.51), subject to (4.52), at $z=\mathrm{Z}(t,\underline{x}_2)$ can conveniently be rewritten thus:

$$\mathrm{Z}(t,\underline{x}_2)=-\frac{1}{g}\left[\frac{\partial\Phi(t,\underline{x})}{\partial t}+\frac{1}{\rho_0}E_{\mathrm{k}}(t,\underline{x})\right]_{z=\mathrm{Z}(t,\underline{x}_2)}=\frac{1}{\rho_0 g}\left[P_{\mathrm{d}}(t,\underline{x})\right]_{z=\mathrm{Z}(t,\underline{x}_2)}, \tag{4.67}$$

which is the *dynamic boundary condition at the upper* (*free*) *surface of the ideal incompressible irrotational liquid layer*, no matter whether the space above that surface is *vacuous*) or whether it is *air-filled*. I regard equation (4.67) as an *implicit definition of the function* Z *in terms of the function* Φ, which will be justified in the sequel by asymptotically solving that equation with respect to Φ.

### 4.8. Kinematic boundary conditions at the bottom and upper surfaces of the liquid layer

Application of the operator $\frac{D}{Dt}$, as defined by (3.14), to both sides of each one of equations (3.1) and (3.2) yields

$$\lim_{\varepsilon\to+0}\left[\frac{Ds_{\mathrm{b}}(\underline{x})}{Dt}\right]_{z=-h(\underline{x}_2)+\varepsilon}=\lim_{\varepsilon\to+0}\left[\frac{Ds_{\mathrm{t}}(t,\underline{x})}{Dt}\right]_{z=\mathrm{Z}(t,\underline{x}_2)-\varepsilon}=0, \tag{4.68}$$

whence

$$\left[\sum_{j=1}^{2}V_j(t,\underline{x})\nabla_j h(\underline{x}_2)+V_3(t,\underline{x})\right]_{z=-h(\underline{x}_2)}=0, \tag{4.69}$$

$$\frac{\partial\mathrm{Z}(t,\underline{x}_2)}{\partial t}+\left[\sum_{j=1}^{2}V_j(t,\underline{x})\nabla_j\mathrm{Z}(t,\underline{x}_2)-V_3(t,\underline{x})\right]_{z=\mathrm{Z}(t,\underline{x}_2)}=0, \tag{4.70}$$

subject to the general definitions (3.15). Equations (4.69) and (4.70) are the *basic kinematic boundary conditions at the bottom (rigid) and upper (free) boundary surfaces of the liquid layer*, respectively. If the fluid flow is incompressible and irrotational then equations (4.69) and (4.70), subject to (4.32), become

$$\left[\sum_{i=1}^{2}\left(\nabla_i \Phi\left(t,\underline{x}\right)\right)\left(\nabla_i h\left(\underline{x}_2\right)\right)+\frac{\partial\Phi\left(t,\underline{x}\right)}{\partial z}\right]_{z=-h\left(\underline{x}_2\right)}=0, \tag{4.71}$$

$$\frac{\partial Z\left(t,\underline{x}_2\right)}{\partial t}+\left[\sum_{i=1}^{2}\left(\nabla_i \Phi\left(t,\underline{x}\right)\right)\left(\nabla_i Z\left(t,\underline{x}_2\right)\right)-\frac{\partial\Phi\left(t,\underline{x}\right)}{\partial z}\right]_{z=Z\left(t,\underline{x}_2\right)}=0, \tag{4.72}$$

which are the *pertinent kinematic boundary conditions* at the bottom and upper boundary surfaces of the liquid layer respectively. If

$$h=C_d\text{, i.e. } h(\underline{x}_2)=d \text{ for each } \underline{x}_2\in\underline{E}_2, \tag{4.73}$$

where ‘ $d$ ’ is a constant, then equation (4.71) reduces to

$$\left[\frac{\partial\Phi\left(t,\underline{x}\right)}{\partial z}\right]_{z=-d}=0. \tag{4.74}$$

From the relevant theoretical considerations and practical experience, one can assume (postulate) that, in the absence of macroscopic currents,

$$\lim_{h_{\mathrm{m}}\to\infty}[\nabla_i \Phi(t,\underline{x})]_{z=-h(\underline{x}_2)}=0 \text{ for each } (t,\underline{x}_2)\in R\times\underline{E}_2 \text{ and each } i\in\omega_{1,3}. \tag{4.75}$$

If (4.73) holds then (4.75) trivially becomes

$$\lim_{d\to\infty}\left[\frac{\partial\Phi\left(t,\underline{x}\right)}{\partial z}\right]_{z=-d}=0 \text{ for each } (t,\underline{x}_2)\in R\times\underline{E}_2. \tag{4.76}$$

By (4.75), equation (4.71) turns into the tautology $0=0$ as $h_{\mathrm{m}}\to\infty$ and thus becomes ineffective. By (4.76), the same applies to equation (4.74) as $d\to\infty$. Hence, the two cases are equivalent.

**Comment 4.3.** In the case of a real, *viscous* fluid, there always exist short-range attractive forces between molecules in the surface of a solid body and molecules in the thin layer of the fluid immediately adjacent to the solid surface. These attractive forces result in adhesion of the adjacent fluid to the solid bottom surface, so that

$$\left[V_i(t,\underline{x})\right]_{z=-h(\underline{x}_2)}=0 \text{ for each } \left(t,\underline{x}_2\right)\in \underline{R\times E}_2 \text{ and each } i\in\omega_{1,3}. \tag{4.77}$$

This relation is the *dynamico-kinematic boundary condition at the bottom surface* of a viscous fluid flow, which comes instead of either condition (4.69) or (4.71).•

### 4.9. A dynamico-kinematic boundary condition at the free surface of the liquid layer

**Theorem 4.1.** For each $(t,\underline{x}_2)\in R\times \underline{E}_2$:

$$\frac{\partial \mathrm{Z}(t,\underline{x}_2)}{\partial t}=-\left[1+\frac{1}{g}\frac{\partial \Phi(t,\underline{x})}{\partial z}+\frac{1}{\rho_0 g}\frac{\partial E_{\mathrm{k}}(t,\underline{x})}{\partial z}\right]^{-1}_{z=\mathrm{Z}(t,\underline{x}_2)} \cdot\frac{1}{g}\left[\frac{\partial \Phi(t,\underline{x})}{\partial t}+\frac{1}{\rho_0}\frac{\partial E_{\mathrm{k}}(t,\underline{x})}{\partial t}\right]_{z=\mathrm{Z}(t,\underline{x}_2)}. \tag{4.78}$$

Equation (4.72) subject to (4.78) is called *dynamico-kinematic boundary condition at the free surface of the liquid layer*.

**Proof:** Equation (4.67) can be developed thus:

$$\mathrm{Z}(t,\underline{x}_2)=-\frac{1}{g}\frac{\partial \Phi(t;\underline{x}_2,\mathrm{Z}(t,\underline{x}_2))}{\partial t}-\frac{1}{\rho_0 g}E_{\mathrm{k}}(t;\underline{x}_2,\mathrm{Z}(t,\underline{x}_2)), \tag{4.79}$$

Therefore,

$$\frac{\partial \mathrm{Z}(t,\underline{x}_2)}{\partial t}=-\frac{1}{g}\left[\frac{\partial \Phi(t;\underline{x}_2,z)}{\partial t}\right]_{z=\mathrm{Z}(t,\underline{x}_2)}-\frac{1}{g}\left[\frac{\partial \Phi(t;\underline{x}_2,z)}{\partial t}\right]_{z=\mathrm{Z}(t,\underline{x}_2)}\frac{\partial \mathrm{Z}(t,\underline{x}_2)}{\partial t} -\frac{1}{\rho_0 g}\left[\frac{\partial E_{\mathrm{k}}(t;\underline{x}_2,z)}{\partial t}\right]_{z=\mathrm{Z}(t,\underline{x}_2)}-\frac{1}{\rho_0 g}\left[\frac{\partial E_{\mathrm{k}}(t;\underline{x}_2,z)}{\partial t}\right]_{z=\mathrm{Z}(t,\underline{x}_2)}\frac{\partial \mathrm{Z}(t,\underline{x}_2)}{\partial t}, \tag{4.79$_1$}$$

whence

$$\frac{\partial \mathrm{Z}(t,\underline{x}_2)}{\partial t}\left[1+\frac{1}{g}\frac{\partial \Phi(t,\underline{x})}{\partial z}+\frac{1}{\rho_0 g}\frac{\partial E_{\mathrm{k}}(t,\underline{x})}{\partial z}\right]_{z=\mathrm{Z}(t,\underline{x}_2)} =-\frac{1}{g}\left[\frac{\partial \Phi(t,\underline{x})}{\partial t}+\frac{1}{\rho_0}\frac{\partial E_{\mathrm{k}}(t,\underline{x})}{\partial t}\right]_{z=\mathrm{Z}(t,\underline{x}_2)} \tag{4.79$_2$}$$

which immediately reduces to (4.78).•

**Comment 4.4.** The set (conjunction) of four equations: (4.34), (4.67), (4.71), and (4.72) will be denoted by '$\mathsf{Q}(\Phi,\mathrm{Z})$', and the set of four equations: (4.34), (4.67), (4.74), and (4.72) by '$\mathsf{Q}_{\mathrm{U}}(\Phi,\mathrm{Z})$', where the subscript 'U' is a capitalized first letter of the word 'uniform'. Likewise, the set of four equations: (4.34), (4.67), (4.75) or (4.76), and (4.72) will be denoted by '$\mathsf{Q}_{\infty}(\Phi,\mathrm{Z})$'. Since, however, the function Z is an unknown, therefore the boundary condition (4.72) is *ineffective*. In order to make such a set of equations effective, it should be supplemented by one or more *additional hypotheses* (*assumptions*), which restrict the class of problems, to which that set applies. One of such hypotheses is stated and discussed in the next section.•

### 4.10. Paradoxes of the Eulerian formalism

Let

$$P_{\mathrm{XY}}(z) \doteqdot \{\underline{x} | \underline{x}_2 \in \underline{E}_2\} \doteqdot \{(\underline{x}_2, z) | \underline{x}_2 \in \underline{E}_2\}, \tag{4.80}$$

i.e. $P_{\mathrm{XY}}(z)$ is the plane perpendicular to the applicate and crossing the latter at a given point *z*. In this case, however, in accordance with (3.5), a given spatial point

$$\underline{x} = (\underline{x}_2, z) \in \underline{E}_2 \times [\mathrm{Z_m}, \mathrm{Z_M}] \tag{4.81}$$

can belong to a fluid particle for some $t \in R$ and it cannot belong to any fluid particles at some other $t \in R$. Consequently, by Definition 3.2, given $t \in R$, the part $P^{+}_{\mathrm{XY}}(t, z)$ of $P_{\mathrm{XY}}(z)$, which is defined by the relation:

$$P^{+}_{\mathrm{XY}}(t, z) \doteqdot P_{\mathrm{XY}}(z) \cap D^{\mathrm{cc}}_{\mathrm{ff}}(-h, \mathrm{Z}(t, )) \neq \varnothing, \tag{4.82}$$

passes through liquid, while the complementary part $P^{-}_{\mathrm{XY}}(t, z)$ of $P_{\mathrm{XY}}(z)$, defined as

$$P^{-}_{\mathrm{XY}}(t, z) \doteqdot P_{\mathrm{XY}}(z) - P^{+}_{\mathrm{XY}}(t, z) = \varnothing, \tag{4.83}$$

is *vacuous* (*immaterial*), so that no liquid characteristics are defined for $(\underline{x}_2, z) \in P^{-}_{\mathrm{XY}}(t, z)$.

In the literature on the Airy (linear) theory of gravity waves on a liquid layer, the boundary condition (4.67) is replaced by the corresponding linear boundary condition, which is evaluated at the coordinate XY-plane $P_{\mathrm{XY}}(0)$, defined by the equation $z = 0$, and not at the exact but unknown free (upper) surface $S_{\mathrm{t}}$ of the liquid layer, defined by the equation $z = \mathrm{Z}(t, \underline{x}_2)$ (see, e.g., Lamb [1932, p. 364], Landau and Lifshitz [1987, p. 32], Mei [1989, p. 8], Dingemans [1997, pp. 39–41]). However, given $t \in R$, the velocity potential $\Phi(t; \underline{x}_2, 0)$ and all related liquid characteristics are defined only on $P^{+}_{\mathrm{XY}}(t, 0)$ and are not defined on $P^{-}_{\mathrm{XY}}(t, 0)$. Therefore, the tricky device of replacing the boundary condition at the exact but unknown disturbed free surface of the liquid layer by an *as if approximate* boundary condition at the liquid free surface in equilibrium is *paradoxical* that can be called the *paradox of the boundary condition at a free liquid surface.*

A problem of fluid flow cannot, as a rule, be solved analytically. Therefore, it is often desirable to average either some governing equations or some fluid characteristics before solving the problem. In this case, the time average $\overline{F(t, \underline{x})}^{t}$ of the fluid characteristic, taking on values of a certain functional form, is supposed to be defined by the universal formula:

$$\overline{F(t,\underline{x})}^t \doteq \lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} F(t,\underline{x})dt \tag{4.84}$$

for any given spatial point $\underline{x}$ in the fluid (see, e.g., Mei [1989, Sec. 10.2, pp. 453–463] or Dingemans [1997, part 1, sec.2.9, pp. 184–215]). However, given $\underline{x}$ subject to (4.81), $F(t,\underline{x})$ is, in accordance with the above-said, defined only on a part of the entire interval in (4.84). Therefore, for any such spatial point, the time average $\overline{F(t,\underline{x})}^t$ as defined by (4.84) *does not exist*. I call this inconsistency of the formula (4.84) the *paradox of time averages.*

Both the above paradoxes are results of describing a fluid flow by Eulerian variables. The first of them will explicitly be solved asymptotically in the next section.

## 4.11. The analytical extension of Φ

Both paradoxes that have been indicated in the previous subsection can be solved (eliminated) by subjecting the velocity potential Φ of a potential (irrotational) fluid flow in a liquid layer as specified in subsection 3.1 to the following *hypothesis*.

**Hypothesis 4.2:** ***The hypothesis of analytical extension of*** Φ. In the case of a potential (irrotational) fluid flow in a liquid layer as specified in subsection 3.1, there exists a *harmonic function* Φ, which satisfies the Laplace equation (4.34) for in the whole time-space, i.e. for all $\langle t,\underline{x}\rangle \in R\times\underline{E}_3$, and which also satisfies all pertinent equations of subsections 4.8 and 4.9, – particularly equations (4.52) or (4.54), (4.71), (4.72), (4.76) and (4.78).•

**Comment 4.5.** To be more specific, Hypothesis 4.2 means that that for each $\langle t,\underline{x}_2\rangle \in R\times\underline{E}_2$: there is the analytical extension (continuation) of the function $\Phi(t,(\underline{x}_2,\ ))$ from the interval $[-h(\underline{x}_2),Z(t,\underline{x}_2)]$ to the whole applicate axis $(-\infty,\infty)$. In this case, the function Φ is denoted by the same letter 'Φ'. Accordingly, the pertinent analytical extension of any bulk function (as $V_i$, $P_\mathrm{d}$, etc) will be denoted by the same symbol as that denoting the original function itself (as '$V_i$', '$P_\mathrm{d}$', etc, respectively).

Hypothesis 4.2 makes explicit the assumption, which remains implicit when the temporal partial derivative or a time average of a bulk characteristic of the fluid flow is unrestrictedly computed at all points of the flow with the help of the corresponding conventional formula as if the characteristic were defined on the whole three-dimensional space at any instant of time; or when the boundary conditions at the free surface of the liquid, i.e. at $z = Z(t, \underline{x}_2)$, are replaced by appropriate approximate boundary conditions at $z = 0$ (see the next section).

Incidentally, every approximate solution for $\Phi$, which is obtained in the literature in the framework of the conventional linear wave theory, proves to be analytical on the whole infinite space. This fact agrees with Hypothesis 4.2, but it does not, however, prove that Hypothesis 4.2 is true, i.e. that it is a theorem of hydrodynamics.

The analytical extension of Φ in question is analogous to the extension of the electrostatic potential, which is tacitly done in solving electrostatic problems by the method of images (see, e.g., Iosilevskii [1978]).•

**Convention 4.1.** In accordance with Hypothesis 4.2 and Comment 4.5,

1) any bulk function (as Φ, $V_i$, $P_{\mathrm{d}}$, etc) will be denoted by the same symbol as that denoting the original function itself (as 'Φ', '$V_i$', '$P_{\mathrm{d}}$', etc, respectively);

2) each bulk relation, which is stated for the liquid layer as a true one, is assumed to be preceded either by the quantifier 'for each $\langle t, \underline{x} \rangle \in R \times \underline{E}_3$:'.•

## 4.12. Corollaries of Hypothesis 4.2: Modified boundary conditions at the free surface

1) It follows from Hypothesis 4.2 that for each $\langle t, \underline{x} \rangle \in R \times \underline{E}_3$:

$$\Phi(t, \underline{x}) = \sum_{m=0}^{\infty} \frac{1}{m!} \Phi^{(m)}(t, \underline{x}_2) z^m \,, \tag{4.85}$$

where

$$\Phi^{(m)}(t, \underline{x}_2) \doteqdot \left[ \frac{\partial^m \Phi(t, \underline{x})}{\partial z^m} \right]_{z=0} \quad \text{for each } m \in \omega_0, \; \Phi^{(0)}(t, \underline{x}_2) \doteqdot \left[ \Phi(t, \underline{x}) \right]_{z=0}, \tag{4.86}$$

and where, as usual, $0! \doteqdot 1$.•

2) By (4.32), (4.85), and (4.86), it follows that

$$V_i(t, \underline{x}) = \nabla_i \Phi(t, \underline{x}) = \sum_{m=0}^{\infty} \frac{1}{m!} \left[ \nabla_i \Phi^{(m)}(t, \underline{x}_2) \right] z^m \text{ for each } i \in \{1, 2\}, \tag{4.87}$$

$$V_3(t, \underline{x}) = \frac{\partial \Phi(t, \underline{x})}{\partial z} = \sum_{m=1}^{\infty} \frac{1}{(m-1)!} \Phi^{(m)}(t, \underline{x}_2) z^{m-1} = \sum_{m=0}^{\infty} \frac{1}{m!} \Phi^{(m+1)}(t, \underline{x}_2) z^m \,, \tag{4.88}$$

$$\frac{\partial \Phi(t, \underline{x})}{\partial t} = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial \Phi^{(m)}(t, \underline{x}_2)}{\partial t} z^m \,. \tag{4.89}$$

3) It follows from (4.40) and (4.85)–(4.89) that

$$E_{\mathrm{k}}(t,\underline{x}) = \sum_{l=0}^{\infty}\frac{1}{l!}E_{\mathrm{k}}^{(l)}(t,\underline{x}_2)z^l\ , \tag{4.90}$$

where

$$\begin{gathered} E_{\mathrm{k}}^{(l)}(t,\underline{x}_2) \stackrel{\rightarrow}{=} \left[\frac{\partial^l E_{\mathrm{k}}(t,\underline{x})}{\partial z^l}\right]_{z=0} \\ = \frac{1}{2}\rho_0\sum_{n=0}^{l}\frac{l!}{(l-n)!n!}\left[\sum_{i=1}^{2}\left(\nabla_i\Phi^{(l-n)}(t,\underline{x}_2)\right)\left(\nabla_i\Phi^{(n)}(t,\underline{x}_2)\right)+\Phi^{(l-n+1)}(t,\underline{x}_2)\Phi^{(n+1)}(t,\underline{x}_2)\right] \\ \text{for each } l\in\omega_0, \end{gathered} \tag{4.91}$$

the understanding being that

$$E_{\mathrm{k}}^{(0)}(t,\underline{x}_2) \stackrel{\rightarrow}{=} \left[E_{\mathrm{k}}(t,\underline{x})\right]_{z=0} = \frac{1}{2}\rho_0\sum_{n=0}^{l}\left[\sum_{i=1}^{2}\left(\nabla_i\Phi^{(0)}(t,\underline{x}_2)\right)^2+\left(\Phi^{(1)}(t,\underline{x}_2)\right)^2\right]. \tag{4.92}$$

Indeed, equation (4.40) can be developed by (4.85)–(4.89) thus:

$$\begin{aligned} E_{\mathrm{k}}(t,\underline{x}) &= \frac{1}{2}\rho_0\sum_{i=1}^{3}\left[\nabla_i\Phi(t,\underline{x})\right]\left[\nabla_i\Phi(t,\underline{x})\right] \\ &= \frac{1}{2}\rho_0\sum_{i=1}^{2}\left[\sum_{m=0}^{\infty}\nabla_i\Phi^{(m)}(t,\underline{x}_2)z^m\right]\left[\sum_{n=0}^{\infty}\nabla_i\Phi^{(n)}(t,\underline{x}_2)z^n\right] \\ &\quad+\frac{1}{2}\rho_0\left[\sum_{m=1}^{\infty}\frac{1}{(m-1)!}\Phi^{(m)}(t,\underline{x}_2)z^{m-1}\right]\left[\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\Phi^{(n)}(t,\underline{x}_2)z^n\right] \\ &= \frac{1}{2}\rho_0\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!n!}\left[\sum_{i=1}^{2}\left(\nabla_i\Phi^{(m)}(t,\underline{x}_2)\right)\left(\nabla_i\Phi^{(n)}(t,\underline{x}_2)\right)+\Phi^{(m+1)}(t,\underline{x}_2)\Phi^{(m+1)}(t,\underline{x}_2)\right]z^{m+n}. \end{aligned} \tag{4.90$_1$}$$

Let $l \stackrel{\rightarrow}{=} m+n$ so that $m=l-n$. Then (4.90$_1$) reduces to

$$E_{\mathrm{k}}(t,\underline{x}) = \frac{1}{2}\rho_0\sum_{l=0}^{\infty}\sum_{n=0}^{l}\frac{1}{(l-n)!n!}\left[\sum_{i=1}^{2}\left(\nabla_i\Phi^{(l-n)}(t,\underline{x}_2)\right)\left(\nabla_i\Phi^{(n)}(t,\underline{x}_2)\right)+\Phi^{(l-n+1)}(t,\underline{x}_2)\Phi^{(n+1)}(t,\underline{x}_2)\right]z^l, \tag{4.90$_2$}$$

which can be rewritten as (4.90) subject to (4.91).

4) By (4.87)–(4.89), expanding the expressions on the left-hand sides of equations (4.67) and (4.70) and (4.72) into the Maclaurin series with respect to powers of ‘ Z ’ yields

$$\begin{aligned} \mathrm{Z}(t,\underline{x}_2) &= \frac{1}{\rho_0 g}\sum_{m=0}^{\infty}\frac{1}{m!}P_{\mathrm{d}}^{(m)}(t,\underline{x}_2)\mathrm{Z}^m(t,\underline{x}_2) \\ &= -\frac{1}{\rho_0 g}\sum_{m=0}^{\infty}\frac{1}{m!}\left[E_{\mathrm{k}}^{(m)}(t,\underline{x}_2)+\rho_0\frac{\partial\Phi^{(m)}(t,\underline{x}_2)}{\partial t}\right]\mathrm{Z}^m(t,\underline{x}_2), \end{aligned} \tag{4.93}$$

$$\frac{\partial Z(t,\underline{x}_2)}{\partial t}+\sum_{m=0}^{\infty}\frac{1}{m!}\left[\sum_{i=1}^{2}\left(\nabla_i\Phi^{(m)}(t,\underline{x}_2)\right)\left(\nabla_i Z(t,\underline{x}_2)\right)-\Phi^{(m+1)}(t,\underline{x}_2)\right]Z^m(t,\underline{x}_2)=0. \tag{4.94}$$

where, in accordance with (4.86) and (4.90),

$$P_{\mathrm{d}}^{(m)}(t,\underline{x}_2)\doteq\left[\frac{\partial^m P_{\mathrm{d}}(t,\underline{x})}{\partial z^m}\right]_{z=0}\text{ for each } m\in\omega_0\,;\; P_{\mathrm{d}}^{(0)}(t,\underline{x}_2)\doteq\left[P_{\mathrm{d}}(t,\underline{x})\right]_{z=0}. \tag{4.95}$$

5) Since

$$\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial\Phi^{(m)}}{\partial t}Z^m=\frac{\partial\Phi^{(0)}}{\partial t}+\sum_{m=1}^{\infty}\frac{1}{m!}\frac{\partial\Phi^{(m)}}{\partial t}Z^m=\frac{\partial\Phi^{(0)}}{\partial t}+\sum_{m=0}^{\infty}\frac{1}{(m+1)!}\frac{\partial\Phi^{(m+1)}}{\partial t}Z^{m+1}, \tag{4.93$_1$}$$

therefore (4.93) can be rewritten as

$$Z(t,\underline{x}_2)+\frac{1}{g}\frac{\partial\Phi^{(0)}(t,\underline{x}_2)}{\partial t}=\mathrm{A}_{\mathrm{d}}(t,\underline{x}_2), \tag{4.96}$$

where

$$\mathrm{A}_{\mathrm{d}}(t,\underline{x}_2)\doteq\sum_{m=0}^{\infty}\frac{1}{m!}\mathrm{H}^{(m)}(t,\underline{x}_2)Z^m(t,\underline{x}_2) \tag{4.97}$$

subject to

$$\mathrm{H}^{(m)}(t,\underline{x}_2)\doteq-\frac{1}{\rho_0 g}\left[E_{\mathrm{k}}^{(m)}(t,\underline{x}_2)+\frac{\rho_0}{m+1}\frac{\partial\Phi^{(m+1)}(t,\underline{x}_2)}{\partial t}Z(t,\underline{x}_2)\right]\text{ for each } m\in\omega_0. \tag{4.98}$$

Equation (4.96) subject to (4.97) and (4.98) is the pertinent *modified dynamic boundary condition at* $z=Z(t,\underline{x}_2)$.

6) Since

$$\sum_{m=0}^{\infty}\frac{1}{m!}\Phi^{(m+1)}Z^m=\Phi^{(1)}+\sum_{m=1}^{\infty}\frac{1}{m!}\Phi^{(m+1)}Z^m=\Phi^{(1)}+\sum_{m=0}^{\infty}\frac{1}{(m+1)!}\Phi^{(m+2)}Z^{m+1}, \tag{4.94$_1$}$$

therefore (4.94) can be rewritten as

$$\frac{\partial Z(t,\underline{x}_2)}{\partial t}-\Phi^{(1)}(t,\underline{x}_2)=\mathrm{A}_{\mathrm{k}}(t,\underline{x}_2), \tag{4.99}$$

where

$$\mathrm{A}_{\mathrm{k}}(t,\underline{x}_2)\doteq\sum_{m=0}^{\infty}\frac{1}{m!}\Xi^{(m)}(t,\underline{x}_2)Z^m(t,\underline{x}_2) \tag{4.100}$$

subject to

$$\Xi^{(m)}(t,\underline{x}_2)\doteq-\sum_{i=1}^{2}\left[\nabla_i\Phi^{(m)}(t,\underline{x}_2)\right]\left[\nabla_i Z(t,\underline{x}_2)\right]+\frac{1}{m+1}\Phi^{(m+2)}(t,\underline{x}_2)Z(t,\underline{x}_2)\text{ for each } m\in\omega_0. \tag{4.101}$$

In this case, by (4.31), (4.32), and (4.86), equation (4.101) at $m = 0$ can be developed thus

$$\begin{aligned}\Xi^{(0)} &\doteq -\sum_{i=1}^{2}\left(\nabla_i \Phi^{(0)}\right)\left(\nabla_i \mathrm{Z}\right) + \Phi^{(2)}\mathrm{Z} = \left(-\sum_{i=1}^{2} V_i \nabla_i \mathrm{Z} + \mathrm{Z}\nabla_3 V_3\right)_{z=0} \\ &= -\sum_{i=1}^{2}\left(V_i \nabla_i \mathrm{Z} + \mathrm{Z}\nabla_i V\right)_{z=0} = -\sum_{i=1}^{2}\left[\nabla_i\left(V_i \mathrm{Z}\right)\right]_{z=0}.\end{aligned} \tag{4.102}$$

Equation (4.99) subject to (4.100) and (4.101) is the pertinent *modified kinematic boundary condition at* $z = \mathrm{Z}(t, \underline{x}_2)$.

7) Subtraction of equation (4.99) from the equation tha is obtained by differentiating both sides of equation (4.96) with respect to '$t$' yields

$$\Phi^{(1)}\left(t, \underline{x}_2\right) + \frac{1}{g}\frac{\partial^2 \Phi^{(0)}\left(t, \underline{x}_2\right)}{\partial t^2} = \left[\frac{\partial \Phi\left(t, \underline{x}\right)}{\partial z} + \frac{1}{g}\frac{\partial^2 \Phi\left(t, \underline{x}\right)}{\partial t^2}\right]_{z=0} = \mathrm{A}\left(t, \underline{x}_2\right), \tag{4.103}$$

where

$$\mathrm{A}\left(t, \underline{x}_2\right) \doteq \frac{\partial \mathrm{A_d}\left(t, \underline{x}_2\right)}{\partial t} - \mathrm{A_K}\left(t, \underline{x}_2\right). \tag{4.104}$$

Equation (4.103) subject to (4.104), is the pertinent *modified dynamico-kinematic boundary condition at* $z = \mathrm{Z}(t, \underline{x}_2)$.

**Comment 4.6.** In accordance with the definitions of Comment 4.4**,** the following notation will be used under Convention 4.1.

a) Equation (4.34) subject to Convention 4.1 will be referred to as (4.34$_+$).

b) The set (conjunction) of four equations: (4.34$_+$), (4.93) subject to (4.95), (4.71), and (4.99) subject to (4.100) and (4.101) will be denoted by '$\mathsf{Q}_+(\Phi, \mathrm{Z})$', and the set of four equations: (4.34$_+$), (4.93) subject to (4.95), (4.74), and (4.99) subject to (4.100) and (4.101) will be denoted by '$\mathsf{Q}_{\mathrm{U}+}(\Phi, \mathrm{Z})$'. Likewise, the set of four equations: (4.34$_+$), (4.93) subject to (4.95), (4.75) or (4.76), and (4.99) subject to (4.100) and (4.101) will be denoted by '$\mathsf{Q}_{\infty+}(\Phi, \mathrm{Z})$'.

c) Analogously, the set (conjunction) of three equations (4.34$_+$), (4.103) subject to (4.104), and (4.71) will be denoted by '$\mathsf{T}_+(\Phi, \mathrm{Z})$', and the set of three equations: (4.34$_+$), (4.103) subject to (4.104), and (4.74) will be denoted by '$\mathsf{T}_{\mathrm{U}+}(\Phi, \mathrm{Z})$'. Likewise, the set of three equations: (4.34$_+$), (4.103) subject to (4.104), and (4.75) or (4.76) will be denoted by '$\mathsf{T}_{\infty+}(\Phi, \mathrm{Z})$'.

Since the function Z is as before an unknown, therefore each one of the three boundary conditions: three equations (4.93) subject to (4.95), (4.99) subject to (4.100) and (4.101), and (4.103) subject to (4.104), is *ineffective*. In order to make such a set of equations effective, it should be supplemented by one or more *additional hypotheses* (*assumptions*), which restrict the class of problems, to which that set applies. One of such hypotheses is stated and discussed in the next section.•

# 5. A recursive asymptotic analysis of basic fields and equations of an irrotational incompressible fluid flow

## 5.1. The hypothesis of recursive asymptotic representations of the free surface of a perturbed liquid layer and of the velocity potential of the pertinent fluid flow

**Hypothesis 5.1.** There exists a real number $\varepsilon \in [0,1)$ such that the following relations hold.

1) For each $\langle t, \underline{x}_2 \rangle \in R \times \underline{E}_2$:

$$\mathrm{Z}(t, \underline{x}_2) \risingdotseq \mathrm{Z}(t; \underline{x}_2, \varepsilon) \sim \mathrm{Z}_{[\infty,1]}(t; \underline{x}_2, \varepsilon) \risingdotseq \sum_{n=1}^{\infty} \varepsilon^n \zeta_{(n)}(t; \underline{x}_2, \varepsilon), \tag{5.1}$$

where '~' is the *sign of [full] asymptotic correspondence* (cf. Erdélyi [1956, pp. 11–14], Olver [1974, pp. 4–8], or Van Dyke [1975, pp. 26–28]), so that

$$\zeta_{(1)}(t, \underline{x}_2) = \lim_{\varepsilon \to +0} \varepsilon^{-1 \cdot} \mathrm{Z}(t; \underline{x}_2, \varepsilon), \tag{5.2}$$

and for each $m \in \omega_2$:

$$\zeta_{(m)}(t, \underline{x}_2) = \lim_{\varepsilon \to +0} \varepsilon^{-m \cdot} \left[\mathrm{Z}(t; \underline{x}_2, \varepsilon) - \mathrm{Z}_{[m-1,1]}(t; \underline{x}_2, \varepsilon)\right], \tag{5.3}$$

subject to

$$\mathrm{Z}_{[m-1,1]}(t; \underline{x}_2, \varepsilon) \risingdotseq \sum_{n=1}^{m-1} \mathrm{Z}_{(n)}(t; \underline{x}_2, \varepsilon) \risingdotseq \sum_{n=1}^{m-1} \varepsilon^n \zeta_{(n)}(t, \underline{x}_2). \tag{5.4}$$

It is understood that

$$\mathrm{Z}_{[1,1]}(t; \underline{x}_2, \varepsilon) \risingdotseq \sum_{n=1}^{1} \mathrm{Z}_{(n)}(t; \underline{x}_2, \varepsilon) = \mathrm{Z}_{(1)}(t; \underline{x}_2, \varepsilon), \tag{5.5}$$

$$\mathrm{Z}_{[\infty,1]}(t; \underline{x}_2, \varepsilon) \risingdotseq \lim_{m \to \infty} \mathrm{Z}_{[m-1,1]}(t; \underline{x}_2, \varepsilon), \tag{5.6}$$

$$\mathrm{Z}_{(n)}(t; \underline{x}_2, \varepsilon) \risingdotseq \varepsilon^n \zeta_{(n)}(t, \underline{x}_2) \text{ for each } n \in \omega_1. \tag{5.7}$$

2) Analogously, for each $\langle t, \underline{x} \rangle \in R \times \underline{E}_3$:

$$\Phi(t,\underline{x}) \doteqdot \Phi(t;\underline{x},\varepsilon) \sim \Phi_{[\infty,1]}(t;\underline{x},\varepsilon) \doteqdot \sum_{n=1}^{\infty} \varepsilon^{n} \phi_{(n)}(t,\underline{x}), \tag{5.8}$$

so that

$$\phi_{(1)}(t,\underline{x}) = \lim_{\varepsilon\to+0} \varepsilon^{-1\cdot} \Phi(t;\underline{x},\varepsilon), \tag{5.9}$$

and for each $m \in \omega_2$:

$$\phi_{(m)}(t,\underline{x}) = \lim_{\varepsilon\to+0} \varepsilon^{-m\cdot} \left[\Phi(t;\underline{x},\varepsilon) - \Phi_{[m-1,1]}(t;\underline{x},\varepsilon)\right], \tag{5.10}$$

subject to

$$\Phi_{[m-1,1]}(t;\underline{x},\varepsilon) \doteqdot \sum_{n=1}^{m-1} \Phi_{(n)}(t;\underline{x},\varepsilon) \doteqdot \sum_{n=1}^{m-1} \varepsilon^{n} \phi_{(n)}(t,\underline{x}). \tag{5.11}$$

It is understood that

$$\Phi_{[1,1]}(t;\underline{x},\varepsilon) \doteqdot \sum_{n=1}^{1} \Phi_{(n)}(t;\underline{x},\varepsilon) = \Phi_{(1)}(t;\underline{x},\varepsilon), \tag{5.12}$$

$$\Phi_{[\infty,1]}(t;\underline{x},\varepsilon) \doteqdot \lim_{m\to\infty} \Phi_{[m-1,1]}(t;\underline{x},\varepsilon), \tag{5.13}$$

$$\Phi_{(n)}(t;\underline{x},\varepsilon) \doteqdot \varepsilon^{n} \phi_{(n)}(t,\underline{x}) \text{ for each } n \in \omega_1. \tag{5.14}$$ •

**Comment 5.1.** In stating Hypothesis 5.1, I have is tacitly assumed that macroscopic currents transferring liquid masses in the layer are absent. In order to take into account such currents, one should have set the lower limit of summation in (5.2) to 0, and not to 1. Accordingly, I regard Hypothesis 5.1 as a *sufficient condition* that an allowable perturbation of the liquid layer can be interpreted as a *single whole water wave*. The real number $\varepsilon$ introduced in Hypothesis 5.1 characterizes the *choppiness* (*dynamic roughness*) of the disturbed water free surface, and therefore the variable '$\varepsilon$' will be properly called the *choppiness parameter*. In theory of asymptotic series, such a parameter is commonly called a *scaling parameter* or *similarity parameter*. In the sequel, '$\varepsilon$' will be defined as: $\varepsilon \doteqdot ka$, where $a$ is the *surface amplitude* of a wave associated with $\Phi_{(1)}(t;\underline{x},\varepsilon)$ and $k$ is the wave number of that wave. After a given problem is solved, one may, when desired, pass to the limit $d \to +\infty$ (or in general $h_{\mathrm{m}} \to +\infty$) in all relevant final formulae.•

## 5.2. General recipes for the asymptotic power expansions of any pertinent bulk functional forms

**Preliminary Remark 5.1.** In this subsection, I shall formulate basic concepts of asymptotic expansions of functions into asymptotic power series in such a form which is most convenient for subsequent applications in the exposition. I shall also make explicit some properties of asymptotic power series, which are not discussed in the standard reference monographs on asymptotic expansions (as Erdélyi [1956], Olver [1974], and Van Dyke [1975]), but which will be most useful in the sequel both for proving theorems and for the adequate interpretation of the theory and its implications.•

**Definition 5.1.** 1) '$F(\tau,\varepsilon)$' is a *placeholder for*, – or, semantically, $F(\tau,\varepsilon)$ *is*, – a *functional form*, which is defined on a certain domain $\mathrm{T}\times[0,1)$, i.e. for each $\tau\in\mathrm{T}$ and each $\varepsilon\in[0,1)$, where $\varepsilon_*$ is a given strictly positive real number or ∞. It is assumed that there exists a natural number $\mu\in\omega_0$ such that for each $\tau\in\mathrm{T}$ and for each $\varepsilon\in[0,1)$:

$$F(\tau,\varepsilon)\sim F_{[\infty,\mu]}(\tau,\varepsilon)\overset{\rightarrow}{=}\sum_{n=\mu}^{\infty}f_{(n)}(\tau)\varepsilon^{n}\,, \tag{5.15}$$

where '~' is the *sign of [full] asymptotic correspondence*, so that

$$f_{(\mu)}(\tau)=\lim_{\varepsilon\to+0}\varepsilon^{-\mu}F(\tau,\varepsilon)\,, \tag{5.16}$$

and for each $m\in\omega_{\mu+1}$:

$$f_{(m)}(\tau)=\lim_{\varepsilon\to+0}\varepsilon^{-m\cdot}\left[F(\tau,\varepsilon)-F_{[m-1,\mu]}(\tau,\varepsilon)\right], \tag{5.17}$$

subject to

$$F_{[m-1,\mu]}(\tau,\varepsilon)\overset{\rightarrow}{=}\sum_{n=\mu}^{m-1}F_{(n)}(\tau,\varepsilon)\overset{\rightarrow}{=}\sum_{n=\mu}^{m-1}f_{(n)}(\tau)\varepsilon^{n}\,. \tag{5.18}$$

It is understood that

$$F_{[\mu,\mu]}(\tau,\varepsilon)\overset{\rightarrow}{=}\sum_{n=\mu}^{\mu}F_{(n)}(\tau,\varepsilon)=F_{(\mu)}(\tau,\varepsilon)\,, \tag{5.19}$$

$$F_{[\infty,\mu]}(\tau,\varepsilon)\overset{\rightarrow}{=}\lim_{m\to\infty}F_{[m-1,\mu]}(\tau,\varepsilon)\,, \tag{5.20}$$

$$F_{(n)}(\tau,\varepsilon)\overset{\rightarrow}{=}f_{(n)}(\tau)\varepsilon^{n}\ \text{ for each }\ n\in\omega_{\mu}\,. \tag{5.21}$$

Also, the conjunction of (5.16) and (5.17) subject to $m\in\omega_{\mu+1}$ is equivalent to (5.17) alone subject to $m\in\omega_{\mu}$, because

$$F_{[\mu-1,\mu]}(\tau,\varepsilon)\overset{\rightarrow}{=}0\,. \tag{5.22}$$

2) Given $\tau \in \mathrm{T}$, if the conjunction of (5.16) and (5.17) holds then the infinite sum '$F_{[\infty,\mu]}(\tau,\varepsilon)$' is said to be the [*full*] *infinite asymptotic series for* '$F(\tau,\varepsilon)$' *in integer powers of* '$\varepsilon$' *in the set* $\omega_\mu$, or briefly the *infinite asymptotic series for* '$F(\tau,\varepsilon)$' *with respect to* '$\varepsilon$', *about the point* $\varepsilon = 0$. This point is said to be the *distinguished point* of the series. The sign '~' is said to be the *sign of asymptotic correspondence*. The variable '$\varepsilon$' is said to be the *scaling* (or *similarity*) *parameter* of each one of the following objects: (i) the asymptotic series '$F_{[\infty,\nu]}(\tau,\varepsilon)$', (ii) the functional form '$F(\tau,\varepsilon)$', (iii) either of the associated functions $F(\tau,\ )$ and $F$ of the functional form'$F(\tau,\varepsilon)$'.

3) For each $n \in \omega_\mu$, for each $\langle \tau,\varepsilon \rangle \in \mathrm{T} \times [0,1)$: the functional form $f_{(n)}(\tau)$ is said to be the *n*th *non-scaled partial asymptotic image of the functional form* $F(\tau,\varepsilon)$ *with respect to* $\varepsilon$, whereas the functional form $F_{(n)}(\tau,\varepsilon)$ is said to be the *n*th *scaled partial asymptotic image of the functional form* $F(\tau,\varepsilon)$ *with respect to* $\varepsilon$. Consequently, the associated function $f_{(n)}$ of the functional form $f_{(n)}(\tau)$ and the associated function $F_{(n)}$ of the functional form $F_{(n)}(\tau,\varepsilon)$ are respectively said to be the *n*th *non-scaled* and *n*th *scaled partial asymptotic images of the associated function F of the functional form* $F(\tau,\varepsilon)$ *with respect to* '$\varepsilon$'.

4) For each $m \in \omega_\mu$, for each $\langle \tau,\varepsilon \rangle \in \mathrm{T} \times [0,1)$: the functional form $F_{[m,\mu]}(\tau,\varepsilon)$ is said to be the *m*th *cumulative asymptotic image of* (or *m*th *asymptotic approximation to*) *the functional form* $F(\tau,\varepsilon)$ *with respect to* $\varepsilon$. In this case, instead of the *prepositive* quantifier "*m*th", either *postpositive* quantifier "*of m*th *order*" or "*of order m*" can alternatively be used before the qualifier "*with respect to* $\varepsilon$". The above definition applies with "function $F_{[m,\mu]}$" and "function *F*" instead of "functional form $F_{[m,\mu]}(\tau,\varepsilon)$" and "functional form $F(\tau,\varepsilon)$" respectively.

5) Given $m \in \omega_\mu$, the above item can *by definition* be formally restated in either of the following two ways.

a) For each $\langle \tau,\varepsilon \rangle \in \mathrm{T} \times [0,1)$: $F(\tau,\varepsilon) \approx F_{[m,\mu]}(\tau,\varepsilon)$ or $F(\tau,\varepsilon) \sim F_{[m,\mu]}(\tau,\varepsilon) + O(\varepsilon^{m+1})$.

b) $F \approx F_{[m,\mu]}$.

The sign '≈', as defined above for each *given* $m \in \omega_\mu$, will be called the *sign of a cut cumulative asymptotic correspondence.* The symbol '$O(\varepsilon^{m+1})$' stands *ad hoc* for the cut reminder of the power asymptotic series, i.e. $O(\varepsilon^{m+1}) \doteqdot F_{[\infty,m+1]}(\tau,\varepsilon)$. If the power series $F_{[\infty,0]}(\tau,\varepsilon)$ is a *Maclaurin* one then I shall write '$o(\varepsilon^{m+1})$' instead of '$O(\varepsilon^{m+1})$', the understanding being that $o(\varepsilon^{m+1})$ is *the remainder of the Maclaurin series, of the order of $\varepsilon^{m+1}$ in Peano's form.*

6) The above five items apply with '*G*', '*g*', and '*v*', and also with '*H*', '*h*', and '*λ*', in place of '*F*', '*f*', and '*μ*' respectively.•

**Comment 5.2.** In the first sentence of Definition 5.1, the description "natural number $\mu \in \omega_0$" can be replaced with this one: "*natural integer* $\mu \in I_{-\infty,\infty}$". In this case, the occurrences of '$\omega_{\mu+1}$' and '$\omega_\mu$' in the item 1 of Definition 5.1 should be replaced with occurrences of '$I_{\mu+1,\infty}$' and '$I_{\mu,\infty}$' respectively. However, one of the main purposes of this paper is to demarcate the difference between the asymptotic power series of a functional form, which is not the Taylor series of that function, and the asymptotic power series of a functional form, which is not its Taylor series. Also, no asymptotic power series relevant to water waves involve any negative powers of a scaling parameter. Therefore, Definition 5.1 is confined to the case, where *μ*, *v*, and *λ* are natural numbers, and not natural integers.•

**Comment 5.3.** In Definition 5.1(1), the function $F(\tau,\ )$ has been assumed to be defined on $[0,1)$ exclusively for the sake of definiteness. The case, where $F(\tau,\ )$ is defined on $[0,-1)$, can be considered analogously. In particular, in this case, the limiting transition '$\varepsilon \to +0$' in (5.16) and (5.17), and also in all other relevant relations below in this section, should be replaced by '$\varepsilon \to -0$'. It is understood that if $F(\tau,\ )$ is defined on $(-1,1)$ and is continuous in a neighborhood of the point $\varepsilon = 0$ then either one of the above two limiting transitions can be replaced by '$\varepsilon \to 0$'.•

**Comment 5.4.** 1) In accordance with Definition 5.1, '$F$', '$f$', '*μ*', 'T', and '*τ*' (e.g.) are ellipses (place-holders), which can be replaced by various specific variables or constants. Particularly, '$\tau$' is an ellipsis for any string of variables as '$t,\underline{x}$', '$t,\underline{x}_2$', '$\underline{x}$', or '$\underline{x}_2$'.

2) Equation (5.20) should, as usual, be understood syntactically, i.e. in the sense that '$F_{[\infty,\mu]}(\tau,\varepsilon)$' is the definiendum, which will be used instead of the definiens '$\sum_{n=\mu}^{\infty} F_{(n)}(\tau,\varepsilon)$'. The problem of convergence of the infinite sum is unimportant in this case.•

**Comment 5.5.** Definition 5.1(1) is in agreement with the conventional general definition of the infinite asymptotic series of a functional form in terms of an infinite sequence of given *gauge* *functional forms* '$\delta_n(\varepsilon)$' instead of '$\varepsilon^n$' subject to $n \in \omega_0$ (see, e.g. Erdélyi [1956, p. 12, equation (5)] or Van Dyke [1975, p. 27, equation (3.11)]). Definition 5.1(1) is also in agreement with the definition of the asymptotic series of a functional form relative to an infinite point as given, e.g., in Smirnov [1964, vol. III, part 1, art. 106, equations (127) and (128)]).•

**Comment 5.6.** The sign '$\approx$', introduced by Definition 5.1(5), should not be confused with the conventional sign '$\cong$' of approximate equality. The latter can, for instance, be defined by the following definition.

Given $\delta \in (0,\ 0.1]$, for each $x \in (-\infty,+\infty)$ and for each $x_0 \in (-\infty,+\infty)$:

$$x \cong x_0 \text{ if and only if } x \in [x_0 - \delta/2\,, x_0 + \delta/2].$$

In this case, the relation '$x \cong x_0$' can also be written as '$x = x_0 + O(\delta)$'.•

**Corollary 5.1.** If '$F(\tau,\ \varepsilon)$' has an infinite asymptotic power series about the point $\varepsilon = 0$ then that series is *unique*.

**Proof:** The corollary immediately follows from Definition 5.1.(1)•

**Corollary 5.2.** It immediately follows from Definition 5.1(1,6) that, for instance, for each $\langle \tau,\varepsilon \rangle \in \mathrm{T} \times [0,1)$: if $H(\tau,\varepsilon) \doteqdot F(\tau,\varepsilon) + G(\tau,\varepsilon)$ then $F_{[\infty,\lambda]}(\tau,\varepsilon) = F_{[\infty,\mu]}(\tau,\varepsilon) \pm G_{[\infty,\nu]}(\tau,\varepsilon)$ subject to $\lambda = \min\{\mu,\nu\}$. In this case,

a) if $\mu > \nu$ than $h_{(n)}(\tau) = g_{(n)}(\tau)$ for each $n \in \omega_{\nu,\mu-1}$;

b) if $\nu > \mu$ than $h_{(n)}(\tau) = f_{(n)}(\tau)$ for each $n \in \omega_{\mu,\nu-1}$;

c) $h_{(n)}(\tau) = f_{(n)}(\tau) + g_{(n)}(\tau)$ for each $n \in \omega_{\max\{\mu,\nu\}}$.•

**Theorem 5.1.** Given a set T, for each $\tau \in \mathrm{T}$: if $F(\tau,\varepsilon) = 0$ for each $\varepsilon \in [0,1)$ then $f_{(m)}(\tau) = 0$ for each $n \in \omega_0$.

**Proof:** Under the hypothesis (antecedent) of the theorem, (5.16)–(5.22) become

$$f_{(0)}(\tau) = 0\,, \tag{5.16$_0$}$$

$$f_{(m)}(\tau) = -\lim_{\varepsilon \to +0} \varepsilon^{-m} F_{[m-1,0]}(\tau,\varepsilon) \text{ for each } m \in \omega_1\,, \tag{5.17$_0$}$$

$$F_{[m-1,0]}(\tau,\varepsilon) \doteqdot \sum_{n=0}^{m-1} F_{(n)}(\tau,\varepsilon) \doteqdot \sum_{n=0}^{m-1} f_{(n)}(\tau)\varepsilon^n\,, \tag{5.18$_0$}$$

$$F_{[0,0]}(\tau,\varepsilon) \doteqdot \sum_{n=0}^{0} F_{(n)}(\tau,\varepsilon) = F_{(0)}(\tau,\varepsilon)\,, \tag{5.19$_0$}$$

$$F_{[\infty,0]}(\tau,\varepsilon) \doteqdot \lim_{m\to\infty} F_{[m-1,0]}(\tau,\varepsilon), \tag{5.20$_0$}$$

$$F_{(n)}(\tau,\varepsilon) \doteqdot f_{(n)}(\tau)\varepsilon^n \text{ for each } n \in \omega_0, \tag{5.21$_0$}$$

$$F_{[-1,0]}(\tau,\varepsilon) \doteqdot 0, \tag{5.22$_0$}$$

respectively. By (5.16$_0$), it follows from (5.19$_0$) and from (5.21$_0$) at $n \doteqdot 0$ that $F_{[0,0]}(\tau,\varepsilon) = 0$. Hence, (5.17$_0$) at $m \doteqdot 1$ yields $f_{(1)}(\tau) = 0$, so that $F_{[1,0]}(\tau,\varepsilon) = 0$, by (5.18$_0$) at $m \doteqdot 2$. Consequently, $f_{(2)}(\tau) = 0$, by (5.17$_0$) at $m \doteqdot 2$; and so on ad infinitum. The validity of the equation $f_{(m)}(\tau) = 0$ for each $m \in \omega_1$ can be proved formally by induction on values of '$m$' as follows. Given $l \in \omega_1$, assume that $f_{(n)}(\tau) = 0$ for each $n \in \omega_{0,l}$. In this case, it follows from (5.18$_0$) at $m \doteqdot l+1$ that $F_{[l,0]}(\tau,\varepsilon) = 0$ for each $\varepsilon \in [0,1)$. Hence, equation (5.17$_0$) at $m = l+1$ yields $f_{(l+1)}(\tau) = 0$. QED.•

**Comment 5.7.** The equation '$F(\tau,\varepsilon) = 0$' can be algebraic, differential, integral, integro-differential, differential-substitutional (as a boundary or initial condition), etc. Therefore, Theorem 5.1 is a basis for constructing various perturbation theories.•

**Corollary 5.3.** Given a set T, for each $\tau \in \mathrm{T}$, for each $\varepsilon \in [0,1)$: if $F(\tau,\varepsilon) = 0$ then $F(\tau,\varepsilon) \sim 0$.

**Proof:** It immediately follows from Theorem 5.1 that if $F(\tau,\varepsilon) = 0$ then

$$F(\tau,\varepsilon) \sim F_{[\infty,0]}(\tau,\varepsilon) \doteqdot \sum_{n=0}^{\infty} 0\varepsilon^n = 0. \tag{5.15$_0$}$$

QED.•

**Comment 5.8.** Substitution of '0' for '$F(\tau,\varepsilon)$' into (5.15$_0$) yields $0 \sim 0$.•

**Definition 5.2.** '$\mathrm{O}_+(\tau,\varepsilon)$', '$\mathrm{O}_-(\tau,\varepsilon)$', '$\mathrm{O}(\tau,\varepsilon)$', and '$\mathrm{o}(\tau)$' are called the *null functional forms*, while their associated functions $\mathrm{O}_+$, $\mathrm{O}_-$, O, and o are called the *null functions*, on $\mathrm{T} \times [0,1)$, $\mathrm{T} \times [0,-1)$, $\mathrm{T} \times (-1,1)$, and T respectively, provided that they are defined as follows.

a) $\mathrm{O}_+(\tau,\varepsilon) \doteqdot F(\tau,\varepsilon) = 0$ for each $\langle \tau,\varepsilon \rangle \in \mathrm{T} \times [0,1)$ subject to Theorem 5.1.

b) $\mathrm{O}_-(\tau,\varepsilon) \doteqdot F(\tau,\varepsilon) = 0$ for each $\langle \tau,\varepsilon \rangle \in \mathrm{T} \times [0,-1)$ subject to the variant of Theorem 5.1 with '$[0,-1)$' in place of '$[0,1)$'.

c) $\mathrm{O}(\tau,\varepsilon) \doteqdot F(\tau,\varepsilon) = 0$ for each $\langle \tau,\varepsilon \rangle \in \mathrm{T} \times (-1,1)$ subject to the variant of Theorem 5.1 with '$(-1,1)$' in place of '$[0,1)$'.

d) $\mathrm{o}(\tau) \doteqdot f_{(n)}(\tau) = 0$ for each $\tau \in \mathrm{T}$ and for each $n \in \omega_0$ subject to any one of the above three variants of Theorem 5.1.

Thus, $\mathrm{O}_+$, $\mathrm{O}_-$, O, and o are the pertinent *specifications* (*restrictions*) of the *universal null function* $C_0$.•

**Definition 5.3.** 1) A functional form '$\Theta_+(\tau,\varepsilon)$', defined on $\mathrm{T} \times (0,\infty)$ so that $\Theta_+ \neq \mathrm{O}_+$, is said to have '$\mathrm{O}_+(\tau,\varepsilon)$' as *its asymptote*, and accordingly the function $\Theta_+$ is said to have $\mathrm{O}_+$ as *its asymptote*, if and only if for each $\langle \tau,\varepsilon \rangle \in \mathrm{T} \times [0,1)$:

$$\Theta_+(\tau,\varepsilon) \sim \sum_{n=0}^{\infty} 0\varepsilon^n = 0, \tag{5.23}$$

i.e. if and only if for each $\tau \in \mathrm{T}$:

$$\theta_{(n)}(\tau) = 0 \text{ for each } n \in \omega_0, \tag{5.24}$$

where '$\theta_{(n)}(\tau)$ is defined by the variants of ($5.16_0$) and ($5.17_0$) subject io ($5.18_0$)–($5.22_0$) with '$\theta$' and '$\Theta_+$' in place of '$f$' and '$F$' respectively.

2) The item 1 applies (a) with '$\Theta_-$' and '$[0,-1)$' or (b) with 'Θ' and '$(-1,1)$' in place of '$\Theta_+$' and '$[0,1)$' respectively.•

**Comment 5.9.** There is an indefinite number of functional forms of each of the three classes defined in Definition 5.3. This is illustrated by the following example.

Let '$a(\tau)$' and '$b(\tau)$' be given real-valued functional forms defined and bounded for each $\tau \in \mathrm{T}$, subject to $a(\tau) > 0$ for each $\tau \in \mathrm{T}$. Then the functional form '$\Theta_+(\tau,\varepsilon)$', defined by

$$\Theta_+(\tau,\varepsilon) \doteqdot b(\tau) e^{-a(\tau)/\varepsilon} \text{ for each } (\tau,\varepsilon) \in \mathrm{T} \times [0,1), \tag{5.25}$$

has the properties indicated in Definition 5.3. Indeed, in this case it follows that for each $m \in \omega_0$:

$$\begin{aligned} \theta_{(m)}(\tau) &= \lim_{\varepsilon \to +0} \varepsilon^{-m} \Theta_+(\tau,\varepsilon) = b(\tau) \lim_{\xi \to +\infty} \xi^m e^{-a(\tau)\xi} \\ &= b(\tau) \lim_{\xi \to +\infty} \left( \frac{d^m \xi^m}{d\xi^m} \right) \left( \frac{d^m e^{a(\tau)\xi}}{d\xi^m} \right)^{-1} = m! [a(\tau)]^{-m} b(\tau) \lim_{\xi \to +\infty} e^{-a(\tau)\xi} = 0, \end{aligned} \tag{5.26}$$

In developing this train of equations for solving the pertinent indeterminate functional form of the type of '$\infty/\infty$', after the substitution $\xi = 1/\varepsilon$ subject to $\varepsilon \in (0,1)$, use has repeatedly been made $m$ times of the relevant versions of the *l'Hospitale rule* (see, e.g., Smirnov [1964, vol. I, art. 65, pp. 153–155]). Given $p \in \omega_0$, given $q \in \omega_0$, the functional form '$b(\tau)\varepsilon^p e^{-a(\tau)/\varepsilon^q}$' has the like property.•

**Corollary 5.4:** ***The inverse of Corollary 5.1*.** Infinitely many mutually different functions may have one and the same asymptotic power series. Hence, a function is not uniquely determined by its asymptotic power series.

**Proof:** Let a functional form '$F(\tau,\varepsilon)$' have an infinite asymptotic power series, for instance on $\mathrm{T}\times[0,1)$, with respect to '$\varepsilon$' about the point $\varepsilon=0$. Then, by Definition 5.3 and Comment 5.9, the functional form '$F(\tau,\varepsilon)+\Theta_{+}(\tau,\varepsilon)$' has the same asymptotic series. QED.•

**Comment 5.10.** The conjunctions of Corollaries 5.1 and 5.3 means that the mapping from the class of functional forms, having asymptotic power series, onto the class of the latter is *strictly surjective*, i.e. *surjective but not bijective*.•

**Theorem 5.2:** ***The asymptotic power series of the direct product of two functions*.** The relevance of the above title to the following theorem is established by the fact that the associated function of the product of two functional forms '$F(\tau,\varepsilon)G(\tau,\varepsilon)$' is often denoted by '$F\otimes G$' and is called "*the direct product of the functions $F$ and $G$*".

Given $\mu\in\omega_0$, given $\nu\in\omega_0$, let for each $\langle\tau,\varepsilon\rangle\in\mathrm{T}\times[0,1)$:

$$F(\tau,\varepsilon)\sim F_{[\infty,\mu]}(\tau,\varepsilon),G(\tau,\varepsilon)\sim G_{[\infty,\nu]}(\tau,\varepsilon)\,, \tag{5.27}$$

where, in accordance with the appropriate variants of (5.15),

$$F_{[\infty,\mu]}(\tau,\varepsilon)=\sum_{m=\mu}^{\infty}f_{(m)}(\tau)\varepsilon^{m},G_{[\infty,\nu]}(\tau,\varepsilon)=\sum_{n=\nu}^{\infty}g_{(n)}(\tau)\varepsilon^{n}\,. \tag{5.28}$$

Then for each $\langle\tau,\varepsilon\rangle\in\mathrm{T}\times[0,1)$:

$$H(\tau,\varepsilon)\stackrel{\rightarrow}{=}F(\tau,\varepsilon)G(\tau,\varepsilon)\sim H_{[\infty,\mu+\nu]}(\tau,\varepsilon)\,, \tag{5.29}$$

where

$$H_{[\infty,\mu+\nu]}(\tau,\varepsilon)\stackrel{\rightarrow}{=}\sum_{l=\mu+\nu}^{\infty}h_{(l)}(\tau)\varepsilon^{l} \tag{5.30}$$

subject to

$$h_{(l)}(\tau)=\sum_{m=\mu}^{l-\nu}f_{(m)}(\tau)g_{(l-m)}(\tau)=\sum_{n=\nu}^{l-\mu}f_{(l-n)}(\tau)g_{(n)}(\tau)\ \text{ for each } l\in\omega_{\mu+\nu}\,. \tag{5.31}$$

**Proof:** By (5.28), it follows that

$$F(\tau,\varepsilon)G(\tau,\varepsilon)\sim F_{[\infty,\mu]}(\tau,\varepsilon)G_{[\infty,\nu]}(\tau,\varepsilon)=\sum_{m=\mu}^{\infty}\sum_{n=\nu}^{\infty}f_{(m)}(\tau)g_{(n)}(\tau)\varepsilon^{m+n}\,. \tag{5.29$_1$}$$

Let $l \risingdotseq m+n$, so that $l \in \omega_{\mu+\nu}$. Consequently, the following two options are possible.

a) If '*l*' is employed as a new variable of summation instead of '*n*', so that $n = l - m$, then the domain of values of the variable '*m*' is determined by the conjunction of two relations: (i) $m \in \omega_\mu$, i.e $\mu \le m < \infty$, and (ii) given $l \in \omega_{\mu+\nu}$, $n = l - m \in \omega_\nu$, i.e. $\nu \le l - m < \infty$ or equivalently $m \le l - \nu$. Hence, given $l \in \omega_{\mu+\nu}$, $\mu \le m < \infty$ and $m \le l - \nu$, so that $\mu \le m \le l - \nu$, i.e. $m \in \omega_{\mu,l-\nu}$.

b) If '*l*' is employed as a new variable of summation instead of '*m*', so that $m = l - n$, then the domain of values of the variable '*n*' is determined by the conjunction of two relations: (i) $n \in \omega_\nu$, i.e. $\nu \le n < \infty$, and (ii) given $l \in \omega_{\mu+\nu}$, $m = l - n \in \omega_\mu$, i.e. $\mu \le l - n < \infty$ or equivalently $n \le l - \mu$. Hence, given $l \in \omega_{\mu+\nu}$, $\nu \le n \le l - \mu$, i.e. $n \in \omega_{\nu,l-\mu}$.

Therefore, the final expression in ($5.29_1$) can be developed as (5.30) subject to (5.31). QED.•

**Comment 5.11.** 1) Theorem 5.2 is of fundamental importance in constructing the recursive asymptotic theory in question. However, this theorem will not, as a rule, be mentioned explicitly either in making relevant statements or in their proofs.

2) In connection with Theorem 5.2, it is worthy of noticing that multiplication of asymptotic series other than power ones does not lead to an asymptotic series (see, e.g., Erdélyi [1956, pp. 17–20]. At the same time, Theorem 5.2 can be generalized somewhat as done below.

Let *μ, M, ν,* and *N* be *natural integers* such that $\mu \in I_{-\infty,\infty}$, $M \in I_{\mu,\infty}$, $\nu \in I_{-\infty,\infty}$, and $N \in I_{\nu,\infty}$. Let for each $\langle \tau, \varepsilon \rangle \in \mathrm{T} \times [0,1)$ (e.g.):

$$F_{[M,\mu]}(\tau,\varepsilon) = \sum_{m=\mu}^{M} f_{(m)}(\tau)\varepsilon^m, G_{[N,\nu]}(\tau,\varepsilon) = \sum_{n=\nu}^{N} g_{(n)}(\tau)\varepsilon^n. \quad (5.32)$$

In this case, the self-evident equality

$$F_{[M,\mu]}(\tau,\varepsilon)G_{[N,\nu]}(\tau,\varepsilon) = \sum_{m=\mu}^{M}\sum_{n=\nu}^{N} f_{(m)}(\tau)g_{(n)}(\tau)\varepsilon^{m+n} \quad (5.33)$$

can be developed in analogy with ($5.29_1$) by letting $l \risingdotseq m+n$, so that $l \in I_{\mu+\nu,M+N}$.

a) If '*l*' is employed as a new variable of summation instead of '*n*' then the domain of values of the variable '*m*' is determined by the conjunction of two relations: (i) $m \in I_{\mu,M}$, i.e. $\mu \le m \le M$, and (ii) given $l \in I_{\mu+\nu,M+N}$, $n = l - m \in I_{\nu,N}$, i.e. $\nu \le l - m \le N$ or equivalently $l - N \le m \le l - \nu$. Hence, given $l \in I_{\mu+\nu,M+N}$, $\mu \le m \le M$ and $l - N \le m \le l - \nu$, so that

$$\max\{\mu, l-N\} \le m \le \min\{M, l-\nu\}, \text{ i.e. } m \in I_{\max\{\mu,l-N\},\min\{M,l-\nu\}}. \tag{5.34}$$

b) If '*l*' is employed as a new variable of summation instead of '*m*' then the domain of values of the variable '*n*' is determined by the conjunction of two relations: (i) $n \in I_{\nu,N}$, i.e. $\nu \le n \le N$, and (ii) given $l \in I_{\mu+\nu,M+N}$, $m = l - n \in I_{\mu,M}$, i.e. $\mu \le l - n \le M$ or equivalently $l - M \le n \le l - \mu$. Hence, given $l \in I_{\mu+\nu,M+N}$, $\nu \le n \le N$ and $l - M \le n \le l - \mu$, so that

$$\max\{\nu, l-M\} \le n \le \min\{N, l-\mu\}, \text{ i.e. } m \in I_{\max\{\nu,l-M\},\min\{N,l-\mu\}}. \tag{5.35}$$

Therefore, the equality (5.33) can be developed thus;

$$F_{[M,\mu]}(\tau,\varepsilon)G_{[N,\nu]}(\tau,\varepsilon) = \sum_{l=\mu+\nu}^{M+N} h_{(l)}(\tau)\varepsilon^l \tag{5.36}$$

subject to

$$h_{(l)}(\tau) = \sum_{m=\max\{\mu,l-N\}}^{\min\{M,l-\nu\}} f_{(m)}(\tau)g_{(l-m)}(\tau) = \sum_{n=\max\{\nu,l-M\}}^{\min\{N,l-\mu\}} f_{(l-n)}(\tau)g_{(n)}(\tau) \text{ for each } l \in I_{\mu+\nu,M+N}. \tag{5.37}$$

As $M\to\infty$ and $N\to\infty$, the equalities (5.36) and (5.37) turn into the pertinent homographs of the equalities (5.30) and (5.31) respectively.•

### 5.3. The Maclaurin series of an analytical function as its asymptotic power series

**Theorem 5.3.** For each $\langle\tau,\varepsilon\rangle \in \mathrm{T}\times(-1,1)$, let the functional form $F(\tau,\varepsilon)$ have partial derivatives with respect to 'ε' of all orders and let $F(\tau,0) \ne 0$. Then

$$F(\tau,\varepsilon) = F_{[\infty,0]}(\tau,\varepsilon) = \sum_{n=0}^{\infty} f_{(n)}(\tau)\varepsilon^n, \tag{5.38}$$

where

$$f_{(n)}(\tau) \risingdotseq \frac{1}{n!}\left[\frac{\partial^n F(\tau,\varepsilon)}{\partial\varepsilon^n}\right]_{\varepsilon=0} \text{ for each } n \in \omega_0, \tag{5.39}$$

subject to

$$f_{(0)}(\tau) \risingdotseq \frac{1}{0!}\left[\frac{\partial^0 F(\tau,\varepsilon)}{\partial\varepsilon^0}\right]_{\varepsilon=0} \risingdotseq F(\tau,0) \ne 0,\ 0! \risingdotseq 1. \tag{5.40}$$

That is to say, $F(\tau,\varepsilon)$ has the Maclaurin series $F_{[\infty,\mu]}(\tau,\varepsilon)$ with respect to 'ε', which is, at the same time, the infinite asymptotic power series of $F(\tau,\varepsilon)$ about the point ε=0.

**Proof:** The theorem is proved from (5.16)–(5.18) with $\mu$=0 and with '$(-1,1)$' in place of '$[0,1)$'. Particularly, (5.16)) with $\mu$=0 becomes

$$f_{(0)}(\tau) = \lim_{\varepsilon \to 0} F(\tau, \varepsilon) = F(\tau, 0) \,. \tag{5.40}$$

At the same time, by (5.18), for each $m \in \omega_1$, the functional form $F_{[m-1,0]}(\tau, \varepsilon)$ is a polynomial of the order $m$-1 with respect to 'ε'. Therefore, for each $m \in \omega_1$:

$$\frac{\partial^m F_{[m-1,\mu]}(\tau, \varepsilon)}{\partial \varepsilon^m} = 0 \tag{5.41}$$

and hence (5.17) yields

$$f_{(m)}(\tau) = \lim_{\varepsilon \to 0} \varepsilon^{-m \cdot} F(\tau, \varepsilon) = \lim_{\varepsilon \to 0} \left[ \left( \frac{d^m \varepsilon^m}{d\varepsilon^m} \right)^{-1} \frac{\partial^m F(\tau, \varepsilon)}{\partial \varepsilon^m} \right] = \frac{1}{m!} \left[ \frac{\partial^m F(\tau, \varepsilon)}{\partial \varepsilon^m} \right]_{\varepsilon = 0} , \tag{5.42}$$

where use of the l'Hospitale rule has been made $m$ times with respect to the variable '$\varepsilon$' for solving the indeterminate functional form of the type of '$0/0$'; it goes without saying that

$$\frac{d^m \varepsilon^m}{d\varepsilon^m} = m! \,. \tag{5.42$_1$}$$ •

**Comment 5.12.** Given $\tau \in \mathrm{T}$, if the functional form $F(\tau, \varepsilon)$ has the Maclaurin series $F_{[\infty,0]}(\tau, \varepsilon)$ of a convergence radius 1 then, in accordance with (5.18) with $\mu$=0, the asymptotic series (5.15) of the pertinent functional form $F(\tau, \varepsilon)$ for each $\langle \tau, \varepsilon \rangle \in \mathrm{T} \times (-1,1)$ can be written as:

$$F(\tau, \varepsilon) \sim F_{[\infty,0]}(\tau, \varepsilon) = F_{[\infty,m]}(\tau, \varepsilon) + F_{[m+1,0]}(\tau, \varepsilon) \tag{5.15$_1$}$$

or as:

$$F(\tau, \varepsilon) = F_{[\infty,m]}(\tau, \varepsilon) + o(\varepsilon^{m+1}) \,, \tag{5.15$_2$}$$

for *any given* $m \in \omega_0$. In this case, $o(\varepsilon^{m+1})$ is *the remainder of the Maclaurin series, of the order of* $\varepsilon^{m+1}$ *in Peano's form.*•

## 5.4. Asymptotic power series of the pertinent basic bulk characteristics and bulk equations of a potential fluid flow

**Preliminary Remark 5.2.** In accordance with Hypothesis 5.1, the functional form $\mathrm{Z}(t, \underline{x}_2) \doteqdot \mathrm{Z}(t; \underline{x}_2, \varepsilon)$ is supposed to be defined for each $\langle t; \underline{x}_2, \varepsilon \rangle \in R \times [\underline{E}_2 \times [0,1)]$, i.e. for each $\langle t, \underline{x}_2 \rangle \in R \times \underline{E}_2$ and each $\varepsilon \in [0,1)$, and to be expandable into an asymptotic power series with respect to ‘ε’ about the point ε=0. At the same time, in accordance with Definitions 3.2–3.4, all asymptotic power series of bulk characteristics and bulk equations of a fluid flow in a perturbed liquid layer, which occur below in this subsection, are or are supposed to be included under either one of the two equivalent conditions:

“For each $t \in R$ and each $\underline{x} \in \underline{E}_3$” and “For each $\langle t, \underline{x} \rangle \in R \times \underline{E}_3$”.

However, $\mathrm{Z}(t; \underline{x}_2, \varepsilon)$ is an unknown. Therefore, all above-mentioned asymptotic power series should be regarded as *conditional*.•

### 5.4.1. The mass continuity equation

Substitution of (5.8) into (4.32) yields for each $i \in \omega_{1,3}$:

$$V_i(t, \underline{x}) \doteqdot V_i(t; \underline{x}, \varepsilon) \sim \sum_{n=1}^{\infty} \varepsilon^n v_{(n)i}(t, \underline{x}) \tag{5.43}$$

subject to

$$v_{(n)i}(t, \underline{x}) \doteqdot \nabla_i \phi_{(n)}(t, \underline{x}) \text{ for each } n \in \omega_1 . \tag{5.44}$$

At the same time, substitution of (5.8) into (4.34) yields

$$0 \sim \sum_{n=1}^{\infty} \varepsilon^n \underline{\nabla}^2 \phi_{(n)}(t, \underline{x}) \text{ for each } n \in \omega_1 . \tag{5.45}$$

Hence, by Theorem 5.1,

$$\Delta \phi_{(n)}(t, \underline{x}) \doteqdot \underline{\nabla}^2 \phi_{(n)}(t, \underline{x}) = 0 \text{ for each } n \in \omega_1 . \tag{5.46}$$

By (5.43), equation (4.3) is equivalent to

$$\underline{\nabla} \cdot \underline{v}_{(n)}(t, \underline{x}) = \sum_{i=1}^{3} \nabla_i v_{(n)i}(t, \underline{x}) = 0 \text{ for each } n \in \omega_1 . \tag{5.47}$$

At the same time, by (5.44), application of the operator $\nabla_i$ to both sides of (5.46) yields

$$\underline{\nabla}^2 v_{(n)i}(t, \underline{x}) = 0 \text{ for each } i \in \omega_{1,3} \text{ and each } n \in \omega_1 . \tag{5.48}$$

Equations (5.48) can also be deduced from (4.35) by (5.43).

### 5.4.2. The unsteady Bernoulli equation.

By the pertinent instance of Theorem 5.2, substitution of (5.8) into (4.40) yields

$$E_{\mathrm{k}}(t,\underline{x}) \doteqdot E_{\mathrm{k}}(t;\underline{x},\varepsilon) \sim \sum_{l=2}^{\infty} \varepsilon^{l} e_{\mathrm{k}(l)}(t,\underline{x}) \tag{5.49}$$

subject to

$$e_{\mathrm{k}(l)}(t,\underline{x}) = \frac{1}{2}\rho_9 \sum_{m=1}^{l-1}\sum_{i=1}^{3} v_{(m)i}(t,\underline{x}) v_{(l-m)i}(t,\underline{x}) \quad \text{for each } l \in \omega_2 . \tag{5.50}$$

Consequently, substitution of (5.8) into (4.50) subject to (4.40)–(4.41) and (4.47)–(4.49) yields:

$$P(t,\underline{x}) \doteqdot P(t;\underline{x},\varepsilon) \sim P_0(t) - \rho_0 gz - \varepsilon\rho_0 \frac{\partial\phi_{(1)}(t,\underline{x})}{\partial t} - \sum_{l=2}^{\infty} \varepsilon^{l} \left[ \rho_0 \frac{\partial\phi_{(l)}(t,\underline{x})}{\partial t} + e_{\mathrm{k}(l)}(t,\underline{x}) \right], \tag{5.51}$$

which is the asymptotic expansion of the local pressure $P(t;\underline{x},\varepsilon)$ at the liquid point $(t,\underline{x})$; $P_0(t)$ is the pressure above the free surface of the liquid layer that either equals 0 or $P_{\mathrm{a}}(t)$. The relation (5.51) can be rewritten as:

$$P(t;\underline{x},\varepsilon) \sim \sum_{l=0}^{\infty} \varepsilon^{l} p_{(l)}(t,\underline{x}) , \tag{5.51$_1$}$$

where

$$\begin{gathered} p_{(0)}(t,\underline{x}) \doteqdot P_0(t) + P_{\mathrm{hs}}(z) = P_0(t) - \rho_0 gz,\ p_{(1)}(t,\underline{x}) \doteqdot p_{\mathrm{d}(1)}(t,\underline{x}) \doteqdot -\rho_0 \frac{\partial\phi_{(1)}(t,\underline{x})}{\partial t}, \\ p_{(l)}(t,\underline{x}) \doteqdot p_{\mathrm{d}(l)}(t,\underline{x}) \doteqdot -\rho_0 \frac{\partial\phi_{(l)}(t,\underline{x})}{\partial t} - e_{\mathrm{k}(l)}(t,\underline{x}) \text{ for each } l \in \omega_2. \end{gathered} \tag{5.51$_2$}$$

In this case, by (4.52),

$$P_{\mathrm{d}}(t,\underline{x}) \doteqdot P_{\mathrm{d}}(t;\underline{x},\varepsilon) \sim \sum_{l=1}^{\infty} \varepsilon^{n} p_{(l)}(t,\underline{x}) \tag{5.51$_3$}$$

subject to the pertinent definitions of (5.51$_2$).

### 5.4.3. The momentum flux density tensor and the momentum flux density continuity equation

By the pertinent instance of Theorem 5.2 and in analogy with (5.49) and (5.50), substitution of (5.43) into (4.56) yields for each $i \in \omega_{1,3}$ and each $j \in \omega_{1,3}$:

$$E_{ij}(t,\underline{x}) \doteqdot E_{ij}(t;\underline{x},\varepsilon) \doteqdot \rho_0 V_i(t;\underline{x},\varepsilon) V_J(t;\underline{x},\varepsilon) \sim \sum_{l=2}^{\infty} \varepsilon^{l} e_{(l)ij}(t,\underline{x}) , \tag{5.52}$$

subject to

$$e_{(l)ij}(t,\underline{x}) = \rho_0 \sum_{m=1}^{l-1}\sum_{i=1}^{3} v_{(m)i}(t,\underline{x}) v_{(l-m)j}(t,\underline{x}) \text{ for each } l \in \omega_2 . \tag{5.53}$$

Particularly, for $l \doteqdot 2$ or $l \doteqdot 3$, equation (5.53) becomes

$$e_{(2)ij}(t,\underline{x}) = \rho_0 v_{(1)i}(t,\underline{x}) v_{(1)j}(t,\underline{x}) , \tag{5.53$_1$}$$

$$e_{(3)ij}(t,\underline{x}) = \rho_0 \left[ v_{(1)i}(t,\underline{x}) v_{(2)j}(t,\underline{x}) + v_{(2)i}(t,\underline{x}) v_{(1)j}(t,\underline{x}) \right]. \tag{5.53$_2$}$$

Substitution of (5.51$_1$) subject to (5.51$_2$) and of (5.52) into equation (4.54) or, alternatively, the above substitution along with substitution of (5.8) into equation (4.55) yields

$$S_{ij}(t,\underline{x}) \doteqdot S_{ij}(t;\underline{x},\varepsilon) \sim \sum_{l=0}^{\infty} \varepsilon^l s_{(l)ij}(t,\underline{x}) , \tag{5.54}$$

where

$$\begin{aligned}
s_{(o)ij}(t,\underline{x}) &\doteqdot P_0(t)\delta_{ij} - \rho_0 gz\left(\delta_{ij} - \delta_{i3}\delta_{j3}\right), \; s_{(1)ij}(t,\underline{x}) \doteqdot p_{\mathrm{d}(1)}(t,\underline{x})\delta_{ij} = -\rho_0 \frac{\partial \phi_{(1)}(t,\underline{x})}{\partial t}\delta_{ij} , \\
s_{(l)ij}(t,\underline{x}) &\doteqdot p_{\mathrm{d}(l)}(t,\underline{x})\delta_{ij} + e_{(l)ij}(t,\underline{x}) = -\left[ \rho_0 \frac{\partial \phi_{(l)}(t,\underline{x})}{\partial t} + e_{\mathrm{k}(l)}(t,\underline{x}) \right]\delta_{ij} + e_{(l)ij}(t,\underline{x}) \\
&= -\rho_0 \frac{\partial \phi_{(l)}(t,\underline{x})}{\partial t}\delta_{ij} + \left[ e_{(l)ij}(t,\underline{x}) - e_{\mathrm{k}(l)}(t,\underline{x})\delta_{ij} \right] \text{ for each } l \in \omega_2.
\end{aligned} \tag{5.55}$$

The successive non-scaled 3×3 tensors, defined by (5.55), satisfy the equations (cf. (4.57) subject to (4.57$_1$)–(4.57$_4$)):

$$\begin{aligned}
\sum_{j=1}^{3} \nabla_j s_{(0)ij}(t,\underline{x}) &\doteqdot \sum_{j=1}^{3} \nabla_j \left[ P_0(t)\delta_{ij} - \rho_0 gz\left(\delta_{ij} - \delta_{i3}\delta_{j3}\right)\right] \\
&= \nabla_i P_0(t) - \rho_0 g\left(\nabla_i z - \delta_{i3}\nabla_3 z\right) = \rho_0 g \delta_{i3}(1-1) = 0,
\end{aligned} \tag{5.56$_0$}$$

$$\sum_{j=1}^{3} \nabla_j s_{(1)ij}(t,\underline{x}) \doteqdot \sum_{j=1}^{3} \nabla_j p_{\mathrm{d}(1)}(t,\underline{x})\delta_{ij} = -\rho_0 \frac{\partial \nabla_i \phi_{(1)}(t,\underline{x})}{\partial t} = -\rho_0 \frac{\partial v_{(1)i}(t,\underline{x})}{\partial t} \tag{5.56$_1$}$$

$$\begin{aligned}
\sum_{j=1}^{3} \nabla_j s_{(l)ij}(t,\underline{x}) &\doteqdot \sum_{j=1}^{3} \nabla_j \left[ p_{\mathrm{d}(l)}(t,\underline{x})\delta_{ij} + e_{(l)ij}(t,\underline{x}) \right] = -\rho_0 \frac{\partial \nabla_i \phi_{(l)}(t,\underline{x})}{\partial t} \\
&- \nabla_i e_{\mathrm{k}(l)}(t,\underline{x}) + \sum_{j=1}^{3} \nabla_j e_{(l)ij}(t,\underline{x}) = -\rho_0 \frac{\partial v_{(l)i}(t,\underline{x})}{\partial t} \text{ for each } l \in \omega_2.
\end{aligned} \tag{5.56$_2$}$$

In developing the final result in equation (5.56$_2$), use has been made of the equation:

$$\sum_{j=1}^{3} \nabla_j e_{(l)ij}(t,\underline{x}) - \nabla_i e_{\mathrm{k}(l)}(t,\underline{x}) = 0 \text{ for each } l \in \omega_2 , \tag{5.57}$$

which follows from (4.57$_4$) by (5.49) and (5.52). Alternatively, (5.57) can be proved straightforwardly as follows.

By (5.8), equations (4.3) and (4.33) yield

$$\underline{\nabla} \cdot \underline{v}_{(n)} = \sum_{j=1}^{3} \nabla_j v_{(n)j} = 0\,, \tag{5.58}$$

$$(\underline{\nabla} \wedge \underline{v}_{(n)})_i \doteqdot \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \nabla_j v_{(n)k} = 0 \text{ for each } i \in \omega_{1,3}\,, \tag{5.59}$$

for each $n \in \omega_1$. Therefore, given $n \in \omega_2$, given $p \in \omega_{1,n-1}$: the following tautology, analogous to (4.36), is established straightforwardly with the help of (4.37) and (5.58):

$$\begin{aligned} \underline{v}_{(n)} \wedge (\underline{\nabla} \wedge \underline{v}_{(n-p)})_i &= \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \varepsilon_{ijk} v_{(n)j} \varepsilon_{klm} \nabla_l v_{(n-p)m} \\ &= \sum_{j=1}^{3} v_{(n)j} \left( \nabla_i v_{(n-p)j} - \nabla_j v_{(n-p)i} \right) = 0. \end{aligned} \tag{5.60}$$

By (5.59) and (5.60), it follows from (5.53) that

$$\sum_{j=1}^{3} \nabla_j e_{(l)ij} = \rho_0 \sum_{j=1}^{3}\sum_{m=1}^{l-1} v_{(l-m)j} \nabla_j v_{(m)i} = \rho_0 \sum_{j=1}^{3}\sum_{m=1}^{l-1} v_{(l-m)j} \nabla_i v_{(m)j} \text{ for each } l \in \omega_2\,. \tag{5.61}$$

Set $m' = l - m$ in (5.61) and then omit all occurrences of the prime on '$m$' in the result. The half-sum of the variant of (5.61) so obtained and of (5.61) itself can be written as

$$\sum_{j=1}^{3} \nabla_j e_{(l)ij} = \frac{1}{2} \rho_0 \nabla_i \sum_{j=1}^{3}\sum_{m=1}^{l-1} v_{(m)j} v_{(l-m)j} = \nabla_i e_{\mathrm{k}(l)} \text{ for each } l \in \omega_2\,, \tag{5.62}$$

where use of (5.50) has been made in writing the final result. Thus, (5.62) is a tautology, and it coincides with (5.57). QED.

Thus, equations ($5.56_1$) and ($5.56_2$) are tautologies

$$\rho_0 \frac{\partial v_{(l)i}(t,\underline{x})}{\partial t} + \sum_{j=1}^{3} \nabla_j s_{(l)ij}(t,\underline{x}) = 0 \quad \text{for each } l \in \omega_1\,, \tag{5.63}$$

which, along with equation ($5.56_0$), are non-scaled successive asymptotic approximations to the tautological equation (4.53) with respect to successive powers of *ka*.

### 5.4.4. The energy continuity equation

By the pertinent instance of Theorem 5.2 with $\mu \doteqdot 1$ and $\nu \doteqdot 1$, substitution of (5.8) and (5.43) into (4.65) yields for each $i \in \omega_{1,3}$

$$Q_{*i}(t,\underline{x}) \doteqdot Q_{*i}\left(t;\underline{x},\varepsilon\right) \sim \sum_{l=2}^{\infty} \varepsilon^l q_{*(l)\mathrm{i}}(t,\underline{x}) \tag{5.64}$$

subject to

$$q_{*(l)\mathrm{i}}(t,\underline{x}) \doteqdot -\rho_0 \sum_{m=1}^{l-1} v_{(m)i}(t,\underline{x}) \frac{\partial \phi_{(l-m)}(t,\underline{x})}{\partial t} = -\rho_0 \sum_{n=1}^{l-1} v_{(l-n)i}(t,\underline{x}) \frac{\partial \phi_{(n)}(t,\underline{x})}{\partial t} \text{ for each } l \in \omega_2 . \quad (5.65)$$

Particularly, for $l \doteqdot 2$ or $l \doteqdot 3$, equation (5.65) yields

$$q_{*(2)i} = -\rho_0 v_{(1)i} \frac{\partial \phi_{(1)}}{\partial t}, \; q_{*(3)i} = -\rho_0 \left( v_{(1)i} \frac{\partial \phi_{(2)}}{\partial t} + v_{(2)i} \frac{\partial \phi_{(1)}}{\partial t} \right). \quad (5.65_1)$$

Now, by (5.43) and (5.52), equation (4.61a) yields;

$$Q_i(t,\underline{x}) \doteqdot Q_i(t;\underline{x},\varepsilon) \sim \sum_{l=1}^{\infty} \varepsilon^l q_{(l)\mathrm{i}}(t,\underline{x}) , \quad (5.66)$$

where

$$q_{(1)\mathrm{i}}(t,\underline{x}) \doteqdot P_0(t) v_{(1)i}(t,\underline{x}), \;\; q_{(l)\mathrm{i}}(t,\underline{x}) = q_{*(l)\mathrm{i}}(t,\underline{x}) + P_0(t) v_{(l)i}(t,\underline{x}) \text{ for each } l \in \omega_2 , \quad (5.67)$$

subjest to (5.44). and (5.53). Substituting (5.49), (5.52), and (5.54) into the pertinent terms of (4.66) and making use of (5.47) yields

$$\frac{\partial e_{\mathrm{k}(l)}(t,\underline{x})}{\partial t} = -\sum_{i=1}^{3} \nabla_i q_{(l)i}(t,\underline{x}) = -\sum_{i=1}^{3} \nabla_i q_{*(l)i}(t,\underline{x}) \text{ for each } l \in \omega_2 . \quad (5.68)$$

# 6. A recursive asymptotic analysis of the boundary conditions

## 6.1. Asymptotic expansions of $\Xi^{(m)}(t,\underline{x}_2)$, $\mathrm{H}^{(m)}(t,\underline{x}_2)$, and $\mathrm{Z}^{m}(t,\underline{x}_2)$

**1.** By (5.8), equations (4.86) yield

$$\Phi^{(m)}(t,\underline{x}_2) \sim \sum_{n=1}^{\infty} \varepsilon^n \phi_{(n)}^{(m)}(t,\underline{x}_2) \text{ for each } m \in \omega_0 , \quad (6.1)$$

where

$$\phi_{(n)}^{(m)}(t,\underline{x}_2) \doteqdot \left[ \frac{\partial^m \phi_{(n)}(t,\underline{x})}{\partial z^m} \right]_{z=0} \text{ for each } m \in \omega_0 \text{ and each } n \in \omega_1 , \quad (6.2)$$

the understanding being that

$$\phi_{(n)}^{(0)}(t,\underline{x}_2) \doteqdot \left[ \phi_{(n)}(t,\underline{x}) \right]_{z=0} \text{ for each } n \in \omega_1 . \quad (6.2_0)$$

**2.** By (4.87) and (4.88), it follows from (5.43) and (5.44) that

$$V_i^{(m)}(t,\underline{x}_2) \doteqdot \nabla_i \Phi^{(m)}(t,\underline{x}_2) \sim \sum_{n=1}^{\infty} \varepsilon^n \nabla_i \phi_{(n)}^{(m)}(t,\underline{x}_2) \doteqdot \sum_{n=1}^{\infty} \varepsilon^n v_{(n)i}^{(m)}(t,\underline{x}_2) \text{ for each } i \in \omega_{1,2},$$

$$V_3^{(m)}(t,\underline{x}_2) \doteqdot \Phi^{(m+!)}(t,\underline{x}_2) \sim \sum_{n=1}^{\infty} \varepsilon^n \phi_{(n)}^{(m+1)}(t,\underline{x}_2) \doteqdot \sum_{n=1}^{\infty} \varepsilon^n v_{(n)3}^{(m)}(t,\underline{x}_2), \tag{6.3}$$

for each $m \in \omega_0$,

where

$$v_{(n)i}^{(m)}(t,\underline{x}_2) \doteqdot \nabla_i \phi_{(n)}^{(m)}(t,\underline{x}_2) = \left[\frac{\partial^m \nabla_i \phi_{(n)}(t,\underline{x})}{\partial z^m}\right]_{z=0} = \left[\frac{\partial^m v_{(n)i}(t,\underline{x})}{\partial z^m}\right]_{z=0} \text{ for each } i \in \omega_{1,2},$$

$$v_{(n)3}^{(m)}(t,\underline{x}_2) \doteqdot \phi_{(n)}^{(m+1)}(t,\underline{x}_2) = \left[\frac{\partial^{m+1} \phi_{(n)}(t,\underline{x})}{\partial z^{m+!}}\right]_{z=0} = \left[\frac{\partial^m v_{(n)3}(t,\underline{x})}{\partial z^m}\right]_{z=0}, \tag{6.4}$$

for each $m \in \omega_0$, and each $n \in \omega_1$.

the understanding being that

$$v_{(n)i}^{(0)}(t,\underline{x}_2) \doteqdot \nabla_i \phi_{(n)}^{(0)}(t,\underline{x}_2) = \left[\nabla_i \phi_{(n)}(t,\underline{x})\right]_{z=0} = \left[v_{(n)i}(t,\underline{x})\right]_{z=0} \text{ for each } i \in \omega_{1,2},$$

$$v_{(n)3}^{(0)}(t,\underline{x}_2) \doteqdot \phi_{(n)}^{(1)}(t,\underline{x}_2) = \left[\frac{\partial \phi_{(n)}(t,\underline{x})}{\partial z}\right]_{z=0} = \left[v_{(n)3}(t,\underline{x})\right]_{z=0}, \tag{6.4$_0$}$$

for each $n \in \omega_1$.

By (5.47) and (5.48), the functional forms $v_{(n)i}^{(m)}(t,\underline{x}_2)$ with $i \in \omega_{1,2}$ and $v_{(n)3}^{(m)}(t,\underline{x}_2)$, as defined by (6.4), are interrelated in the following ways:

$$v_{(n)i}^{(m)}(t,\underline{x}_2) \doteqdot \nabla_i \phi_{(n)}^{(m)}(t,\underline{x}_2) = \left[\frac{\partial^m \nabla_i \phi_{(n)}(t,\underline{x})}{\partial z^m}\right]_{z=0} = \nabla_i \left[\frac{\partial^{m-1} \nabla_3 \phi_{(n)}(t,\underline{x})}{\partial z^{m-1}}\right]_{z=0}$$

$$= \nabla_i \left[\frac{\partial^{m-1} v_{(n)3}(t,\underline{x})}{\partial z^{m-1}}\right]_{z=0} = \nabla_i v_{(n)3}^{(m-1)}(t,\underline{x}_2) \tag{6.4$_1$}$$

for each $i \in \omega_{1,2}$, each $m \in \omega_1$, and each $n \in \omega_1$,

$$v_{(n)3}^{(m)}(t,\underline{x}_2) = \left[\frac{\partial^m v_{(n)3}(t,\underline{x})}{\partial z^m}\right]_{z=0} = \left[\frac{\partial^{m-1} \nabla_3 v_{(n)3}(t,\underline{x})}{\partial z^{m-1}}\right]_{z=0}$$

$$= -\left[\frac{\partial^{m-1}}{\partial z^{m-1}} \sum_{i=1}^{2} \nabla_i v_{(n)i}(t,\underline{x})\right]_{z=0} = -\sum_{i=1}^{2} \nabla_i \left[\frac{\partial^{m-1} v_{(n)i}(t,\underline{x})}{\partial z^{m-1}}\right]_{z=0} = -\sum_{i=1}^{2} \nabla_i v_{(n)i}^{(m-1)}(t,\underline{x}_2) \tag{6.4$_2$}$$

for each $m \in \omega_1$, and each $n \in \omega_1$.

$$
\begin{aligned}
v_{(n)3}^{(m)}(t,\underline{x}_2) &= \left[\frac{\partial^m v_{(n)3}(t,\underline{x})}{\partial z^m}\right]_{z=0} = \left[\frac{\partial^{m-2}}{\partial z^{m-2}}\nabla_3{}^2 v_{(n)3}(t,\underline{x})\right]_{z=0} \\
&= -\left[\frac{\partial^{m-2}}{\partial z^{m-2}}\sum_{i=1}^{2}\nabla_i{}^2 v_{(n)3}(t,\underline{x})\right]_{z=0} = -\sum_{i=1}^{2}\nabla_i{}^2\left[\frac{\partial^{m-2} v_{(n)3}(t,\underline{x})}{\partial z^{m-2}}\right]_{z=0} \\
&= -\sum_{i=1}^{2}\nabla_i{}^2 v_{(n)3}^{(m-2)}(t,\underline{x}_2) \text{ for each } m\in\omega_2, \text{ and each } n\in\omega_1.
\end{aligned}
\tag{6.4$_3$}
$$

**3.** By (4.91) and (4.92), it follows from (5.49) and (5.50) that

$$
E_{\mathrm{k}}^{(m)}(t,\underline{x}_2) \sim \sum_{n=2}^{\infty}\varepsilon^n e_{\mathrm{k}(n)}^{(m)}(t,\underline{x}_2) \text{ for each } m\in\omega_0,
\tag{6.5}
$$

where

$$
\begin{gathered}
e_{\mathrm{k}(n)}^{(m)}(t,\underline{x}_2) \doteqdot \left[\frac{\partial^m e_{\mathrm{k}(n)}(t,\underline{x})}{\partial z^m}\right]_{z=0} \text{ for each } m\in\omega_0,\ e_{\mathrm{k}(n)}^{(0)}(t,\underline{x}_2) \doteqdot \left[e_{\mathrm{k}(n)}(t,\underline{x})\right]_{z=0}, \\
\text{for each } n\in\omega_2.
\end{gathered}
\tag{6.6}
$$

**4.** By Theorem 5.2 and in analogy with (5.49) and (5.50), substitution of (5.1), (6.1), and (6.5) into (4.98) yields

$$
\mathrm{H}^{(m)}(t,\underline{x}_2) \sim \sum_{n=2}^{\infty}\varepsilon^n \eta_{(n)}^{(m)}(t,\underline{x}_2) \text{ for each } m\in\omega_0,
\tag{6.7}
$$

where

$$
\begin{aligned}
\eta_{(n)}^{(m)}(t,\underline{x}_2) &\doteqdot -\frac{1}{\rho_0 g}\left[e_{\mathrm{k}(n)}^{(m)}(t,\underline{x}_2) + \frac{\rho_0}{m+1}\sum_{p=1}^{n-1}\frac{\partial\phi_{(p)}^{(m+1)}(t,\underline{x}_2)}{\partial t}\zeta_{(n-p)}(t,\underline{x}_2)\right] \\
&= -\frac{1}{\rho_0 g}\left[e_{\mathrm{k}(n)}^{(m)}(t,\underline{x}_2) + \frac{\rho_0}{m+1}\sum_{p=1}^{n-1}\frac{\partial v_{(p)3}^{(m)}(t,\underline{x}_2)}{\partial t}\zeta_{(n-p)}(t,\underline{x}_2)\right] \\
&\qquad \text{for each } m\in\omega_0, \text{ and each } n\in\omega_2.
\end{aligned}
\tag{6.8}
$$

Particularly, equation (6.8) at $m=0$ becomes

$$
\begin{aligned}
\eta_{(n)}^{(0)}(t,\underline{x}_2) &\doteqdot -\frac{1}{\rho_0 g}\left[e_{\mathrm{k}(n)}^{(0)}(t,\underline{x}_2) + \rho_0\sum_{p=1}^{n-1}\frac{\partial\phi_{(p)}^{(1)}(t,\underline{x}_2)}{\partial t}\zeta_{(n-p)}(t,\underline{x}_2)\right] \\
&= -\frac{1}{\rho_0 g}\left[e_{\mathrm{k}(n)}^{(0)}(t,\underline{x}_2) + \rho_0\sum_{p=1}^{n-1}\frac{\partial v_{(p)3}^{(0)}(t,\underline{x}_2)}{\partial t}\zeta_{(n-p)}(t,\underline{x}_2)\right] \text{ for each } n\in\omega_2
\end{aligned}
\tag{6.8$_0$}
$$

(cf. (6.4$_0$)).

**5.** In the same way, substitution of (5.1) and (6.1) into (4.101) yields

$$\Xi^{(m)}(t,\underline{x}_2) \sim \sum_{n=2}^{\infty} \varepsilon^n \xi_{(n)}^{(m)}(t,\underline{x}_2) \text{ for each } m \in \omega_0, \tag{6.9}$$

where

$$\begin{aligned}\xi_{(n)}^{(m)}(t,\underline{x}_2) &\doteq \sum_{p=1}^{n-1}\left[-\sum_{i=1}^{2}\left(\nabla_i \phi_{(p)}^{(m)}(t,\underline{x}_2)\right)\left(\nabla_i \zeta_{(n-p)}(t,\underline{x}_2)\right) + \frac{1}{m+1}\phi_{(p)}^{(m+2)}(t,\underline{x}_2)\zeta_{(n-p)}(t,\underline{x}_2)\right] \\ &= \sum_{p=1}^{n-1}\left[-\sum_{i=1}^{2} v_{(p)i}^{(m)}(t,\underline{x}_2)\left(\nabla_i \zeta_{(n-p)}(t,\underline{x}_2)\right) + \frac{1}{m+1} v_{(p)3}^{(m+1)}(t,\underline{x}_2)\zeta_{(n-p)}(t,\underline{x}_2)\right] \end{aligned} \tag{6.10}$$

$$\text{for each } m \in \omega_0, \text{ and each } n \in \omega_2,$$

subject to (6.4) and (6.4$_0$). Particularly, by (5.44), (5.47), and (6.2), equation (6.10) at $m = 0$ reduces to

$$\begin{aligned}\xi_{(n)}^{(0)}(t,\underline{x}_2) &= \sum_{p=1}^{n-1}\left[-\sum_{i=1}^{2} v_{(p)i}^{(0)}(t,\underline{x}_2)\left(\nabla_i \zeta_{(n-p)}(t,\underline{x}_2)\right) + v_{(p)3}^{(1)}(t,\underline{x}_2)\zeta_{(n-p)}(t,\underline{x}_2)\right] \\ &= -\sum_{p=1}^{n-1}\sum_{i=1}^{2}\nabla_i\left[v_{(p)i}^{(0)}(t,\underline{x}_2)\zeta_{(n-p)}(t,\underline{x}_2)\right] = -\sum_{i=1}^{2}\nabla_i\sum_{p=1}^{n-1} v_{(p)i}^{(0)}(t,\underline{x}_2)\zeta_{(n-p)}(t,\underline{x}_2)\end{aligned} \tag{6.10$_0$}$$

$$\text{for each } n \in \omega_2,$$

because

$$v_{(p)3}^{(1)}(t,\underline{x}_2) = \left[\frac{\partial v_{(p)3}(t,\underline{x})}{\partial z}\right]_{z=0} = -\left[\sum_{i=1}^{2}\nabla_i v_{(p)i}(t,\underline{x})\right]_{z=0} = -\sum_{i=1}^{2}\nabla_i v_{(p)i}^{(0)}(t,\underline{x}) \tag{6.11}$$

$$\text{for each } p \in \omega_1.$$

by the pertinent instances of definitions (6.4).

**6.** In accordance with (5.1), given $m \in \omega_2$:

$$\mathrm{Z}^m(t,\underline{x}_2) \sim \prod_{\mu=1}^{m}\sum_{n_\mu=1}^{\infty}\varepsilon^{n_\mu}\zeta_{(n_\mu)}(t,\underline{x}_2), \tag{6.12}$$

the understanding being that

$$\begin{aligned}&\prod_{\mu=1}^{m}\sum_{n_\mu=1}^{\infty}\varepsilon^{n_\mu}\zeta_{(n_\mu)}(t,\underline{x}_2) \\ &= \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_{m-1}=1}^{\infty}\sum_{n_m=1}^{\infty}\varepsilon^{n_1+n_2+\ldots+n_m}\zeta_{(n_1)}(t,\underline{x}_2)\zeta_{(n_2)}(t,\underline{x}_2)\ldots\zeta_{(n_{m-1})}(t,\underline{x}_2)\zeta_{(n_m)}(t,\underline{x}_2) \\ &= \sum_{n_m=1}^{\infty}\sum_{n_{m-1}=1}^{\infty}\cdots\sum_{n_2=1}^{\infty}\sum_{n_1=1}^{\infty}\varepsilon^{n_1+n_2+\ldots+n_m}\zeta_{(n_m)}(t,\underline{x}_2)\zeta_{(n_{m-1})}(t,\underline{x}_2)\ldots\zeta_{(n_2)}(t,\underline{x}_2)\zeta_{(n_1)}(t,\underline{x}_2).\end{aligned} \tag{6.12$_1$}$$

Let us introduce $m$ new variables of summation '$l_1$', ..., '$l_m$' instead of '$n_1$', ..., '$n_m$' with the help of the equations

$$l_1 \doteqdot n_1,\ l_\mu \doteqdot \sum_{\nu=1}^{\mu} n_\nu = l_{\mu-1} + n_\mu \text{ for each } \mu \in \omega_{1,m}, \tag{6.13}$$

so that

$$n_1 = l_1,\ n_\mu = l_\mu - l_{\mu-1} \text{ for each } \mu \in \omega_{1,m}. \tag{6.13$_1$}$$

By (6.13$_1$), the train of equations (6.12$_1$) can be developed thus:

$$\begin{aligned}&\prod_{\mu=1}^{m}\sum_{n_\mu=1}^{\infty}\varepsilon^{n_\mu}\zeta_{(n_\mu)}(t,\underline{x}_2)\\ &= \sum_{l_m=m}^{\infty}\varepsilon^{l_m}\sum_{l_{m-1}=m-1}^{l_m-1}\dots\sum_{l_2=2}^{l_3-1}\sum_{l_1=1}^{l_2-1}\zeta_{(l_m-l_{m-1})}(t,\underline{x}_2)\zeta_{(l_{m-1}-l_{m-2})}(t,\underline{x}_2)\dots\zeta_{(l_2-l_1)}(t,\underline{x}_2)\zeta_{(l_1)}(t,\underline{x}_2).\end{aligned} \tag{6.12$_2$}$$

Hence, (6.12) reduces to

$$\mathrm{Z}^m(t,\underline{x}_2) \sim \sum_{l_m=m}^{\infty}\varepsilon^{l_m}\zeta_{(l_m)}^{<m>}(t,\underline{x}_2) \text{ for each } m \in \omega_1, \tag{6.14}$$

where

$$\begin{gathered}\zeta_{(l_m)}^{<m>}(t,\underline{x}_2) \doteqdot \sum_{l_{m-1}=m-1}^{l_m-1}\dots\sum_{l_2=2}^{l_3-1}\sum_{l_1=1}^{l_2-1}\zeta_{(l_m-l_{m-1})}(t,\underline{x}_2)\zeta_{(l_{m-1}-l_{m-2})}(t,\underline{x}_2)\dots\zeta_{(l_2-l_1)}(t,\underline{x}_2)\zeta_{(l_1)}(t,\underline{x}_2)\\ \text{for each } m \in \omega_1 \text{ and for each } l_m \in \omega_m,\end{gathered} \tag{6.15}$$

the understanding being that

$$\zeta_{(l)}^{<m>}(t,\underline{x}_2) \doteqdot 0 \text{ for each } m \in \omega_1 \text{ and for each } l \in \omega_{0,m-1}. \tag{6.15$_0$}$$

Relation (6.14) subject (6.15) regarded as a result of applying Theorem 5.2 with $\mu = \nu = 1$ to (6.12) $m-1$ times.. Particularly, for each $m \in \omega_{1,3}$, equation (6.15) becomes

$$\zeta_{(l_1)}^{<1>}(t,\underline{x}_2) \doteqdot \zeta_{(l_1)}(t,\underline{x}_2) \text{ for } m \doteqdot 1 \text{ and for each } l_1 \in \omega_1, \tag{6.16}$$

$$\zeta_{(l_2)}^{<2>}(t,\underline{x}_2) \doteqdot \sum_{l_1=1}^{l_2-1}\zeta_{(l_2-l_1)}(t,\underline{x}_2)\zeta_{(l_1)}(t,\underline{x}_2) \text{ for } m \doteqdot 2 \text{ and for each } l_2 \in \omega_2, \tag{6.17}$$

$$\begin{gathered}\zeta_{(l_3)}^{<3>}(t,\underline{x}_2) \doteqdot \sum_{l_2=2}^{l_3-1}\sum_{l_1=1}^{l_2-1}\zeta_{(l_3-l_2)}(t,\underline{x}_2)\zeta_{(l_2-l_1)}(t,\underline{x}_2)\zeta_{(l_1)}(t,\underline{x}_2)\\ \text{for } m \doteqdot 3 \text{ and for each } l_3 \in \omega_3,\end{gathered} \tag{6.18}$$

respectively. In turn, equation (6.17) for each $l_2 \in \omega_{2,4}$, e.g., yields

$$\zeta_{(2)}^{<2>} = \zeta_{(1)}^2,\ \zeta_{(3)}^{<2>} = 2\zeta_{(1)}\zeta_{(2)},\ \zeta_{(4)}^{<2>} = 2\zeta_{(1)}\zeta_{(3)} + \zeta_{(2)}^2, \tag{6.19}$$

whereas equation (6.18) for each $l_3 \in \omega_{3,5}$, e.g., yields

$$\zeta_{(3)}^{<3>}=\zeta_{(1)}^{3},\ \zeta_{(4)}^{<3>}=3\zeta_{(2)}\zeta_{(1)}^{2},\ \zeta_{(5)}^{<3>}=3\zeta_{(1)}\left(\zeta_{(1)}\zeta_{(3)}+\zeta_{(2)}^{2}\right). \tag{6.20}$$

In general, given $m\in\omega_1$, it follows from (6.15) that

$$\zeta_{(m)}^{<m>}(t,\underline{x}_2)=\zeta_{(l1)}{}^{m}(t,\underline{x}_2)\doteq\left[\zeta_{(l1)}(t,\underline{x}_2)\right]^{m}. \tag{6.21}$$

## 6.2. Asymptotic expansions of $\mathrm{A_d}$, $\mathrm{A_k}$, and $\mathrm{A}$

**1.** By (6.7), (6.9), and (6.14), it follows that for each $m\in\omega_1$:

$$\begin{aligned}\mathrm{H}^{(m)}(t,\underline{x}_2)\mathrm{Z}^{m}(t,\underline{x}_2)&\sim\left[\sum_{n=2}^{\infty}\varepsilon^{n}\eta_{(n)}^{(m)}(t,\underline{x}_2)\right]\left[\sum_{l_m=m}^{\infty}\varepsilon^{l_m}\zeta_{(l_m)}^{<m>}(t,\underline{x}_2)\right]\\&=\sum_{l_m=m}^{\infty}\sum_{n=2}^{\infty}\varepsilon^{l_m+n}\eta_{(n)}^{(m)}(t,\underline{x}_2)\zeta_{(l_m)}^{<m>}(t,\underline{x}_2),\end{aligned} \tag{6.22}$$

$$\begin{aligned}\Xi^{(m)}(t,\underline{x}_2)\mathrm{Z}^{m}(t,\underline{x}_2)&\sim\left[\sum_{n=2}^{\infty}\varepsilon^{n}\xi_{(n)}^{(m)}(t,\underline{x}_2)\right]\left[\sum_{l_m=m}^{\infty}\varepsilon^{l_m}\zeta_{(l_m)}^{<m>}(t,\underline{x}_2)\right]\\&=\sum_{l_m=m}^{\infty}\sum_{n=2}^{\infty}\varepsilon^{l_m+n}\xi_{(n)}^{(m)}(t,\underline{x}_2)\zeta_{(l_m)}^{<m>}(t,\underline{x}_2),\end{aligned} \tag{6.23}$$

which are variants of each other with ⟨'H','η'⟩ and ⟨'Ξ','ξ'⟩ exchanged.

The final expression in (6.22) or (6.23) can be developed further in analogy with item b of the proof of Theorem 5.2 as follows. Let $l\doteq l_m+n$, so that $l\in\omega_{m+2}$, because $l=m+2$ when $l_m=m$ and $n=2$ If '$l$' is employed as a new variable of summation instead of '$l_m$', so that $l_m=l-n$, then the domain of values of the variable '$n$' is determined by the conjunction of two relations: (i) $n\in\omega_2$, i.e. $2\le n<\infty$, and (ii) $n=l-m$ at $l_m=m$. Hence, $2\le n\le l-m$, i.e. $n\in\omega_{2,l-m}$. Therefore, (6.22) and (6.23) become

$$\mathrm{H}^{(m)}(t,\underline{x}_2)\mathrm{Z}^{m}(t,\underline{x}_2)\sim\sum_{l=m+2}^{\infty}\sum_{n=2}^{l-m}\varepsilon^{l}\eta_{(n)}^{(m)}(t,\underline{x}_2)\zeta_{(l-n)}^{<m>}(t,\underline{x}_2), \tag{6.24}$$

$$\Xi^{(m)}(t,\underline{x}_2)\mathrm{Z}^{m}(t,\underline{x}_2)\sim\sum_{l=m+2}^{\infty}\sum_{n=2}^{l-m}\varepsilon^{l}\xi_{(n)}^{(m)}(t,\underline{x}_2)\zeta_{(l-n)}^{<m>}(t,\underline{x}_2), \tag{6.25}$$

for each $m\in\omega_1$.

By (6.7)–(6.10$_0$), (6.24), and (6.25), it follows from (4.97) and (4.100) that

$$\mathrm{A_d}(t,\underline{x}_2)=\mathrm{H}^{(0)}(t,\underline{x}_2)+\tilde{\mathrm{A}}_{\mathrm{d}}(t,\underline{x}_2), \tag{6.26}$$

where

$$\mathrm{H}^{(0)}(t,\underline{x}_2)\sim\sum_{l=2}^{\infty}\varepsilon^l\eta_{(l)}^{(0)}(t,\underline{x}_2), \qquad (6.27)$$

$$\tilde{\mathrm{A}}_{\mathrm{d}}(t,\underline{x}_2)\,\bar{\bar{=}}\sum_{m=1}^{\infty}\frac{1}{m!}\mathrm{H}^{(m)}(t,\underline{x}_2)\mathrm{Z}^m(t,\underline{x}_2)\sim\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{l=m+2}^{\infty}\sum_{n=2}^{l-m}\varepsilon^l\xi_{(n)}^{(m)}(t,\underline{x}_2)\zeta_{(l-n)}^{<m>}(t,\underline{x}_2), \qquad (6.28)$$

and that

$$\mathrm{A}_{\mathrm{k}}(t,\underline{x}_2)=\Xi^{(0)}(t,\underline{x}_2)+\tilde{\mathrm{A}}_{\mathrm{k}}(t,\underline{x}_2), \qquad (6.29)$$

where

$$\Xi^{(0)}(t,\underline{x}_2)\sim\sum_{l=2}^{\infty}\varepsilon^l\xi_{(l)}^{(0)}(t,\underline{x}_2), \qquad (6.30)$$

$$\tilde{\mathrm{A}}_{\mathrm{k}}(t,\underline{x}_2)\,\bar{\bar{=}}\sum_{m=1}^{\infty}\frac{1}{m!}\Xi^{(m)}(t,\underline{x}_2)\mathrm{Z}^m(t,\underline{x}_2)\sim\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{l=m+2}^{\infty}\sum_{n=2}^{l-m}\varepsilon^l\xi_{(n)}^{(m)}(t,\underline{x}_2)\zeta_{(l-n)}^{<m>}(t,\underline{x}_2). \qquad (6.31)$$

In (6.27) and (6.30), the variable of summation '$l$' has been employed instead of '$n$' that is employed in ($6.8_0$) and ($6.10_0$).

Let '$q$', defined as $q\,\bar{\bar{=}}\,l-m$, be a new variable of summation to be employed in (6.28) and (6.31) instead of '$m$'. Therefore, (i) $q$=2 when $l=m+2$ and (ii) $q\,\bar{\bar{=}}\,l-1$ when $m$=1, so that $q\in\omega_{2,l-1}$. At the same time, since $l=m+q$, therefore $l=3$ if $m$=1 and $q$=2, so that $l\in\omega_3$. Also, $m=l-q$. Hence, (6.28) and (6.31) reduce to

$$\tilde{\mathrm{A}}_{\mathrm{d}}(t,\underline{x}_2)\sim\sum_{l=3}^{\infty}\varepsilon^l\sum_{q=2}^{l-1}\frac{1}{(l-q)!}\sum_{n=2}^{q}\eta_{(n)}^{(l-q)}(t,\underline{x}_2)\zeta_{(l-n)}^{<l-q>}(t,\underline{x}_2), \qquad (6.32)$$

$$\tilde{\mathrm{A}}_{\mathrm{k}}(t,\underline{x}_2)\sim\sum_{l=3}^{\infty}\varepsilon^l\sum_{q=2}^{l-1}\frac{1}{(l-q)!}\sum_{n=2}^{q}\xi_{(n)}^{(l-q)}(t,\underline{x}_2)\zeta_{(l-n)}^{<l-q>}(t,\underline{x}_2). \qquad (6.33)$$

**Comment 6.1.** Relations (6.32) and (6.33) are of fundamental importance for the recursive theory in progress. Therefore, to be doubly sure that relations (6.32) and (6.33) are deduced below somewhat differently. The final expression in (6.22) or (6.23) can alternatively be developed further in analogy with item b of the proof of Theorem 5.2 as follows. Let $q\,\bar{\bar{=}}\,l_m-m+n$, so that $q\in\omega_2$, because $q$=2 when $l_m=m$ and $n$=2. If '$q$' is employed as a new variable of summation instead of '$l_m$', so that $l_m=m-n+q$, then the domain of values of the variable '$n$' is determined by the conjunction of two relations: (i) $n\in\omega_2$, i.e. $2\le n<\infty$, and (ii) $n=q$ at $l_m=m$. Hence, $2\le n\le q$, i.e. $n\in\omega_{2,q}$. Therefore, (6.22) and (6.23) become

$$\mathrm{H}^{(m)}(t,\underline{x}_2)\mathrm{Z}^m(t,\underline{x}_2) \sim \sum_{q=2}^{\infty}\sum_{n=2}^{q}\varepsilon^{m+q}\eta_{(n)}^{(m)}(t,\underline{x}_2)\zeta_{(m-n+q)}^{<m>}(t,\underline{x}_2), \quad (6.24_1)$$

$$\Xi^{(m)}(t,\underline{x}_2)\mathrm{Z}^m(t,\underline{x}_2) \sim \sum_{q=2}^{\infty}\sum_{n=2}^{q}\varepsilon^{m+q}\xi_{(n)}^{(m)}(t,\underline{x}_2)\zeta_{(m-n+q)}^{<m>}(t,\underline{x}_2), \quad (6.25_1)$$

for each $m \in \omega_1$.In this case, relations (6.26), (6.27), (6.29), and (6.30) hold, whereas the relations

$$\widetilde{\mathrm{A}}_{\mathrm{d}}(t,\underline{x}_2) \stackrel{\rightharpoonup}{\equiv} \sum_{m=1}^{\infty}\frac{1}{m!}\mathrm{H}^{(m)}(t,\underline{x}_2)\mathrm{Z}^m(t,\underline{x}_2) \sim \sum_{m=1}^{\infty}\frac{1}{m!}\sum_{q=2}^{\infty}\sum_{n=2}^{q}\varepsilon^{m+q}\eta_{(n)}^{(m)}(t,\underline{x}_2)\zeta_{(m-n+q)}^{<m>}(t,\underline{x}_2), \quad (6.28_1)$$

$$\widetilde{\mathrm{A}}_{\mathrm{k}}(t,\underline{x}_2) \stackrel{\rightharpoonup}{\equiv} \sum_{m=1}^{\infty}\frac{1}{m!}\Xi^{(m)}(t,\underline{x}_2)\mathrm{Z}^m(t,\underline{x}_2) \sim \sum_{m=1}^{\infty}\frac{1}{m!}\sum_{q=2}^{\infty}\sum_{n=2}^{q}\varepsilon^{m+q}\xi_{(n)}^{(m)}(t,\underline{x}_2)\zeta_{(m-n+q)}^{<m>}(t,\underline{x}_2). \quad (6.31_1)$$

come instead of (6.28) and (6.31).

Let '$l$' defined as $l \stackrel{\rightharpoonup}{\equiv} m+q$ be a new variable of summation to be employed in ($6.28_1$) and ($6.31_1$) instead of '$m$'. If $m$=1 and $q$=2 then $l$=3, so that $l \in \omega_3$. At the same time, since $q = l - m$, therefore $q = l - 1$ if $m$=1, so that $q \in \omega_{2,l-1}$. Also, $m = l - q$. Hence, ($6.28_1$) and ($6.31_1$) reduce to (6.32) and (6.33) as expected.•

By (6.27) subject to (6.28) and (6.32), relation (6.26) implies that

$$\mathrm{A}_{\mathrm{d}}(t,\underline{x}_2) \sim \sum_{l=2}^{\infty}\mathrm{A}_{\mathrm{d}(l)}(t,\underline{x}_2) = \sum_{l=2}^{\infty}\varepsilon^{l}\alpha_{\mathrm{d}(l)}(t,\underline{x}_2), \quad (6.34)$$

where

$$\begin{aligned}\alpha_{\mathrm{d}(2)}(t,\underline{x}_2) \stackrel{\rightharpoonup}{\equiv} \eta_{(z)}^{(0)}(t,\underline{x}_2) &= -\frac{1}{\rho_0 g}\left[e_{\mathrm{k}(2)}^{(0)}(t,\underline{x}_2) + \rho_0\frac{\partial\phi_{(1)}^{(1)}(t,\underline{x}_2)}{\partial t}\zeta_{(1)}(t,\underline{x}_2)\right] \\ &= -\frac{1}{\rho_0 g}\left[e_{\mathrm{k}(2)}^{(0)}(t,\underline{x}_2) + \rho_0\frac{\partial v_{(1)3}^{(0)}(t,\underline{x}_2)}{\partial t}\zeta_{(1)}(t,\underline{x}_2)\right] \\ &= -\frac{1}{\rho_0 g}\left[e_{\mathrm{k}(2)}(t,\underline{x}) + \rho_0\frac{\partial v_{(1)3}(t,\underline{x})}{\partial t}\zeta_{(1)}(t,\underline{x}_2)\right]_{z=0},\end{aligned} \quad (6.35)$$

$$\alpha_{\mathrm{d}(l)}(t,\underline{x}_2) \stackrel{\rightharpoonup}{\equiv} \eta_{(l)}^{(0)}(t,\underline{x}_2) + \sum_{q=2}^{l-1}\frac{1}{(l-q)!}\sum_{n=2}^{q}\eta_{(n)}^{(l-q)}(t,\underline{x}_2)\zeta_{(l-n)}^{<l-q>}(t,\underline{x}_2) \text{ for each } l \in \omega_3, \quad (6.36)$$

subject to the pertinent variants of (6.8), ($6.8_0$), and (6.15). Particularly, in developing the final expression in (6.35), use of equation ($6.8_0$) at $n = 2$ has been made.

Similarly, by (6.30) and (6.33), relation (6.29) implies that

$$\mathrm{A}_{\mathrm{k}}(t,\underline{x}_2)\sim\sum_{l=2}^{\infty}\mathrm{A}_{\mathrm{k}(l)}(t,\underline{x}_2)=\sum_{l=2}^{\infty}\varepsilon^l\alpha_{\mathrm{k}(l)}(t,\underline{x}_2), \tag{6.37}$$

where

$$\alpha_{\mathrm{k}(2)}(t,\underline{x}_2)\doteqdot\xi_{(2)}^{(0)}(t,\underline{x}_2)=-\sum_{i=1}^{2}\nabla_i\left[v_{(1)i}^{(0)}(t,\underline{x}_2)\zeta_{(1)}(t,\underline{x}_2)\right]=-\sum_{i=1}^{2}\nabla_i\left[v_{(1)i}(t,\underline{x})\zeta_{(1)}(t,\underline{x}_2)\right]_{z=0}, \tag{6.38}$$

$$\alpha_{\mathrm{k}(l)}(t,\underline{x}_2)\doteqdot\xi_{(l)}^{(0)}(t,\underline{x}_2)+\sum_{q=2}^{l-1}\frac{1}{(l-q)!}\sum_{n=2}^{q}\xi_{(n)}^{(l-q)}(t,\underline{x}_2)\zeta_{(l-n)}^{<l-q>}(t,\underline{x}_2)\text{ for each } l\in\omega_3, \tag{6.39}$$

subject to the pertinent variants of (6.10), ($6.10_0$), and (6.15). Particularly, in developing the final expression in (6.38), use of equation ($6.10_0$) at $n=2$ has been made.

Relations (6.34) and (6.37) imply that

$$\alpha_{\mathrm{d}(1)}(t,\underline{x}_2)=\alpha_{\mathrm{k}(1)}(t,\underline{x}_2)\doteqdot 0\,. \tag{6.40}$$

**2.** At $l$=3, equations (6.36) and (6.39) become

$$\alpha_{\mathrm{d}(3)}(t,\underline{x}_2)\doteqdot\eta_{(3)}^{(0)}(t,\underline{x}_2)+\eta_{(2)}^{(1)}(t,\underline{x}_2)\zeta_1(t,\underline{x}_2), \tag{6.41}$$

$$\alpha_{\mathrm{k}(3)}(t,\underline{x}_2)\doteqdot\xi_{(3)}^{(0)}(t,\underline{x}_2)+\xi_{(2)}^{(1)}(t,\underline{x}_2)\zeta_1(t,\underline{x}_2). \tag{6.42}$$

where use of (6.16) at $l_1=1$ has been made. In this case, at $m=0$ and $n=3$, equation (6.8) becomes

$$\begin{aligned}\eta_{(3)}^{(0)}(t,\underline{x}_2)&=-\frac{1}{\rho_0 g}\left[e_{\mathrm{k}(3)}^{(0)}(t,\underline{x}_2)+\rho_0\sum_{p=1}^{2}\frac{\partial\phi_{(p)}^{(1)}(t,\underline{x}_2)}{\partial t}\zeta_{(3-p)}(t,\underline{x}_2)\right]\\&=-\frac{1}{\rho_0 g}\left[e_{\mathrm{k}(3)}^{(0)}(t,\underline{x}_2)+\rho_0\sum_{p=1}^{2}\frac{\partial v_{(p)3}^{(0)}(t,\underline{x}_2)}{\partial t}\zeta_{(3-p)}(t,\underline{x}_2)\right]\\&=-\frac{1}{\rho_0 g}\left[e_{\mathrm{k}(3)}^{(0)}(t,\underline{x}_2)+\rho_0\left(\frac{\partial v_{(1)3}^{(0)}(t,\underline{x}_2)}{\partial t}\zeta_{(2)}(t,\underline{x}_2)+\frac{\partial v_{(2)3}^{(0)}(t,\underline{x}_2)}{\partial t}\zeta_{(1)}(t,\underline{x}_2)\right)\right],\end{aligned} \tag{6.43}$$

whereas equation (6.10) reduces to

$$\begin{aligned}\xi_{(3)}^{(0)}(t,\underline{x}_2)&=-\sum_{i=1}^{2}\nabla_i\sum_{p=1}^{2}\left[v_{(p)i}^{(0)}(t,\underline{x}_2)\zeta_{(3-p)}(t,\underline{x}_2)\right]\\&=-\sum_{i=1}^{2}\nabla_i\left[v_{(1)i}^{(0)}(t,\underline{x}_2)\zeta_{(2)}(t,\underline{x}_2)+v_{(2)i}^{(0)}(t,\underline{x}_2)\zeta_{(1)}(t,\underline{x}_2)\right]\end{aligned} \tag{6.44}$$

via ($6.10_0$) at $n=3$. At the same time, at $m=1$ and $n=2$, equations (6.8) and (6.10) become

$$\eta_{(2)}^{(1)}(t,\underline{x}_2) \stackrel{\rightharpoonup}{\equiv} -\frac{1}{\rho_0 g}\left[e_{\mathrm{k}(2)}^{(1)}(t,\underline{x}_2)+\frac{1}{2}\rho_0\frac{\partial\phi_{(1)}^{(2)}(t,\underline{x}_2)}{\partial t}\zeta_{(1)}(t,\underline{x}_2)\right]$$
$$= -\frac{1}{\rho_0 g}\left[e_{\mathrm{k}(2)}^{(1)}(t,\underline{x}_2)+\frac{1}{2}\rho_0\frac{\partial v_{(1)3}^{(1)}(t,\underline{x}_2)}{\partial t}\zeta_{(1)}(t,\underline{x}_2)\right], \tag{6.45}$$

$$\xi_{(2)}^{(1)}(t,\underline{x}_2) \stackrel{\rightharpoonup}{\equiv} -\sum_{i=1}^{2}\left[\nabla_i\phi_{(1)}^{(1)}(t,\underline{x}_2)\right]\left[\nabla_i\zeta_{(1)}(t,\underline{x}_2)\right]+\frac{1}{2}\phi_{(1)}^{(3)}(t,\underline{x}_2)\zeta_{(1)}(t,\underline{x}_2)$$
$$= -\sum_{i=1}^{2}\left[\nabla_i v_{(1)3}^{(0)}(t,\underline{x}_2)\right]\left[\nabla_i\zeta_{(1)}(t,\underline{x}_2)\right]-\frac{1}{2}\sum_{i=1}^{2}\nabla_i^{\,2}v_{(1)3}^{(0)}(t,\underline{x}_2)\zeta_{(1)}(t,\underline{x}_2), \tag{6.46}$$

because

$$\nabla_i\phi_{(1)}^{(1)}(t,\underline{x}_2) = \nabla_i v_{(1)3}^{(0)}(t,\underline{x}),\ \phi_{(1)}^{(3)}(t,\underline{x}_2) = v_{(1)3}^{(2)}(t,\underline{x}_2) = -\sum_{i=1}^{2}\nabla_i^{\,2}v_{(1)3}^{(0)}(t,\underline{x}_2), \tag{6.46$_1$}$$

by the pertinent instances of (6.4), (6.4$_1$), and (6.4$_3$). By (6.46), it follows that

$$\xi_{(2)}^{(1)}(t,\underline{x}_2)\zeta_{(1)}(t,\underline{x}_2)$$
$$= -\frac{1}{2}\left[\sum_{i=1}^{2}\left[\nabla_i v_{(1)3}^{(0)}(t,\underline{x}_2)\right]\left[\nabla_i\zeta_{(1)}^{2}(t,\underline{x}_2)\right]+\left[\sum_{i=1}^{2}\nabla_i^{\,2}v_{(1)3}^{(0)}(t,\underline{x}_2)\right]\zeta_{(1)}^{2}(t,\underline{x}_2)\right]$$
$$= -\frac{1}{2}\sum_{i=1}^{2}\nabla_i\left[\left(\nabla_i v_{(1)3}^{(0)}(t,\underline{x}_2)\right)\zeta_{(1)}^{2}(t,\underline{x}_2)\right]. \tag{6.47}$$

By (6.43) and (6.45), equation (6.41) becomes

$$\alpha_{\mathrm{d}(3)}(t,\underline{x}_2) = -\frac{1}{\rho_0 g}\left[e_{\mathrm{k}(3)}^{(0)}(t,\underline{x}_2)+\rho_0\sum_{p=1}^{2}\frac{\partial v_{(p)3}^{(0)}(t,\underline{x}_2)}{\partial t}\zeta_{(3-p)}(t,\underline{x}_2)\right]$$
$$-\frac{1}{\rho_0 g}\left[e_{\mathrm{k}(2)}^{(1)}(t,\underline{x}_2)+\frac{1}{2}\rho_0\frac{\partial v_{(1)3}^{(1)}(t,\underline{x}_2)}{\partial t}\zeta_{(1)}(t,\underline{x}_2)\right]\zeta_1(t,\underline{x}_2)$$
$$= -\frac{1}{\rho_0 g}\left[e_{\mathrm{k}3}(t,\underline{x})+\rho_0\sum_{p=1}^{2}\frac{\partial v_{(p)3}(t,\underline{x})}{\partial t}\zeta_{(3-p)}(t,\underline{x}_2)\right]_{x=0}$$
$$-\frac{1}{\rho_0 g}\left[\frac{\partial^2 e_{\mathrm{k}(2)}(t,\underline{x})}{\partial z}+\frac{1}{2}\rho_0\frac{\partial^2 v_{(1)3}(t,\underline{x})}{\partial t\partial z}\zeta_{(1)}(t,\underline{x}_2)\right]_{x=0}\zeta_1(t,\underline{x}_2), \tag{6.48}$$

whereas by (6.44) and (6.47), equation (6.42) reduces to

$$\alpha_{\mathrm{k}(3)}(t,\underline{x}_2) = -\sum_{i=1}^{2}\nabla_i\left[\sum_{p=1}^{2}\left[v_{(p)i}^{(0)}(t,\underline{x}_2)\zeta_{(3-p)}(t,\underline{x}_2)\right]+\frac{1}{2}\zeta_{(1)}^{2}(t,\underline{x}_2)\nabla_i v_{(1)3}^{(0)}(t,\underline{x}_2)\right]$$
$$= -\sum_{i=1}^{2}\nabla_i\left[\sum_{p=1}^{2}\left[v_{(p)i}(t,\underline{x})\zeta_{(3-p)}(t,\underline{x}_2)\right]+\frac{1}{2}\zeta_{(1)}^{2}(t,\underline{x}_2)\nabla_i v_{(1)3}(t,\underline{x})\right]_{z=0}. \tag{6.49}$$

**3.** By (6.34) and (6.37), it follows from (4.104) that

$$\mathrm{A}\left(t,\underline{x}_2\right) \sim \sum_{l=2}^{\infty} \mathrm{A}_{(l)}\left(t,\underline{x}_2\right) = \sum_{l=2}^{\infty} \varepsilon^l \alpha_{(l)}\left(t,\underline{x}_2\right), \tag{6.50}$$

where

$$\alpha_{(l)}\left(t,\underline{x}_2\right) \stackrel{\cdot}{\equiv} \frac{\partial \alpha_{\mathrm{d}(l)}\left(t,\underline{x}_2\right)}{\partial t} - \alpha_{\mathrm{k}(l)}\left(t,\underline{x}_2\right) \text{ for each } \; l \in \omega_2 \,. \tag{6.51}$$

## 6.3. Infinite recursive sequences of dynamic, kinematic, and dynamico-kinematic boundary conditions at the free surface

**Corollary 6.1.** By Theorem 5.1, equations (4.96), (4.99), and (4.104), subject to (5.1), (5.8), (6.34), (6.37), and (6.50), reduce to the following three infinite recursive sequences of equations with successive $l \in \omega_1$:

$$\zeta_{(l)}\left(t,\underline{x}_2\right) + \frac{1}{g}\frac{\partial \phi_{(l)}^{(0)}\left(t,\underline{x}_2\right)}{\partial t} = \alpha_{\mathrm{d}(l)}\left(t,\underline{x}_2\right), \tag{6.52}$$

$$\frac{\partial \zeta_{(l)}\left(t,\underline{x}_2\right)}{\partial t} - \phi_{(l)}^{(1)}\left(t,\underline{x}_2\right) = \alpha_{\mathrm{k}(l)}\left(t,\underline{x}_2\right), \tag{6.53}$$

$$\phi_{(l)}^{(1)}\left(t,\underline{x}_2\right) + \frac{1}{g}\frac{\partial^2 \phi_{(l)}^{(01}\left(t,\underline{x}_2\right)}{\partial t^2} = \alpha_{(l)}\left(t,\underline{x}_2\right), \tag{6.54}$$

subject to

$$\alpha_{\mathrm{d}(1)}\left(t,\underline{x}_2\right) = \alpha_{\mathrm{k}(1)}\left(t,\underline{x}_2\right) = \alpha_{(1)}\left(t,\underline{x}_2\right) \stackrel{\cdot}{\equiv} 0 \,, \tag{6.55}$$

by (6.40) and (6.51). It is understood that

$$\frac{\partial \phi_{(l)}^{(0)}\left(t,\underline{x}_2\right)}{\partial t} \stackrel{\cdot}{\equiv} \left[\frac{\partial \phi_{(l)}\left(t,\underline{x}\right)}{\partial t}\right]_{z=0}, \tag{6.56}$$

$$\phi_{(l)}^{(1)}\left(t,\underline{x}_2\right) \stackrel{\cdot}{\equiv} \left[\frac{\partial \phi_{(l)}\left(t,\underline{x}\right)}{\partial z}\right]_{z=0}, \tag{6.57}$$

$$\frac{\partial^2 \phi_{(l)}^{(01}\left(t,\underline{x}_2\right)}{\partial t^2} \stackrel{\cdot}{\equiv} \left[\frac{\partial^2 \phi_{(l)}\left(t,\underline{x}\right)}{\partial t^2}\right]_{z=0}. \tag{6.58}$$

Therefore, equations (6.52)-(6.54) with ascending $l \in \omega_1$, subject to (6.55), are three recursive asymptotic sequences of *boundary conditions at* $z = 0$, – *dynamic, kinematic,* and *dynamico-kinematic*, respectively.•

**Corollary 6.2.** Given $l \in \omega_2$, the value of each of the three functional forms $\alpha_{\mathrm{d}(l)}(t, \underline{x}_2)$, $\alpha_{\mathrm{k}(l)}(t, \underline{x}_2)$, and $\alpha_{(l)}(t, \underline{x}_2)$ can be expressed (i) in terms of the values of some spatial, temporal, or spatiotemporal derivatives at $z$=0 of the functional form $\phi_{(n)}(t, \underline{x})$ with all $n \in \omega_{1,l-1}$ and (ii) ultimately, in terms of those of the functional form $\phi_{(1)}(t, \underline{x})$.

**Proof:** i) Item (i) of the corollary follows from the definitions of $\alpha_{\mathrm{K}(l)}$, $\alpha_{\mathrm{D}(l)}$, and $\alpha_{(l)}$ as given by (6.35), (6.36), (6.38), (6.39), and (6.51). This general recursive property is illustrated by (6.35) (or (6.57)), (6.38), (6.48), and (6.49).

ii) Item (ii) of the corollary follows from the Hypothesis 5.1 implies that the functional form $\phi_{(l)}(t, \underline{x})$ with any $l \in \omega_2$ is ultimately expressible in terms of functional form $\phi_{(1)}(t, \underline{x})$.•

**Corollary 6.3.** With allowance for (6.55), equation (6.52) can be written as these two:

$$\zeta_{(1)}(t, \underline{x}_2) = -\frac{1}{g}\frac{\partial \phi_{(1)}^{(0)}(t, \underline{x}_2)}{\partial t} \text{ for } l = 1, \tag{6.52$_1$}$$

$$\zeta_{(l)}(t, \underline{x}_2) = \alpha_{\mathrm{d}(l)}(t, \underline{x}_2) - \frac{1}{g}\frac{\partial \phi_{(l)}^{(0)}(t, \underline{x}_2)}{\partial t} \text{ for each } l \in \omega_2, \tag{6.52$_2$}$$

whereas equation (6.53) can be written as these two:

$$\frac{\partial \zeta_{(1)}(t, \underline{x}_2)}{\partial t} = \phi_{(1)}^{(1)}(t, \underline{x}_2) \text{ for } l = 1, \tag{6.53$_1$}$$

$$\frac{\partial \zeta_{(l)}(t, \underline{x}_2)}{\partial t} = \alpha_{\mathrm{k}(l)}(t, \underline{x}_2) + \phi_{(l)}^{(1)}(t, \underline{x}_2) \text{ for each } l \in \omega_2, \tag{6.53$_2$}$$

From (6.52$_1$) and (6.52$_2$), subject to (6.56), it follows that Corollary 6.2 applies with '$\zeta_{(l)}(t, \underline{x}_2)$' in place of '$\alpha_{\mathrm{d}(l)}(t, \underline{x}_2)$', while from (6.53$_1$) and (6.53$_2$), subject to (6.57), it follows that Corollary 6.2 applies with '$\frac{\partial \zeta_{(1)}(t, \underline{x}_2)}{\partial t}$' in place of '$\alpha_{\mathrm{k}(l)}(t, \underline{x}_2)$'. The latter conclusion also follows from the former one, for differentiating both sides of each one of equations (6.52$_1$) and (6.52$_2$) yields

$$\frac{\partial \zeta_{(1)}(t, \underline{x}_2)}{\partial t} = -\frac{1}{g}\frac{\partial^2 \phi_{(1)}^{(0)}(t, \underline{x}_2)}{\partial t^2} \text{ for } l = 1, \tag{6.52$_3$}$$

$$\frac{\partial \zeta_{(l)}(t, \underline{x}_2)}{\partial t} = \frac{\partial \alpha_{\mathrm{d}(l)}(t, \underline{x}_2)}{\partial t} - \frac{1}{g}\frac{\partial^2 \phi_{(l)}^{(0)}(t, \underline{x}_2)}{\partial t^2} \text{ for each } l \in \omega_2. \tag{6.52$_4$}$$•

**Comment 6.2.** Comparison of the pair of equation (6.53$_1$) and (6.53$_2$) and the pair of equation (6.52$_3$) and (6.52$_4$) yields

$$\frac{\partial\zeta_{(1)}(t,\underline{x}_2)}{\partial t}=\phi_{(1)}^{(1)}(t,\underline{x}_2)=-\frac{1}{g}\frac{\partial^2\phi_{(1)}^{(0)}(t,\underline{x}_2)}{\partial t^2}\ \text{for}\ l=1, \tag{6.59}$$

$$\frac{\partial\zeta_{(l)}(t,\underline{x}_2)}{\partial t}=\alpha_{\mathrm{k}(l)}(t,\underline{x}_2)+\phi_{(l)}^{(1)}(t,\underline{x}_2)=\frac{\partial\alpha_{\mathrm{d}(l)}(t,\underline{x}_2)}{\partial t}-\frac{1}{g}\frac{\partial^2\phi_{(l)}^{(0)}(t,\underline{x}_2)}{\partial t^2}\ \text{for each}\ l\in\omega_2, \tag{6.60}$$

whence

$$\phi_{(1)}^{(1)}(t,\underline{x}_2)+\frac{1}{g}\frac{\partial^2\phi_{(1)}^{(0)}(t,\underline{x}_2)}{\partial t^2}\doteq\left[\frac{\partial\phi_{(1)}(t,\underline{x})}{\partial z}+\frac{1}{g}\frac{\partial^2\phi_{(1)}(t,\underline{x})}{\partial t^2}\right]_{z=0}=0\ \text{for}\ l=1, \tag{6.61}$$

$$\begin{aligned}\phi_{(l)}^{(1)}(t,\underline{x}_2)+\frac{1}{g}\frac{\partial^2\phi_{(l)}^{(0)}(t,\underline{x}_2)}{\partial t^2}&\doteq\left[\frac{\partial\phi_{(l)}(t,\underline{x})}{\partial z}+\frac{1}{g}\frac{\partial^2\phi_{(l)}(t,\underline{x})}{\partial t^2}\right]_{z=0}\\&=\frac{\partial\alpha_{\mathrm{d}(l)}(t,\underline{x}_2)}{\partial t}-\alpha_{\mathrm{k}(l)}(t,\underline{x}_2)\doteq\alpha_{(l)}(t,\underline{x}_2)\ \text{for each}\ l\in\omega_2,\end{aligned} \tag{6.62}$$

in agreement with (6.54) subject to (6.51) and (6.55)–(6.58).•

**Comment 6.3.** By (6.52$_1$), equations (6.35) and (6.38) can alternatively be written as

$$\alpha_{\mathrm{d}(2)}(t,\underline{x}_2)\doteq\eta_{(z)}^{(0)}(t,\underline{x}_2)=-\frac{1}{\rho_0 g}\left[e_{\mathrm{k}(2)}^{(0)}(t,\underline{x}_2)-\frac{\rho_0}{g}\frac{\partial\phi_{(1)}^{(1)}(t,\underline{x}_2)}{\partial t}\frac{\partial\phi_{(1)}^{(0)}(t,\underline{x}_2)}{\partial t}\right], \tag{6.35$_+$}$$

$$\alpha_{\mathrm{k}(2)}(t,\underline{x}_2)\doteq\xi_{(2)}^{(0)}(t,\underline{x}_2)=\frac{1}{g}\sum_{i=1}^{2}\nabla_i\left[v_{(1)i}^{(0)}(t,\underline{x}_2)\frac{\partial\phi_{(1)}^{(0)}(t,\underline{x}_2)}{\partial t}\right]. \tag{6.38$_+$}$$

Consequently, substitution of (6.52$_1$) and (6.52$_2$) at $l$=2 subject to (6.35$_+$) into (6.48) and (6.49) allows eliminating $\zeta_{(1)}(t,\underline{x}_2)$ and $\zeta_{(1)}(t,\underline{x}_2)$ from $\alpha_{\mathrm{d}(3)}(t,\underline{x}_2)$ and $\alpha_{\mathrm{k}(3)}(t,\underline{x}_2)$, and thus representing the two latter in terms of $\alpha_{\mathrm{d}(2)}(t,\underline{x}_2)$ and the other pertinent functional forms that are preserved.•

## 6.4. An infinite recursive sequence of kinematic boundary conditions at the bottom surface

Substituting (5.8) into (4.71), and then making use of Theorem 5.1 yields the following infinite sequence of kinematic boundary conditions at $z=-h(\underline{x}_2)$:

$$\left[\sum_{i=1}^{2}[\nabla_i h(\underline{x}_2)][\nabla_i\phi_{(l)}(t,\underline{x})]+\frac{\partial\phi_{(l)}(t,\underline{x})}{\partial z}\right]_{z=-h(\underline{x}_2)}=0\ \text{for each}\ l\in\omega_1, \tag{6.58}$$

where the variable '$l$' is employed in place of '$n$' for convenience in making subsequent statements. If (4.73) holds then (6.58) turns into

$$\left[\frac{\partial \phi_{(l)}(t,\underline{x})}{\partial z}\right]_{z=-d} = 0 \text{ for each } l \in \omega_1. \tag{6.59}$$

Alternatively, (6.59) can be deduced directly from (4.74). Accordingly, if (4.76) holds then (6.59) turns into

$$\lim_{d\to\infty}\left[\frac{\partial \phi_{(l)}(t,\underline{x})}{\partial z}\right]_{z=-d} = 0 \text{ for each } l \in \omega_1. \tag{6.60}$$

It goes without saying that each one of equations (6.58)–(6.60) holds for each $(t,\underline{x}_2) \in R \times \underline{E}_2$.

# 7. A general recursive asymptotic wave problem for a liquid layer of a uniform depth

## 7.1. Basic equations constituting the problem

**Corollary 7.1.** In the case of a liquid layer of a uniform depth $d$, in accordance with (5.8) and (6.50), application of the pertinent instances of Theorem 5.1 to the following three equations: (i) (4.34) extended to all $\underline{x} \in \underline{E}_3$; (ii) (4.74); and (iii) (4.103) subject to (4.104), (6.50), and (6.51) results in the following infinite recursive sequence of *non-scaled two-plane boundary value problems* for the partial velocity potentials $\phi_{(l)}$ of successive asymptotic approximations with ascending $l \in \omega_1$:

$$\Delta \phi_{(l)}(t,\underline{x}) = 0, \tag{7.1}$$

$$\left[\frac{\partial \phi_{(l)}(t,\underline{x})}{\partial z}\right]_{z=-d} = 0, \tag{7.2}$$

$$\left[\frac{\partial \phi_{(l)}(t,\underline{x})}{\partial z} + \frac{1}{g}\frac{\partial^2 \phi_{(l)}(t,\underline{x})}{\partial t^2}\right]_{z=0} = \alpha_{(l)}(t,\underline{x}_2), \tag{7.3}$$

where

$$\alpha_{(1)}(t,\underline{x}_2) \doteqdot 0, \tag{7.4}$$

in accordance with (6.55). Equation (7.1) is the variant of (5.46) with '$l$' in place of '$n$', whereas equation (7.2) is just an occurrence (token) of (6.59).•

**Definition 7.1.** 1) In compliance with the nomenclature introduced in Comment 4.6(c), the triple of equations (7.1)–(7.3) subject to (7.4) will be denoted by '$\mathsf{T}_{\mathrm{u}(l)}\left(\phi_{(l)}\right)$' for each given value of '$l$' in the set $\omega_1$; the subscript 'u' is the first letter of the word 'uniform' (cf. Comment 4.4). All the above equations are valid under Hypothesis 4.2. Therefore, in contrast to the symbol '$\mathsf{T}_{\mathrm{U}+}(\Phi, \mathrm{Z})$' and all other relevant symbols that have been introduced in Comment 4.6(c), the symbol '$\mathsf{T}_{\mathrm{u}(l)}\left(\phi_{(l)}\right)$' does not carry a subscript '+' that is indicative of Hypothesis 4.2.

2) Given $m \in \omega_1$, the finite sequence of triples $\mathsf{T}_{\mathrm{u}(l)}\left(\phi_{(l)}\right)$ with ascending $l \in \omega_{1,m}$ will be denoted by '$\left\langle \mathsf{T}_{\mathrm{u}(1)}\left(\phi_{(1)}\right), \mathsf{T}_{\mathrm{u}(2)}\left(\phi_{(2)}\right), \ldots, \mathsf{T}_{\mathrm{u}(m)}\left(\phi_{(m)}\right)\right\rangle$' or briefly by '$\left\langle \mathsf{T}_{\mathrm{u}(l)}\left(\phi_{(l)}\right)\right\rangle_{l \in \omega_{1,m}}$'. Accordingly, either one of the strings '$\left\langle \mathsf{T}_{\mathrm{u}(1)}\left(\phi_{(1)}\right), \mathsf{T}_{\mathrm{u}(2)}\left(\phi_{(2)}\right), \ldots\right\rangle$' and '$\left\langle \mathsf{T}_{\mathrm{u}(l)}\left(\phi_{(l)}\right)\right\rangle_{l \in \omega_{1,\infty}}$' stands for an infinite sequence of triples $\mathsf{T}_{\mathrm{u}(l)}\left(\phi_{(l)}\right)$ with ascending $l \in \omega_1$.

3) The version of $\mathsf{T}_{\mathrm{u}(l)}\left(\phi_{(l)}\right)$ with (6.60) in place of (7.2) will be denoted by '$\mathsf{T}_{\infty(l)}\left(\phi_{(l)}\right)$' and be called a *non-scaled two-plane boundary value problem* for the partial velocity potential $\phi_{(l)}$ of successive asymptotic approximation with ascending $l \in \omega_1$. The denotata of '$\mathsf{T}_{\infty(l)}\left(\phi_{(l)}\right)$' and of the variants with '∞' in place of 'u', of all symbols introduced in the previous item apply to a liquid semi-space $z \leq 0$.

4) The version of $\mathsf{T}_{\mathrm{u}(l)}\left(\phi_{(l)}\right)$ with (6.58) in place of (7.2) will be denoted by '$\mathsf{T}_{(l)}\left(\phi_{(l)}\right)$' and be called a *non-scaled two-surface boundary value problem* for the partial velocity potential $\phi_{(l)}$ of successive asymptotic approximation with ascending $l \in \omega_1$. The denotata of '$\mathsf{T}_{(l)}\left(\phi_{(l)}\right)$' and of the variants with '$\mathsf{T}_{(l)}$' in place of '$\mathsf{T}_{\mathrm{u}(l)}$', of all symbols introduced in the previous item apply to a liquid layer with a non-uniform (variable) depth described by the functional form '$-h\left(\underline{x}_2\right)$'. the same time, with an arbitrary depth function *h*, the triple $\mathsf{T}_{(1)}\left(\phi_{(1)}\right)$ is unsolvable analytically, – to say nothing of $\mathsf{T}_{(l)}\left(\phi_{(l)}\right)$ with $l \in \omega_2$. Therefore, I shall hereafter treat mainly of $\mathsf{T}_{\mathrm{u}(l)}\left(\phi_{(l)}\right)$, whereas $\mathsf{T}_{\infty(l)}\left(\phi_{(l)}\right)$, being the limit of $\mathsf{T}_{(l)}\left(\phi_{(l)}\right)$ as $h_m \to \infty$,will be regarded as the limit of $\mathsf{T}_{\mathrm{u}(l)}\left(\phi_{(l)}\right)$ as $d \to \infty$.•

**Corollary 7.2.** In accordance with the general place-holding formula (5.21), multiplying both sides of each one of equations (7.1)–(7.4) by '$\varepsilon^l$' results in the following upper case variants of those equations:

$$\Delta\Phi_{(l)}(t,\underline{x},\varepsilon) = 0\,, \tag{7.5}$$

$$\left[\frac{\partial\Phi_{(l)}(t,\underline{x},\varepsilon)}{\partial z}\right]_{z=-d} = 0\,, \tag{7.6}$$

$$\left[\frac{\partial\Phi_{(l)}(t,\underline{x},\varepsilon)}{\partial z} + \frac{1}{g}\frac{\partial^2\Phi_{(l)}(t,\underline{x},\varepsilon)}{\partial t^2}\right]_{z=0} = \mathrm{A}_{(l)}(t,\underline{x}_2,\varepsilon), \tag{7.7}$$

where

$$\mathrm{A}_{(1)}(t,\underline{x}_2,\varepsilon) \doteqdot 0\,. \tag{7.8}$$

It goes without saying that

$$\Phi_{(l)}(t,\underline{x},\varepsilon) \doteqdot \phi_{(l)}(t,\underline{x})\varepsilon^l,\ \mathrm{A}_{(l)}(t,\underline{x}_2,\varepsilon) \doteqdot \alpha_{(l)}(t,\underline{x}_2)\varepsilon^l \text{ for each } l \in \omega_1\,. \tag{7.9}$$

In this case, it is also understood that for each $l \in \omega_2$ ‘$\mathrm{A}_{(l)}(t,\underline{x}_2,\varepsilon)$’ can be expressed as the variant of ‘$\alpha_{(l)}(t,\underline{x}_2)$’, in which all constituent functional forms are replaced with their upper case variants in accordance with the pertinent instances of the place-holding formula (5.21) Accordingly, ‘$\varepsilon$’ becomes a hidden parameter of ‘$\mathrm{A}_{(l)}(t,\underline{x}_2,\varepsilon)$’.•

**Definition 7.2.** Definition 7.1 applies with “(7.5)”–“(7.8)” and ‘Φ in place of “(7.1)”–“(7.4)” and ‘$\phi$’ respectively.•

## 7.2. Reduction of $\mathsf{T}_{\mathrm{u}(l)}\left(\phi_{(l)}\right)$ for $l \in \omega_1$

**1**. I shall seek a solution of the Laplace equation (7.1) at any $l \in \omega_1$ by the method of separation of variables in the form:

$$\phi_{(l)}\left(t, \underline{x}\right) \doteqdot \theta_{(l)}(z)\psi_{(l)}\left(t, \underline{x}_2\right) \text{ for each } l \in \omega_1, \tag{7.10}$$

so that

$$\theta_{(l)}(z)\Delta_2\psi_{(l)}\left(t, \underline{x}_2\right) + \frac{d^2\theta_{(l)}(z)}{dz^2}\psi_{(l)}\left(t, \underline{x}_2\right) = 0, \tag{7.11}$$

where

$$\Delta_2 \doteqdot \Delta_2\left(\underline{x}_2\right) \doteqdot \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \doteqdot \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{7.12}$$

It follows from (7.11) that

$$\frac{1}{\psi_{(l)}\left(t, \underline{x}_2\right)}\Delta_2\psi_{(l)}\left(t, \underline{x}_2\right) = -\frac{1}{\theta_{(l)}(z)}\frac{d^2\theta_{(l)}(z)}{dz^2} = -k_{(l)}^2, \tag{7.13}$$

where $k_{(l)} > 0$ is a real-valued constant. Therefore,

$$\Delta_2\psi_{(l)}\left(t, \underline{x}_2\right) + k_{(l)}^2\psi_{(l)}\left(t, \underline{x}_2\right) = 0, \tag{7.14}$$

$$\frac{d^2\theta_{(l)}(z)}{dz^2} - k_{(l)}^2\theta_{(l)}(z) = 0. \tag{7.15}$$

It follows from (7.15) that

$$\theta_{(l)}(z) = c_{(l)1}e^{k_{(n)}z} + c_{(l)2}e^{-k_{(l)}z}, \tag{7.16}$$

where $c_{(l)1}$ and $c_{(l)2}$ are arbitrary real-valued constants and $k_{(l)} > 0$ is a *strictly positive real number*. Substitution of (7.8) subject to (7.16) into (7.2) yields:

$$c_{(l)1}e^{-k_{(l)}d} = c_{(l)2}e^{k_{(\mathrm{ln})}d}, \tag{7.17}$$

so that (7.16) can be developed as:

$$\begin{aligned} \theta_{(l)}(z) &= c_{(l)2}\left[e^{k_{(l)}(2d+z)} + e^{-k_{(l)}z}\right] = c_{(l)2}e^{k_{(l)}d}\left[e^{k_{(l)}(z+d)} + e^{-k_{(l)}(z+d)}\right] \\ &= 2c_{(l)2}e^{k_{(l)}d}\cosh k_{(l)}(z+d) = \frac{\cosh k_{(l)}(z+d)}{\cosh k_{(l)}d}, \end{aligned} \tag{7.16$_1$}$$

where I have set

$$c_{(l)2} = \frac{e^{-k_{(l)}d}}{2\cosh k_{(l)}d}. \tag{7.17$_1$}$$

Thus, (7.16$_1$) can ultimately be written as:

$$\Delta_2\psi_{(l)}(t,\underline{x}_2)+k_{(l)}^2\psi_{(l)}(t,\underline{x}_2)=0\,, \tag{7.14}$$

$$\theta_{(l)}(z)=\frac{\cosh k_{(l)}(z+d)}{\cosh k_{(l)}d}\ \text{ for each } l\in\omega_1\,, \tag{7.18}$$

so that

$$\left[\frac{d\theta_{(l)}(z)}{dz}\right]_{x=-d}=0\,. \tag{7.19}$$

Owing to (7.19), the functional form $\phi_{(l)}(t,\underline{x})$, defined by (7.10) subject to (7.18), satisfies the boundary condition (7.2) automatically.

At the same time, substitution of (7.10) into the left-hand side of equation (7.3) yields:

$$\left[\frac{d\theta_{(l)}(z)}{dz}\psi_{(l)}(t,\underline{x}_2)+\frac{1}{g}\theta_{(l)}(z)\frac{\partial^2\psi_{(l)}(t,\underline{x}_2)}{\partial t^2}\right]_{z=0}=\alpha_{(l)}(t,\underline{x}_2)\ \text{ for each } l\in\omega_1\,, \tag{7.20}$$

whereas it follows from (7.18) that

$$\theta_{(l)}(0)=1\ \ \text{(a)},\quad \left[\frac{d\theta_{(l)}(z)}{dz}\right]_{x=0}=k_{(l)}\tanh k_{(l)}d\quad \text{(b)}. \tag{7.21}$$

By (7.4) and (7.21), relation (7.20) becomes

$$\Omega^2(k_{(1)})\psi_{(1)}(t,\underline{x}_2)+\frac{\partial^2\psi_{(1)}(t,\underline{x}_2)}{\partial t^2}=0\ \text{ for } l=1, \tag{7.22}$$

$$\frac{1}{g}\left[\Omega^2(k_{(l)})\psi_{(l)}(t,\underline{x}_2)+\frac{\partial^2\psi_{(l)}(t,\underline{x}_2)}{\partial t^2}\right]=\alpha_{(l)}(t,\underline{x}_2)\ \text{ for each } l\in\omega_2\,, \tag{7.23}$$

where

$$\Omega^2(k_{(l)})\doteqdot gk_{(l)}\tanh k_{(l)}d\ \text{ for each } l\in\omega_1\,. \tag{7.24}$$

**2.** Thus, under the definition (7.10), the triple of equations (7.1)–(7.3) subject to (7.4) reduces to the quadruple (conjunction of four) equations (7.14), (7.18), (7.22), and (7.23), subject to (7.24), – the conjunction set that involves an infinite set of *arbitrary* strictly positive real numbers $k_{(l)}>0$ with $l\in\omega_1$. However, Hypothesis 5.1 implies that the functional form $\phi_{(l)}(t,\underline{x})$ with any $l\in\omega_2$ is ultimately expressible in terms of the functional form $\phi_{(1)}(t,\underline{x})$. Therefore, once a real number $k_{(1)}>0$ is selected, the following relation must hold:

$$k_{(l)}=k_{(1)}\doteqdot k>0\ \text{ for each } l\in\omega_2\,, \tag{7.30}$$

the understanding being that the variable '$k_{(1)}$' is thereby abbreviated as '$k$'. Hence, the above quadruple of equations reduces to the following one for each $l \in \omega_1$:

$$\Delta_2 \psi_{(l)}(t, \underline{x}_2) + k^2 \psi_{(l)}(t, \underline{x}_2) = 0 \text{ for each } l \in \omega_1, \tag{7.31}$$

$$\theta_{(l)}(z) = \theta(z) \doteq \theta(z, k) \doteq \theta(z, k, d) \doteq \frac{\cosh k(z+d)}{\cosh kd} \text{ for each } l \in \omega_1, \tag{7.32}$$

$$\Omega^2(k)\psi_{(1)}(t, \underline{x}_2) + \frac{\partial^2 \psi_{(1)}(t, \underline{x}_2)}{\partial t^2} = 0 \text{ for } l = 1, \tag{7.33}$$

$$\frac{1}{g}\left[\Omega^2(k)\psi_{(l)}(t, \underline{x}_2) + \frac{\partial^2 \psi_{(l)}(t, \underline{x}_2)}{\partial t^2}\right] = \alpha_{(l)}(t, \underline{x}_2) \text{ for each } l \in \omega_2, \tag{7.34}$$

where

$$\Omega^2(k) \doteq [\Omega(k)]^2 \doteq gk\tanh kd\,. \tag{7.35}$$

Consequently, in accordance with (7.32) and (7.35), equations (7.19) and (7.21)

$$\left[\frac{d\theta(z)}{dz}\right]_{x=-d} = 0\,. \tag{7.36}$$

$$\theta(0) = 1 \;\text{ (a)}, \quad \left[\frac{d\theta_{(l)}(z)}{dz}\right]_{x=0} = k\tanh kd \doteq \Omega^2(k) \;\text{ (b)}. \tag{7.37}$$

The *quantity*, i.e. the *dimensional strictly positive real number of a dimension* "time$^{-1}$", $\Omega(k)$, defined as

$$\Omega(k) \doteq \Omega(k, d) \doteq \sqrt{gk\tanh kd} > 0\,, \tag{7.38}$$

is called a *main initial cyclic eigenfrequency of gravity waves on the liquid layer*, whereas the *functional form* '$\Omega(k, d)$' or its definies '$\sqrt{gk\tanh kd}$' is called the *dispersion functional form of those waves*.

**Theorem 7.1.**

$$\Delta_2 \alpha_{(l)}(t, \underline{x}_2) + k^2 \alpha_{(l)}(t, \underline{x}_2) = 0 \text{ for each } l \in \omega_2, \tag{7.39}$$

**Proof:** Equations (7.31) and (7.34) can conveniently be written as:

$$\hat{K}(\underline{x}_2, k)\psi_{(l)}(t, \underline{x}_2) = 0 \text{ for each } l \in \omega_1 \tag{7.40}$$

subject to

$$\hat{K}(\underline{x}_2, k) \doteq \Delta_2(\underline{x}_2) + k^2 \doteq \Delta_2 + k^2 \tag{7.41}$$

(cf. (7.12)) and as

$$\hat{L}(t,\Omega(k))\psi_{(l)}(t,\underline{x}_2) = g\alpha_{(l)}(t,\underline{x}_2) \text{ for each } l \in \omega_2\,, \tag{7.42}$$

subject to

$$\hat{L}(t,\Omega(k)) \doteq \frac{\partial^2}{\partial t^2} + \Omega^2(k). \tag{7.43}$$

Differentiation of both sides of equation (7.40) with respect to '$t$' twice yields

$$\hat{K}(\underline{x}_2,k)\frac{\partial^2\psi_{(l)}(t,\underline{x}_2)}{\partial t^2} = 0 \text{ for each } l \in \omega_1 \tag{7.40$_1$}$$

Therefore, applying of the operator $\hat{K}(\underline{x}_2,k)$ to both sides of equation (7.42) and then making use of equations (7.43), (7.40$_1$), and (7.40) in this order, one obtains

$$\begin{aligned} &\hat{K}(\underline{x}_2,k)\hat{L}(t,\Omega(k))\psi_{(l)}(t,\underline{x}_2) \\ &= \hat{K}(\underline{x}_2,k)\left[\frac{\partial^2\psi_{(l)}(t,\underline{x}_2)}{\partial t^2} + \Omega^2(k)\psi_{(l)}(t,\underline{x}_2)\right] = 0 \\ &= g\hat{K}(\underline{x}_2,k)\alpha_{(l)}(t,\underline{x}_2) \text{ for each } l \in \omega_2, \end{aligned} \tag{7.40$_2$}$$

whence

$$\hat{K}(\underline{x}_2,k)\alpha_{(l)}(t,\underline{x}_2) = 0 \text{ for each } l \in \omega_2\,. \tag{7.44}$$

QED.•

### 7.3. Particular solutions of equation (7.34) for $l \in \omega_2$

**Theorem 7.2.** Given a time instant $t_0 \in R$, for each $\langle t,\underline{x}_2\rangle \in R \times \underline{E}_2$, the functional form $\psi_{(l)*}(t,\underline{x}_2,t_0)$, defined as:

$$\psi_{(l)*}(t,\underline{x}_2,t_0) \doteq \frac{g}{\Omega(k)}\int_{t_0}^{t} \alpha_{(l)}(t',\underline{x}_2)\sin[\Omega(k)(t-t')]\,dt' \text{ for each } l \in \omega_2\,., \tag{7.45}$$

is a unique particular solution of the equation

$$\Omega^2(k)\psi_{(l)*}(t,\underline{x}_2,t_0) + \frac{\partial^2\psi_{(l)*}(t,\underline{x}_2,t_0)}{\partial t^2} = g\alpha_{(l)}(t,\underline{x}_2) \text{ for each } l \in \omega_2\,, \tag{7.46}$$

being the pertinent variant of equation (7.34), – the solution that automatically satisfies its *identifying conditions*:

$$\psi_{(l)*}(t_0,\underline{x}_2,t_0) = 0 \text{ (a) and } \left[\frac{\partial\psi_{(l)*}(t,\underline{x}_2,t_0)}{\partial t}\right]_{t=t_0} = 0 \text{ (b) for each } l \in \omega_1\,. \tag{7.47}$$

Since equation (7.45) holds both for $t \geq t_0$ and for $t \leq t_0$, therefore I call equations (7.47) "*identifying conditions*", thus avoiding calling them "*initial conditions*".

**Proof:** Equation (7.45) will be *deduced* from a lemma to be stated and *proved deductively* in the next subsection. Meanwhile, I shall verify the validity of (7.45) and (7.47) by the following straightforward calculations. Equation (7.47a) subject to (7.45) at $t = t_0$ is self-evident, while differentiating both sides of (7.45) with respect to *t* in accordance with the Leibnitz rule of differentiation of an integral with variable limits yields

$$\frac{\partial \psi_{(l)*}(t, \underline{x}_2, t_0)}{\partial t} = \frac{g}{\Omega(k)}\left[\alpha_{(l)}(t', \underline{x}_2)\sin[\Omega(k)(t - t')]\right]_{t'=t} + g\int_{t_0}^{t} \alpha_{(l)}(t', \underline{x}_2)\cos[\Omega(k)(t - t')]\,dt' = g\int_{t_0}^{t} \alpha_{(l)}(t', \underline{x}_2)\cos[\Omega(k)(t - t')]\,dt', \tag{7.45$_1$}$$

whence (7.47b) follows immediately. At the same time, another differentiation of both sides of (7.45$_1$) with respect to *t* yields

$$\begin{aligned}\frac{\partial^2 \psi_{(l)*}(t, \underline{x}_2, t_0)}{\partial t^2} &= g\left[\alpha_{(l)}(t', \underline{x}_2)\cos[\Omega(k)(t - t')]\right]_{t'=t} \\ &\quad - g\Omega(k)\int_{t_0}^{t} \alpha_{(l)}(t', \underline{x}_2)\sin[\Omega(k)(t - t')]\,dt' \qquad (7.45_2) \\ &= g\alpha_{(l)}(t, \underline{x}_2) - g\Omega(k)\int_{t_0}^{t} \alpha_{(l)}(t', \underline{x}_2)\sin[\Omega(k)(t - t')]\,dt'.\end{aligned}$$

Substituting (7.45) and (7.45$_2$) into the left-hand side of equation (7.46), one obtains

$$\begin{aligned}\Omega^2(k)\psi_{(l)*}(t, \underline{x}_2, t_0) + \frac{\partial^2 \psi_{(l)*}(t, \underline{x}_2, t_0)}{\partial t^2} &= g\Omega(k)\int_{t_0}^{t} \alpha_{(l)}(t', \underline{x}_2)\sin[\Omega(k)(t - t')]\,dt' \\ &\quad + g\alpha_{(l)}(t, \underline{x}_2) - g\Omega(k)\int_{t_0}^{t} \alpha_{(l)}(t', \underline{x}_2)\sin[\Omega(k)(t - t')]\,dt' = g\alpha_{(l)}(t, \underline{x}_2),\end{aligned} \tag{7.46$_1$}$$

thus proving (7.46).•

**Comment 7.1.** If $t_1 \in R$, denoted by '$t_1$', is a time instant distinct from the time instant $t_0 \in R$ that has been denoted by '$t_0$' then Theorem 7.2 holds with '$t_1$' in place '$t_0$'. That is to say, for each $\langle t, \underline{x}_2 \rangle \in R \times \underline{E}_2$, the functional form $\psi_{(l)*}(t, \underline{x}_2, t_1)$, defined as:

$$\psi_{(l)*}(t, \underline{x}_2, t_1) \stackrel{=}{\rightarrow} \frac{g}{\Omega(k)}\int_{t_1}^{t} \alpha_{(l)}(t', \underline{x}_2)\sin[\Omega(k)(t - t')]\,dt' \text{ for each } l \in \omega_2, \tag{7.45a}$$

is a unique particular solution of the equation

$$\Omega^2(k)\psi_{(l)*}(t,\underline{x}_2,t_1)+\frac{\partial^2\psi_{(l)*}(t,\underline{x}_2,t_1)}{\partial t^2}=g\alpha_{(l)}(t,\underline{x}_2) \text{ for each } l\in\omega_2, \tag{7.46a}$$

being the pertinent variant of equation (7.34), – the solution that automatically satisfies its *identifying conditions*:

$$\psi_{(l)*}(t_1,\underline{x}_2,t_1)=0 \text{ (a) and } \left[\frac{\partial\psi_{(l)*}(t,\underline{x}_2,t_1)}{\partial t}\right]_{t=t_1}=0 \text{ (b) for each } l\in\omega_1, \tag{7.47a}$$

and that therefore and differs from the solution $\psi_{(l)*}(t,\underline{x}_2,t_0)$ given by (7.45).

Subtraction of equation (7.46a) from equation (7.46) yields

$$\left[\Omega^2(k)+\frac{\partial^2}{\partial t^2}\right]\left[\psi_{(l)*}(t,\underline{x}_2,t_0)-\psi_{(l)*}(t,\underline{x}_2,t_1)\right]=0 \text{ for each } l\in\omega_2, \tag{7.46b}$$

which is a variant of the homogeneous equation (7.33). At the same time, by (7.45) and (7.45a), it follows that

$$\begin{aligned}\psi_{(l)*}(t,\underline{x}_2,t_0)-\psi_{(l)*}(t,\underline{x}_2,t_1)&=\frac{g}{\Omega(k)}\int_{t_0}^{t}\alpha_{(l)}(t',\underline{x}_2)\sin[\Omega(k)(t-t')]\,dt'\\ -\frac{g}{\Omega(k)}\int_{t_1}^{t}\alpha_{(l)}(t',\underline{x}_2)\sin[\Omega(k)(t-t')]\,dt'&=\frac{g}{\Omega(k)}\int_{t_0}^{t}\alpha_{(l)}(t',\underline{x}_2)\sin[\Omega(k)(t-t')]\,dt'\\ +\frac{g}{\Omega(k)}\int_{t}^{t_1}\alpha_{(l)}(t',\underline{x}_2)\sin[\Omega(k)(t-t')]\,dt'&=\frac{g}{\Omega(k)}\int_{t_0}^{t_1}\alpha_{(l)}(t',\underline{x}_2)\sin[\Omega(k)(t-t')]\,dt'\\ &\text{for each } l\in\omega_2,\end{aligned} \tag{7.45b}$$

whence

$$\frac{\partial\left[\psi_{(l)*}(t,\underline{x}_2,t_0)-\psi_{(l)*}(t,\underline{x}_2,t_1)\right]}{\partial t}=g\int_{t_0}^{t_1}\alpha_{(l)}(t',\underline{x}_2)\cos[\Omega(k)(t-t')]\,dt', \tag{7.45b$_1$}$$

$$\frac{\partial^2\left[\psi_{(l)*}(t,\underline{x}_2,t_0)-\psi_{(l)*}(t,\underline{x}_2,t_1)\right]}{\partial t^2}=-g\Omega(k)\int_{t_0}^{t_1}\alpha_{(l)}(t',\underline{x}_2)\sin[\Omega(k)(t-t')]\,dt'. \tag{7.45b$_2$}$$

By (7.45) and (7.45b$_2$), the expression on the left-hand side of equation (7.46b) vanishes as expected.

Assume that a certain functional form $\psi_{(1)}(t,\underline{x}_2)$, satisfying both equation (7.31) at $l$=1 and equation (7.33), is found. Then it follows from (7.46b) that

$$\psi_{(l)*}(t,\underline{x}_2,t_0)-\psi_{(l)*}(t,\underline{x}_2,t_1)=c_{(l)}\psi_{(1)}(t,\underline{x}_2) \text{ for each } l\in\omega_1, \tag{7.45c}$$

where $c_{(l)}$ is a certain real number other than 0. Thus, to any given solution $\psi_{(1)}(t,\underline{x}_2)$ of equation (7.33), there corresponds *continuum* of particular solutions $\psi_{(l)*}(t,\underline{x}_2,t_0)$ of equations (7.45) that are determined by values of the variable '$t_0$' in *continuum* (*uncountable set*) *of real numbers, R.* This result contradicts Hypothesis 5.1, by the argument similar to that used in item 2 of the previous subsection for deducing (7.30).

At the same time, there is a general philosophical principle of "saving thoughts", the original version of which is known under the name "*Ockham's razor*" or "*The Ockham's razor principle*", after the English Scholastic philosopher William of Ockham or Occam (A.D. ca1300–ca1349). This principle says that *entities should not be multiplied unless necessary*. Consequently, Ockham's razor is the most general groundwork for formulating *axiomatic theories,* because in setting up any particular axiomatic theory, it suggests that, for avoidance contradictions, the number of axioms (permanent postulates, permanent hypotheses) should be as small as possible. Therefore, in accordance with Ockham's razor, I shall adopt the following additional hypothesis, being the instance of Theorem 7.2 for $t_0 = 0$.•

**Hypothesis 7.1.** For each $\langle t,\underline{x}_2\rangle \in R\times \underline{E}_2$, the functional form $\psi_{(l)}(t,\underline{x}_2)$, defined as:

$$\psi_{(l)}(t,\underline{x}_2) \doteqdot \psi_{(l)*}(t,\underline{x}_2,0) \doteqdot \frac{g}{\Omega(k)}\int_0^t \alpha_{(l)}(t',\underline{x}_2)\sin[\Omega(k)(t-t')]\,dt' \text{ for each } l\in\omega_2, \quad (7.48)$$

is the only pertinent *full solution* of equation (7.34), – the solution that automatically satisfies its *identifying conditions*:

$$\psi_{(l)}(0,\underline{x}_2) = 0 \text{ (a) and } \left[\frac{\partial\psi_{(l)}(t,\underline{x}_2)}{\partial t}\right]_{t=0} = 0 \text{ (b) for each } l\in\omega_1. \quad (7.49)$$

In this case, the qualifier "*full*" to "*solution*" means that $\psi_{(l)}(t,\underline{x}_2)$ is actually defined for each $l\in\omega_2$ as:

$$\psi_{(l)}(t,\underline{x}_2) \doteqdot c_{(l)}\psi_{(1)}(t,\underline{x}_2) + \psi_{(l)*}(t,\underline{x}_2,0) \text{ subject to } c_{(l)} \doteqdot 0. \quad (7.48a)$$•

### 7.4. An auxiliary identifying value problem of a forced motion of an abstract unit one-dimensional harmonic oscillator

**Lemma 7.1.** Let a real-valued functional form ‘ $f(t)$ ’ of time variable ‘ $t$ ’, which is assumed to be defined for all $t \in R$ and whose values are assumed to be absolutely integrable on any finite interval of *R*. Then, given a real number $\omega$>0, for each $t \in R$ the functional form ‘ $u(t)$ ’, defined as:

$$u(t) \fallingdotseq \frac{1}{\omega}\int_0^t f(t')\sin\omega(t-t')\,dt', \tag{7.50}$$

is a unique particular solution of the equation

$$\ddot{u}(t) + \omega^2 u(t) = f(t), \tag{7.51}$$

which satisfies its *identifying conditions*:

$$u(0) = 0 \text{ (a) and } \dot{u}(0) = 0 \text{ (b);} \tag{7.52}$$

it is understood that

$$\dot{u}(t) \fallingdotseq \frac{du(t)}{dt},\ \ddot{u}(t) \fallingdotseq \frac{d^2u(t)}{dt^2}. \tag{7.53}$$

Just as (7.45), equation (7.50) holds both for $t \geq t_0$ and for $t \leq t_0$. Therefore, equations (7.52) are called “*identifying conditions*”, and not “*initial conditions*”.

Most naturally, equation (7.51) can be interpreted as one that describes, at each given time instant *t*, the one-dimensional displacement $u(t)$ of a point material particle, or of the mass center of a material solid body, of unit mass from its equilibrium position $u(0)=0$ under the action both of the internal recovering force $-\omega^2 u(t)$ with a stiffness coefficient (per unit mass) $\omega^2$ and of the external force $f(t)$. The imaginary one-dimensional system of the material point particle, or of the material body, of unit mass, acted upon only by the recovering force $-\omega^2 u(t)$, is capable of executing one-dimensional free vibrations with cyclic frequency $\omega > 0$, and is therefore called an *abstract unit one-dimensional harmonic oscillator* (cf. Landau and Lifshitz [1988, p. 58]).

**Proof:** 1) It is understood that Theorem 7.2 and Hypothesis 7.1 are instance of this lemma. Therefore, in analogy with the proof of Theorem 7.2, the validity of equations (7.50) and (7.52) can be verified by the following straightforward calculations. Equation (7.52a) subject to (7.50) at $t=0$ is self-evident, while differentiating both sides of (7.50) with respect to *t* yields

$$\frac{du(t)}{dt} = \frac{1}{\omega}\left[f(t')\sin\omega(t-t')\right]_{t'=t} + \int_0^t f(t')\cos\omega(t-t')\,dt' = \int_0^t f(t')\cos\omega_*(t-t')\,dt', \tag{7.50$_1$}$$

whence (7.52b) follows immediately. At the same time, another differentiation of both sides of ($7.50_1$) with respect to *t* yields

$$\frac{d^2u(t)}{dt^2} = \left[ f(t')\cos\omega(t-t') \right]_{t'=t} - \omega\int_{t_0}^{t} f(t')\sin\omega(t-t')\,dt'$$
$$= f(t) - \omega\int_{0}^{t} f(t')\sin\omega_*(t-t')\,dt'. \qquad (7.50_2)$$

Substituting (7.50) and ($7.50_2$) into the left-hand side of equation (7.51) subject to (7.53) yields s

$$\frac{d^2u(t)}{dt^2} + \omega^2 u(t)$$
$$= f(t) - \omega\int_{0}^{t} f(t')\sin\omega(t-t')\,dt' + \omega\int_{0}^{t} f(t')\sin\omega_*(t-t')\,dt' = f(t), \qquad (7.51_1)$$

thus proving (7.51). Still, in order to establish the foundations of equation (7.50), it is instructive to *deduce* it from (7.51)–(7.53) by the pertinent instance of the *method of variation of parameters* (see Ellsgolts [1980, pp. 122–126, especially Example 3, pp. 125–126]}, which is done below.

2) The general solution of the homogeneous equation

$$\ddot{y}(t) + \omega^2 y(t) = 0\,, \qquad (7.54)$$

adjoint of (7.51), has the form

$$y(t) = \sum_{i=1}^{2} c_i(t)\,y_i(t) = c_1\cos\omega t + c_2\sin\omega t\,, \qquad (7.55)$$

the understanding being that $y_1(t)$ and $y_2(t)$, defined as

$$y_1(t) \doteqdot \cos\omega t \text{ and } y_2(t) \doteqdot \sin\omega t\,, \qquad (7.56)$$

are fundamental solutions of (7.54), while '$c_1$' and '$c_2$' are arbitrary real-valued constants. In applying the method of variation of parameters to the nonhomogeneous equation (7.51), the solution of the latter is sought in the form

$$u(t) = \sum_{i=1}^{2} c_i(t)\,y_i(t) = c_1(t)\cos\omega t + c_2(t)\sin\omega t\,, \qquad (7.57)$$

so that two unknown functions $c_1(t)$ and $c_2(t)$ are introduced in place of the one unknown function $u(t)$. Since the former two functions have to satisfy only one equation (7.51) subject to (7.57), it can be required that they should satisfy some other additional equation. The latter is chosen to have the form

$$\sum_{i=1}^{2} \dot{c}_i(t) y_i(t) = \dot{c}_1(t) \cos \omega t + \dot{c}_2(t) \sin \omega t = 0 , \tag{7.58}$$

so that

$$\dot{u}(t) = \sum_{i=1}^{2} c_i(t) \dot{y}_i(t) + \sum_{i=1}^{2} \dot{c}_i(t) y_i(t) = \sum_{i=1}^{2} c_i(t) \dot{y}_i(t) = -\omega c_1(t) \sin \omega t + \omega c_2(t) \cos \omega t . \tag{7.59}$$

That is, owing to (7.57), $\dot{u}(t)$ has the same form that it would have in the case of constant $c_1$ and $c_2$. By (7.55) and (7.56), it follows from (7.59) that

$$\begin{aligned} \ddot{u}(t) &= \sum_{i=1}^{2} c_i(t) \dot{y}_i(t) + \sum_{i=1}^{2} \dot{c}_i(t) y_i(t) = \sum_{i=1}^{2} \dot{c}_i(t) \dot{y}_i(t) + \sum_{i=1}^{2} c_i(t) \ddot{y}_i(t) \\ &= \sum_{i=1}^{2} \dot{c}_i(t) \dot{y}_i(t) - \omega^2 \sum_{i=1}^{2} c_i(t) y_i(t) = -\omega \dot{c}_1(t) \sin \omega t + \omega \dot{c}_2(t) \cos \omega t - \omega^2 u(t), \end{aligned} \tag{7.60}$$

Consequently, by (7.55) and (7.60), equation (7.51) becomes

$$-\omega \dot{c}_1(t) \sin \omega t + \omega \dot{c}_2(t) \cos \omega t = f(t) . \tag{7.61}$$

Solving the system of two linear algebraic equations (7.58) and (7.60) with respect to $\dot{c}_1(t)$ and $\dot{c}_2(t)$ yields

$$\dot{c}_1(t) = -\frac{1}{\omega} f(t) \sin \omega t, \ \dot{c}_2(t) = \frac{1}{\omega} f(t) \cos \omega t = f(t) , \tag{7.62}$$

whence

$$c_1(t) = -\frac{1}{\omega} \int_0^t f(\tau) \sin \omega \tau d\tau + \bar{c}_1, \ c_2(t) = \frac{1}{\omega} \int_0^t f(\tau) \cos \omega \tau d\tau + \bar{c}_2 , \tag{7.63}$$

'$\bar{c}_1$' and '$\bar{c}_1$' being arbitrary real-valued constants. Equation (7.57) subject to (7.63) becomes

$$\begin{aligned} u(t) = -\frac{\cos \omega t}{\omega} \int_0^t f(t') \sin \omega t' dt' + \frac{\sin \omega t}{\omega} \int_0^t f(t') \cos \omega t' dt' \\ + \bar{c}_1 \cos \omega t + \bar{c}_2 \sin \omega t \end{aligned} \tag{7.64}$$

or

$$\begin{aligned} u(t) &= \frac{1}{\omega} \int_0^t [\sin \omega t \cos \omega t' - \cos \omega t \sin \omega t'] f(t') dt' + \bar{c}_1 \cos \omega t + \bar{c}_2 \sin \omega t \\ &= \frac{1}{\omega} \int_0^t \sin \omega (t - t') f(t') dt' + \bar{c}_1 \cos \omega t + \bar{c}_2 \sin \omega t, \end{aligned} \tag{7.65}$$

where use of the pertinent variant of the first one of the following two general equations has been made:

$$\begin{aligned} \sin(\alpha \pm \beta) &= \sin\alpha \cos\beta \pm \cos\alpha \sin\beta \ \text{(a)}, \\ \cos(\alpha \pm \beta) &= \cos\alpha \cos\beta \mp \sin\alpha \sin\beta \ \text{(b)}. \end{aligned} \tag{7.66}$$

In accordance with (7.50) and ($7.50_1$), it follows from (7.65) that

$$u(0) = \bar{c}_1 \ \text{(a)}, \ \dot{u}(0) = \bar{c}_2 \ \text{(b)}. \tag{7.67}$$

Therefore, $u(t)$ given by (7.65) satisfies equations (7.52) if and only if

$$\bar{c}_1 = \bar{c}_2 = 0. \tag{7.68}$$

The lemma is established.•

**Comment 7.**2. With $\omega = 0$, equation (5.33) subject to (5.34) becomes

$$\frac{d^2u(t)}{dt^2} = f(t) \ \text{for each} \ t \in R. \tag{7.69}$$

The general solution of this equation can be obtained straightforwardly by two successive integrations with respect to *t* thus:

$$u(t) = \int_0^t dt'' \int_0^{t''} dt' f(t') + \bar{c}_1 + \bar{c}_2 t \tag{7.70}$$

subject to (7.67). At the same time, as $\omega \to 0$, equation (7.65) becomes

$$u(t) = \int_0^t (t - t') f(t') dt' + \bar{c}_1 \tag{7.71}$$

subject to (7.67,a) and (7.52,b). Making use of the Leibnitz rule of differentiation of an integral with variable limits, it can readily be verified by the pertinent straightforward calculations that $u(t)$, given by (7.71), satisfies equation (7.69) indeed; namely:

$$\frac{du(t)}{dt} = \left[(t - t') f(t')\right]_{t'=t} + \int_0^t f(t') dt' = \int_0^t f(t') dt', \tag{$7.71_1$}$$

$$\frac{d^2u(t)}{dt^2} = \frac{d}{dt} \int_0^t f(t') dt' = f(t) \tag{$7.71_2$}$$

(cf. ($7.50_1$) and ($7.50_2$)); equation (7.52,b) follows from ($7.71_1$) immediately.•

## 7.5. Reduction $\mathrm{T}_{\mathrm{U}(l)}\left(\Phi_{(l)}\right)$ and its solutions for $l \in \omega_1$

Multiplying both sides of equation (7.10) subject to (7.32) by $\varepsilon^l$ yields:

$$\Phi_{(l)}(t,\underline{x}) \doteqdot \theta(z)\Psi_{(l)}(t,\underline{x}_2) \text{ for each } l \in \omega_1, \tag{7.72}$$

where

$$\Psi_{(l)}(t,\underline{x}_2) \doteqdot \psi_{(l)}(t,\underline{x}_2)\varepsilon^l \text{ for each } l \in \omega_1, \tag{7.73}$$

in addition to (7.9). In this case, equation (7.14) is replaced with this one:

$$\Delta_2\Psi_{(l)}(t,\underline{x}_2) + k_{(l)}^2\Psi_{(l)}(t,\underline{x}_2) = 0 . \tag{7.74}$$

Consequently, under the definition (7.72), the following three equations subject to (7.8) come instead of equations (7.31), (7.33), and (7.34) respectively:

$$\Delta_2\Psi_{(l)}(t,\underline{x}_2) + k^2\Psi_{(l)}(t,\underline{x}_2) = 0 \text{ for each } l \in \omega_1, \tag{7.75}$$

$$\mathit{\Omega}^2(k)\Psi_{(1)}(t,\underline{x}_2) + \frac{\partial^2\Psi_{(1)}(t,\underline{x}_2)}{\partial t^2} = 0 \text{ for } l = 1, \tag{7.76}$$

$$\frac{1}{g}\left[\mathit{\Omega}^2(k)\Psi_{(l)}(t,\underline{x}_2) + \frac{\partial^2\Psi_{(l)}(t,\underline{x}_2)}{\partial t^2}\right] = \mathrm{A}_{(l)}(t,\underline{x}_2) \text{ for each } l \in \omega_2 . \tag{7.77}$$

Likewise, all of the rest of subsections 7.2 and 7.3 apply with the upper case letter variants of functional variables in place of the lower case ones.

### 7.6. A general recursive asymptotic wave problem for a liquid semi-infinite space

A liquid semi-infinite space $z \le 0$ can be regarded as the limiting case of a gravity wave of a short wavelength $\lambda \doteqdot 2\pi/k$ on a liquid layer of a uniform depth $d$, such that $kd \to \infty$. Therefore, all formulas of subsection ns 7.2, 7.3, and 7.5 that involve tokens of the functional form '$\theta(z)$' defined by (7.32) hold if those tokens are replaced with tokens of the functional form '$\theta_\infty(z)$' defined as:

$$\theta_\infty(z) \doteqdot \theta_\infty(z,k) \doteqdot \lim_{kd\to\infty}\theta(z,k,d) = \lim_{kd\to\infty}\frac{\cosh k(z+d)}{\cosh kd} = \lim_{kd\to\infty}\frac{e^{k(z+d)} + e^{-k(z+d)}}{e^{kd} + e^{-kd}} = e^{kz} \tag{7.78}$$

subject to $z \le 0$. In this case, it follows from (7.38) that

$$\mathit{\Omega}_\infty(k) \doteqdot \lim_{kd\to\infty}\mathit{\Omega}(k,d) = \lim_{kd\to\infty}\sqrt{gk\tanh kd} = \sqrt{gk} > 0 , \tag{7.79}$$

# 8. A progressive or standing plane monochromatic water wave as the first non-vanishing approximation of the recursive theory

## 8.1. Fundamental solutions of the set of equations (7.31) at *l*=1 and (7.33)

The basic Eulerian equation of motion (4.24), an asymptotic solution of which is under the study, is nonlinear. Therefore, its solutions, whatever they could be, do not satisfy a principle of superposition. For instance, if $\underline{V}^{(1)}$ and $\underline{V}^{(2)}$ are two different solutions of equation (4.24) then the sum $\underline{V}^{(1)} + \underline{V}^{(2)}$ does not satisfy that equation. Therefore, the functional form $\psi_{(1)}(t, \underline{x}_2)$, which determines both the scaled velocity potential of the first approximation $\phi_{(1)}(t, \underline{x})$ (see equation (7.10) subject to (7.32)), and the non-scaled one $\Phi_{(1)}(t, \underline{x})$ (see equations (7.72) subject to (7.73), for *l*=1), is sought as a solution of c In compliance with Definition 7.1(1), this set of equations will be denoted by '$\mathsf{D}(\psi_{(1)})$'. The set of equation (7.75) at *l*=1 and of equation (7.76) is the pertinent variant of the above set and it will therefore be denoted by '$\mathsf{D}(\Psi_{(1)})$'. The set $\mathsf{D}(\psi_{(1)})$ implies that $\psi_{(1)}(t, \underline{x}_2)$ should be interpreted as a *single plane monochromatic wave of a wave number* $k > 0$ (introduced by equations (7.13) and (7.30)) *and of the cyclic eigenfrequency* $\Omega(k)$, defined by equation (7.38). Accordingly, neither $\psi_{(1)}(t, \underline{x}_2)$ nor $\Psi_{(1)}(t, \underline{x}_2)$ can be sought, e.g., in the form of any Fourier integral.

Both equation (7.31) at *l*=1 and equation (7.33) are linear in $\psi_{(1)}(t, \underline{x}_2)$ and homogeneous. Fundamental real solutions of the former equation are given by these two *trigonometric* functional forms:

$$\cos \underline{k}_2 \cdot \underline{x}_2 \text{ and } \sin \underline{k}_2 \cdot \underline{x}_2 , \tag{8.1}$$

where

$$\underline{k}_2 \doteqdot \langle k_1, k_2 \rangle \doteqdot \langle k_x, k_y \rangle \tag{8.2}$$

subject to

$$|\underline{k}_2|^2 \doteqdot \underline{k}_2{}^2 \doteqdot k_1{}^2 + k_2{}^2 = k^2 , \tag{8.3}$$

and also where

$$\underline{x}_2 \doteqdot \langle x_1, x_2 \rangle \doteqdot \langle x, y \rangle . \tag{8.4}$$

Fundamental real solutions of equation (7.33) are given by these two *trigonometric* functional forms:

$$\cos \Omega(k) t \text{ and } \sin \Omega(k) t , \tag{8.5}$$

subject to (7.38). Functional forms (8.1) can be combined with functional forms (8.5) so as to create various fundamental trigonometric functional forms satisfying $\mathsf{D}(\psi_{(1)})$, i.e. both equations: (7.31) at $l$=1 and (7.33) simultaneously. To be specific, each one of the following four fundamental trigonometric functional forms satisfies $\mathsf{D}(\psi_{(1)})$ and is descriptive of a *plane standing gravity wave*:

$$\begin{aligned} &\chi_{1,1}(t,\underline{x}_2,\underline{k}_2) \stackrel{=}{=} \sin\Omega(k)t \sin\underline{k}_2\cdot\underline{x}_2 \quad \text{(a)},\\ &\chi_{1,-1}(t,\underline{x}_2,\underline{k}_2) \stackrel{=}{=} \sin\Omega(k)t \cos\underline{k}_2\cdot\underline{x}_2 \quad \text{(b)},\\ &\chi_{-1,1}(t,\underline{x}_2,\underline{k}_2) \stackrel{=}{=} \cos\Omega(k)t \sin\underline{k}_2\cdot\underline{x}_2 \quad \text{(c)},\\ &\chi_{-1,-1}(t,\underline{x}_2,\underline{k}_2) \stackrel{=}{=} \cos\Omega(k)t \cos\underline{k}_2\cdot\underline{x}_2 \quad \text{(d)}, \end{aligned} \tag{8.6}$$

whereas each one of the following four fundamental trigonometric functional forms also satisfies $\mathsf{D}(\psi_{(1)})$ but it is descriptive of a *plane progressive gravity wave*:

$$\begin{aligned} &\chi_1^-(t,\underline{x}_2,\underline{k}_2) \stackrel{=}{=} \sin[\Omega(k)t-\underline{k}_2\cdot\underline{x}_2] \quad \text{(a)},\\ &\chi_1^+(t,\underline{x}_2,\underline{k}_2) \stackrel{=}{=} \sin[\Omega(k)t+k_2\cdot\underline{x}_2] \quad \text{(b)},\\ &\chi_{-1}^-(t,\underline{x}_2,\underline{k}_2) \stackrel{=}{=} \cos[\Omega(k)t-\underline{k}_2\cdot\underline{x}_2] \quad \text{(c)},\\ &\chi_{-1}^+(t,\underline{x}_2,\underline{k}_2) \stackrel{=}{=} \cos[\Omega(k)t+\underline{k}_2\cdot\underline{x}_2] \quad \text{(d)}. \end{aligned} \tag{8.7}$$

The first or second or only occurrence of the subscript '$_1$' or '$_{-1}$' in a definiendum of definitions (8.6) and (8.7) is indicative of the respective occurrence of 'sin' or 'cos' in the definiens. The definientia of definitions (8.7) have been deduced from (8.1) and (8.5) by the instances of equations (7.66) with

$$\alpha \stackrel{=}{=} \Omega(k)t \text{ and } \beta \stackrel{=}{=} \underline{k}_2\cdot\underline{x}_2. \tag{8.8}$$

In addition to the appropriate fundamental trigonometric functional form or forms selected from (8.7) and (8.8), $\psi_{(1)}(t,\underline{x}_2)$ may in principle involve an *arbitrary* real-valued constant factor. Still, this factor will be *particularized* in accordance with the following considerations.

### 8.2. A dimension factor of the velocity potential

In accordance with (5.8) and (5.14), the functional form $\Phi_{(1)}(t,\underline{x})$ is the a scaled *velocity* potential in the first, linear approximation with respect to the *dimensionless* scaling parameter $\varepsilon\in[0,1)$. Therefore, $\Phi_{(1)}(t,\underline{x})$ should have a dimension of $\mathrm{l}^2/\mathrm{t}$, where 'l' stands for 'length' and 't' for 'time'. Each one of the pertinent functional forms $\Psi_{(l)}(t,\underline{x}_2)$, $\phi_{(1)}(t,\underline{x})$, and $\psi_{(1)}(t,\underline{x}_2)$ must have the same dimension; that is, symbolically

$$\left[\Phi_{(1)}(t,\underline{x})\right]=\left[\Psi_{(l)}(t,\underline{x}_2)\right]=\left[\phi_{(1)}(t,\underline{x})\right]=\left[\psi_{(l)}(t,\underline{x}_2)\right]=\mathrm{l}^2/\mathrm{t}\,, \tag{8.9}$$

where a pair of square brackets [ ] stands for the dimension of the expression that it encloses. At the same time, it will be assumed that $\varepsilon \doteqdot ka$, where '*a*' is an arbitrary real-valued constant, whose value can be particularized when desired and be called the *effective amplitude of* $\Phi_{(1)}(t,\underline{x})$ and hence *that of* $\Psi_{(l)}(t,\underline{x}_2)$; the sense of the name "effective amplitude" will be explicated before long. Since $[k]=\mathrm{l}^{-1}$, therefore $[a]=\mathrm{l}$. Either $\phi_{(1)}(t,\underline{x})$ or $\psi_{(1)}(t,\underline{x}_2)$ does not involve $\varepsilon$ and therefore it does not involve *a*. Therefore, in order to provide $\psi_{(1)}(t,\underline{x}_2)$ and hence $\phi_{(1)}(t,\underline{x})$ with the necessary dimension, I shall supplement the entire pure trigonometric functional form, involved in each one of the two functional forms $\psi_{(1)}(t,\underline{x}_2)$ and $\phi_{(1)}(t,\underline{x})$, with a certain constant *dimension factor* $\gamma(k)$ to be composed of some appropriate parameterrs available in the problem. Namely, I define that factor thus:

$$\gamma(k)\doteqdot\frac{g}{k\Omega(k)}=\sqrt{\frac{g}{k^3\tanh kd}}>0\text{, so that }\left[\gamma(k)\right]=\mathrm{l}^2/\mathrm{t}\,. \tag{8.10}$$

Consequently,

$$\gamma_\infty(k)\doteqdot\lim_{d\to\infty}\gamma(k)=\lim_{d\to\infty}\frac{g}{k\Omega(k)}=\lim_{d\to\infty}\sqrt{\frac{g}{k^3\tanh kd}}=\sqrt{\frac{g}{k^3}}=\frac{g}{k\Omega_\infty(k)}>0\,. \tag{8.11}$$

(cf. (7.79)).

Beside the dimension factor $\gamma(k)$, some entire pure trigonometric functional forms can for convenience be supplemented by a *sign factor* – or +. Criteria for the choice of both the dimension factor and the sign factor are relevant to the sense of '*a*', i.e. to the sense of the name "effective amplitude", so that they will be explicated in due course before long.

## 8.3. A plane monochromatic standing or progressive gravity wave on a liquid layer as the first approximation of the recursive asymptotic series

### 8.3.1. General definitions

**1.** In accordance with the two previous subsections, the solution of the set of equations (7.31) at *l*=1 and (7.33) will be selected either as one of the following four, descriptive of *standing* waves:

$$\psi_{\mu,\nu}(t,\underline{x}_2,\underline{k}_2)\doteqdot-\mu\gamma(k)\chi_{\mu,\nu}(t,\underline{x}_2,\underline{k}_2)\text{ for }\langle\mu,\nu\rangle\in\{1,-1\}\times\{1,-1\}, \tag{8.12}$$

subject to (8.6) and (8.10), or as one of the following four, descriptive of *progressive* waves:

$$\psi_\mu^\lambda(t,\underline{x}_2,\underline{k}_2) \stackrel{=}{=} -\mu\gamma(k)\chi_\mu^\lambda(t,\underline{x}_2,\underline{k}_2) \text{ for } \langle\mu,\lambda\rangle \in \{1,-1\}\times\{+,-\}, \tag{8.13}$$

subject to (8.7) and (8.10). That is, once either $\mu$ and $\nu$ in (8.12) or $\mu$ and $\lambda$ in (8.13) are selected, I shall set either

$$\psi_{(1)}(t,\underline{x}_2) \stackrel{=}{=} \psi_{\mu,\nu}(t,\underline{x}_2,\underline{k}_2) = -\mu\gamma(k)\chi_{\mu,\nu}(t,\underline{x}_2,\underline{k}_2) \tag{8.14}$$

or

$$\psi_{(1)}(t,\underline{x}_2) \stackrel{=}{=} \psi_\mu^\lambda(t,\underline{x}_2,\underline{k}_2) = -\mu\gamma(k)\chi_\mu^\lambda(t,\underline{x}_2,\underline{k}_2) \tag{8.15}$$

respectively. By (7.10) subject to (7.32), I shall also set either

$$\phi_{(1)}(t,\underline{x}) \stackrel{=}{=} \phi_{\mu,\nu}(t,\underline{x},\underline{k}_2) \stackrel{=}{=} \theta(z)\psi_{\mu,\nu}(t,\underline{x}_2,\underline{k}_2) \tag{8.16}$$

in the case of (8.14) or

$$\phi_{(1)}(t,\underline{x}) \stackrel{=}{=} \phi_\mu^\lambda(t,\underline{x},\underline{k}_2) \stackrel{=}{=} \theta(z)\psi_\mu^\lambda(t,\underline{x}_2,\underline{k}_2) \tag{8.17}$$

in the case of (8.15). In accordance with (5.8), (5.14), (7.72), and (7.73), $\Phi_{(1)}(t,\underline{x})$ and $\Psi_{(1)}(t,\underline{x}_2)$ are defined correspondingly either as

$$\Phi_{(1)}(t,\underline{x}) \stackrel{=}{=} \theta(z)\Psi_{(1)}(t,\underline{x}_2) \stackrel{=}{=} \varepsilon\phi_{\mu,\nu}(t,\underline{x},\underline{k}_2) \stackrel{=}{=} \varepsilon\theta(z)\psi_{\mu,\nu}(t,\underline{x}_2,\underline{k}_2), \tag{8.18}$$

so that

$$\Psi_{(1)}(t,\underline{x}_2) \stackrel{=}{=} \varepsilon\psi_{\mu,\nu}(t,\underline{x}_2,\underline{k}_2), \tag{8.19}$$

– in the case of (8.14), or as

$$\Phi_{(1)}(t,\underline{x}) \stackrel{=}{=} \theta(z)\Psi_{(1)}(t,\underline{x}_2) \stackrel{=}{=} \varepsilon\phi_\mu^\lambda(t,\underline{x},\underline{k}_2) \stackrel{=}{=} \varepsilon\theta(z)\psi_\mu^\lambda(t,\underline{x}_2,\underline{k}_2), \tag{8.20}$$

so that

$$\Psi_{(1)}(t,\underline{x}_2) \stackrel{=}{=} \varepsilon\psi_\mu^\lambda(t,\underline{x}_2,\underline{k}_2), \tag{8.21}$$

– in the case of (8.15) (cf. Lamb [1932, Arts 228, 229]).

**2.** By (7.10) subject to (7.32), equation ($6.52_1$) subject to (6.56) becomes

$$\zeta_{(1)}(t,\underline{x}_2) = -\frac{1}{g}\frac{\partial\phi_{(1)}^{(0)}(t,\underline{x}_2)}{\partial t} = -\frac{1}{g}\frac{\partial\psi_{(1)}(t,\underline{x}_2)}{\partial t} \tag{8.22}$$

and hence, by (5.1) and (5.7),

$$Z_{(1)}(t,\underline{x}_2) = \varepsilon\zeta_{(1)}(t,\underline{x}_2) = ak\zeta_{(1)}(t,\underline{x}_2) = -\frac{ak}{g}\frac{\partial\psi_{(1)}(t,\underline{x}_2)}{\partial t}. \tag{8.23}$$

Consequently, by (8.14) subject to (8.6), equation (8.22) can be specified thus:

$$\begin{aligned}\zeta_{(1)}(t,\underline{x}_2) &\doteqdot \zeta_{\mu,\nu}(t,\underline{x}_2,\underline{k}_2) \doteqdot -\frac{1}{g}\frac{\partial\psi_{\mu,\nu}(t,\underline{x}_2,\underline{k}_2)}{\partial t} = \frac{\mu\gamma(k)}{g}\frac{\partial\chi_{\mu,\nu}(t,\underline{x}_2,\underline{k}_2)}{\partial t} \\ &= \frac{\mu}{k\Omega(k)}\frac{\partial\chi_{\mu,\nu}(t,\underline{x}_2,\underline{k}_2)}{\partial t} = \frac{1}{k}\chi_{-\mu,\nu}(t,\underline{x}_2,\underline{k}_2),\end{aligned} \tag{8.24}$$

and hence equation (8.23) yields:

$$\mathrm{Z}_{(1)}(t,\underline{x}_2) \doteqdot \mathrm{Z}_{\mu,\nu}(t,\underline{x}_2,\underline{k}_2) \doteqdot ak\zeta_{\mu,\nu}(t,\underline{x}_2,\underline{k}_2) = a\chi_{-\mu,\nu}(t,\underline{x}_2,\underline{k}_2). \tag{8.25}$$

By (8.6), the last equation particularly means that

$$\begin{aligned}&\mathrm{Z}_{1,1}(t,\underline{x}_2,\underline{k}_2) \doteqdot a\cos\Omega(k)t\,\sin\underline{k}_2\cdot\underline{x}_2 \quad \text{(a)},\\ &\mathrm{Z}_{1,-1}(t,\underline{x}_2,\underline{k}_2) \doteqdot a\cos\Omega(k)t\,\cos\underline{k}_2\cdot\underline{x}_2 \quad \text{(b)},\\ &\mathrm{Z}_{-1,1}(t,\underline{x}_2,\underline{k}_2) \doteqdot a\sin\Omega(k)t\,\sin\underline{k}_2\cdot\underline{x}_2 \quad \text{(c)},\\ &\mathrm{Z}_{-1,-1}(t,\underline{x}_2,\underline{k}_2) \doteqdot a\sin\Omega(k)t\,\cos\underline{k}_2\cdot\underline{x}_2 \quad \text{(d)},\end{aligned} \tag{8.25$_1$}$$

while equation (8.24) is particularized by the variant of (8.25$_1$) with '$\zeta$' and '$k^{-1}$' in place of 'Z' and '$a$' respectively. Analogously, by (8.15) subject to (8.7), equation (8.22) can be specified thus:

$$\begin{aligned}\zeta_{(1)}(t,\underline{x}_2) &\doteqdot \zeta^{\lambda}_{\mu}(t,\underline{x}_2,\underline{k}_2) \doteqdot -\frac{1}{g}\frac{\partial\psi^{\lambda}_{\mu}(t,\underline{x}_2,\underline{k}_2)}{\partial t} = \frac{\mu\gamma(k)}{g}\frac{\partial\chi^{\lambda}_{\mu}(t,\underline{x}_2,\underline{k}_2)}{\partial t} \\ &= \frac{\mu}{k\Omega(k)}\frac{\partial\chi^{\lambda}_{\mu}(t,\underline{x}_2,\underline{k}_2)}{\partial t} = \frac{1}{k}\chi^{\lambda}_{-\mu}(t,\underline{x}_2,\underline{k}_2),\end{aligned} \tag{8.26}$$

and hence equation (8.23) yields:

$$\mathrm{Z}_{(1)}(t,\underline{x}_2) \doteqdot \mathrm{Z}^{\lambda}_{\mu}(t,\underline{x}_2,\underline{k}_2) \doteqdot ak\zeta^{\lambda}_{\mu}(t,\underline{x}_2,\underline{k}_2) = a\chi^{\lambda}_{-\mu}(t,\underline{x}_2,\underline{k}_2). \tag{8.27}$$

By (8.7), the last equation particularly means that

$$\begin{aligned}&\mathrm{Z}^{-}_{1}(t,\underline{x}_2,\underline{k}_2) \doteqdot a\cos[\Omega(k)t-\underline{k}_2\cdot\underline{x}_2] \quad \text{(a)},\\ &\mathrm{Z}^{+}_{1}(t,\underline{x}_2,\underline{k}_2) \doteqdot a\cos[\Omega(k)t+\underline{k}_2\cdot\underline{x}_2] \quad \text{(b)},\\ &\mathrm{Z}^{-}_{-1}(t,\underline{x}_2,\underline{k}_2) \doteqdot a\sin[\Omega(k)t-\underline{k}_2\cdot\underline{x}_2] \quad \text{(c)},\\ &\mathrm{Z}^{+}_{-1}(t,\underline{x}_2,\underline{k}_2) \doteqdot a\sin[\Omega(k)t+\underline{k}_2\cdot\underline{x}_2] \quad \text{(d)},\end{aligned} \tag{8.27$_1$}$$

while equation (8.26) is particularized by the variant of (8.27$_1$) with '$\zeta$' and '$k^{-1}$' in place of 'Z' and '$a$' respectively. Thus, in accordance with (8.25$_1$) and (8.27$_1$), $a$ is the *amplitude of displacement of the free surface of the liquid layer* due to either a standing gravity wave or a progressive gravity wave. This interpretation of '$a$' is preserved also in the case of a semi-infinite liquid space $z\leq 0$, provided that '$\Omega(k)$' defined by (7.38) is replaced with '$\Omega_{\infty}(k)$' defined by (7.79).

**3.** Once a non-scaled velocity potential $\phi_{(1)}(t,\underline{x})$ or the respective scaled velocity potential $\Phi_{(1)}(t,\underline{x})$ is selected, – the former from those defined by (8.16) and (8.17 ) and the latter from those defined by (8.18) and (8.20), – all pertinent characteristics of the fluid flow can be calculated.

### 8.3.2. A progressive plane monochromatic gravity water wave

**Concention 8.1.** 1) For the sake of definiteness and without loss of generality, I shall henceforth confine myself to the following non-scaled velocity potential of a *plane progressive monochromatic gravity water wave*:

$$\phi_{(1)}(t,\underline{x}) \doteqdot \phi_\mu^-(t,\underline{x},\underline{k}_2) = \theta(z)\psi_\mu^-(t,\underline{x}_2,\underline{k}_2) = -\mu\gamma(k)\theta(z)\chi_\mu^-(t,\underline{x}_2,\underline{k}_2) \tag{8.28}$$

and hence to the respective scaled one:

$$\Phi_{(1)}(t,\underline{x}) \doteqdot \varepsilon\phi_\mu^-(t,\underline{x},\underline{k}_2) = \varepsilon\theta(z)\psi_\mu^-(t,\underline{x}_2,\underline{k}_2) = -\mu ak\gamma(k)\theta(z)\chi_\mu^-(t,\underline{x}_2,\underline{k}_2), \tag{8.29}$$

subject to (8.7,a,c) and hence subject to some particular $\mu \in \{1,-1\}$ and subject to $\lambda$=–. It is understood that (8.28) and (8.29) can alternatively be specified for and $\lambda$=+, whereas

$$\chi_\mu^+(t,\underline{x}_2,\underline{k}_2) = \chi_\mu^-(t,\underline{x}_2,-\underline{k}_2). \tag{8.28$_+$}$$

2) In order to make statements relevant to both versions of $\phi_{(1)}(t,\underline{x})$ or $\Phi_{(1)}(t,\underline{x})$, I set

$$\tau_1 \doteqdot \sin,\ \ \tau_{-1} \doteqdot \cos, \tag{8.30$\bullet$}$$

**1.** Under Convention 8.1, for each $\mu \in \{1,-1\}$:

$$\chi_\mu^-(t,\underline{x}_2,\underline{k}_2) = \tau_\mu(\Omega(k)t - k_2\cdot\underline{x}_2), \tag{8.31}$$

$$\frac{\partial\chi_\mu^-(t,\underline{x}_2,\underline{k}_2)}{\partial t} = \frac{\partial\tau_\mu(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2)}{\partial t} = \mu\Omega(k)\tau_{-\mu}(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2), \tag{8.32}$$

$$\frac{\partial\chi_\mu^-(t,\underline{x}_2,\underline{k}_2)}{\partial x_i} = \frac{\partial\tau_\mu(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2)}{\partial x_i} = -\mu k_i\tau_{-\mu}(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2) \quad \text{for each } i \in \{1,2\}. \tag{8.33}$$

Consequently, by (7.32), (8.10), and (8.30), equation (8.28) (e.g.) becomes

$$\begin{aligned}\phi_{(1)}(t,\underline{x}) &\doteqdot \phi_\mu^-(t,\underline{x},\underline{k}_2) = \theta(z)\psi_\mu^-(t,\underline{x}_2,\underline{k}_2) = -\mu\gamma(k)\theta(z)\chi_\mu^-(t,\underline{x}_2,\underline{k}_2)\\ &= -\frac{\mu g}{k\Omega(k)}\theta(z)\tau_\mu(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2)\\ &= -\frac{\mu}{k}\sqrt{\frac{g}{k\tanh kd}}\frac{\cosh k(z+d)}{\cosh kd}\tau_\mu(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2),\end{aligned} \tag{8.34}$$

whence, in view of (8.33),

$$\begin{aligned}v_{(1)i}(t,\underline{x}) &= \nabla_i\phi_{(1)}(t,\underline{x}) = \frac{\partial\phi_{(1)}(t,\underline{x})}{\partial x_i} = -\mu\gamma(k)\theta(z)\frac{\partial\tau_\mu(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2)}{\partial x_i}\\ &= \frac{k_i}{k}\sqrt{\frac{g}{k\tanh kd}}\frac{\cosh k(z+d)}{\cosh kd}\tau_{-\mu}(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2) \text{ for each } i \in \{1,2\},\end{aligned} \tag{8.35a}$$

$$v_{(1)3}(t,\underline{x})=\nabla_3\phi_{(1)}(t,\underline{x})=\frac{\partial\phi_{(1)}(t,\underline{x})}{\partial x_3}=-\mu\gamma(k)\frac{\partial\theta(z)}{\partial z}\tau_\mu(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2)$$
$$=-\mu\sqrt{\frac{g}{k\tanh kd}}\frac{\sinh k(z+d)}{\cosh kd}\tau_\mu(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2)\text{ for } i=3. \qquad (8.35b)$$

Particularly, equations (8.34), (8.35a), and (8.35b) at $z$=0 become

$$\phi_{(1)}^{(0)}(t,\underline{x}_2)\doteqdot[\phi_{(1)}(t,\underline{x})]_{z=0}=-\frac{\mu}{k}\sqrt{\frac{g}{k\tanh kd}}\tau_\mu(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2), \qquad (8.34')$$

$$v_{(1)i}^{(0)}(t,\underline{x}_2)\doteqdot[v_{(1)i}(t,\underline{x})]_{z=0}=\frac{k_i}{k}\sqrt{\frac{g}{k\tanh kd}}\tau_{-\mu}(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2)\text{ for each } i\in\{1,2\}, \qquad (8.35a')$$

$$v_{(1)3}^{(0)}(t,\underline{x}_2)\doteqdot[v_{(1)3}(t,\underline{x})]_{z=0}=-\mu\sqrt{\frac{g\tanh kd}{k}}\tau_\mu(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2)\text{ for } i=3. \qquad (8.35b')$$

At the same time, equation (8.26) at $\lambda$=– becomes

$$\zeta_{(1)}(t,\underline{x}_2)\doteqdot\zeta_\mu^-(t,\underline{x}_2,\underline{k}_2)=\frac{1}{k}\chi_{-\mu}^-(t,\underline{x}_2,\underline{k}_2)=\frac{1}{k}\tau_{-\mu}(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2). \qquad (8.36)$$

**2.** By (8.3), (8.35a), and (8.35b), equation (5.50) at $l$=2 can be developed thus:

$$e_{k(2)}(t,\underline{x})=\frac{1}{2}\rho_9\sum_{i=1}^{3}(v_{(1)i}(t,\underline{x}))^2=\frac{1}{2}\rho_9\left[\sum_{i=1}^{2}(v_{(1)i}(t,\underline{x}))^2+(v_{(1)3}(t,\underline{x}))^2\right]$$
$$=\frac{\rho_9 g}{2k\tanh kd}\left[\left(\frac{\cosh k(z+d)}{\cosh kd}\right)^2\tau_{-\mu}{}^2(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2)\right. \qquad (8.37)$$
$$\left.+\left(\frac{\sinh k(z+d)}{\cosh kd}\right)^2\tau_\mu{}^2(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2)\right].$$

At the same time, equations (7.66) with $\beta$=$\alpha$ yield

$$\sin 2\alpha=2\sin\alpha\cos\alpha\ \text{(a)},\ \cos2\alpha=\cos^2\alpha-\sin^2\alpha\ \text{(b)},\ 1=\cos^2\alpha+\sin^2\alpha\ \text{(c)}, \qquad (8.38)$$

whereas the half-difference and half-sum of (8.38c) and (8.38b) yield

$$2\sin^2\alpha=1-\cos2\alpha\ \text{(a)},\ 2\cos^2\alpha=1+\cos2\alpha\ \text{(b)}. \qquad (8.39)$$

By (8.30), the pertinent instances of (8.39) become

$$\tau_{\pm\mu}{}^2(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2)=\frac{1}{2}[1\mp\mu\cos2(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2)]\text{ for each }\mu\in\{1,-1\}. \qquad (8.40)$$

Since

$$\sinh\alpha=\frac{1}{2}\left(e^{\alpha}-e^{-\alpha}\right)=-i\sin i\alpha \quad \text{(a)}, \quad \cosh\alpha=\frac{1}{2}\left(e^{\alpha}+e^{-\alpha}\right)=\cos i\alpha \quad \text{(b)}, \tag{8.41}$$

therefore equations (8.38) and (8.39) are equivalent to the following ones of hyperbolic trigonometry:

$$\begin{gathered}\sinh 2\alpha=2\sinh\alpha\cosh\alpha \;\; \text{(a)}, \;\; \cosh 2\alpha=\cosh^2\alpha+\sinh^2\alpha \;\; \text{(b)},\\ 1=\cosh^2\alpha-\sinh^2\alpha \;\; \text{(c)},\end{gathered} \tag{8.42}$$

$$2\sinh^2\alpha=\cosh 2\alpha-1 \;\; \text{(a)}, 2\cosh^2\alpha=\cosh 2\alpha+1 \;\; \text{(b)}, \tag{8.43}$$

respectively. It is also noteworthy that

$$2\tanh kd\cosh^2 kd=\frac{2\sinh kd\cosh^2 kd}{\cosh kd}=2\sinh kd\cosh kd=\sinh 2kd\,, \tag{8.44}$$

by the instance of (8.41a) with $\alpha \doteqdot kd$ .

Making use of (8.40) and of the instances of (8.42b) and (8.42c) with $\alpha \doteqdot k(z+d)$ in that order, and also making use of the train of equations (8.44), equation (8.37) can be developed thus:

$$\begin{aligned} e_{\mathrm{k}(2)}(t,\underline{x})=&\frac{\rho_9 g}{2k\sinh 2kd}\left\{\cosh^2 k(z+d)\left[1+\mu\cos 2\left(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\right)\right]\right.\\ &\left.+\sinh^2 k(z+d)\left[1-\mu\cos 2\left(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\right)\right]\right\}\\ =&\frac{\rho_9 g}{2k\sinh 2kd}\left[\cosh 2k(z+d)+\mu\cos 2\left(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\right)\right].\end{aligned} \tag{8.45}$$

Particularly, equation (8.45) at $z$=0 becomes

$$e_{\mathrm{k}(2)}^{(0)}(t,\underline{x}_2)\doteqdot\left[e_{\mathrm{k}(2)}(t,\underline{x})\right]_{z=0}=\frac{\rho_9 g}{2k\sinh 2kd}\left[\cosh 2kd+\mu\cos 2\left(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\right)\right]. \tag{8.45'}$$

**3.** Let

$$k_2 \doteqdot k_y \doteqdot 0\,,\ \text{so that}\ \left|k_1\right| \doteqdot \left|k_x\right| \doteqdot k>0\,. \tag{8.46}$$

Then, by (8.44), equations (8.34), (8.35a), and (8.35b) reduce to

$$\begin{aligned} \phi_{(1)}\left(t,\underline{x}\right)&\doteqdot\overline{\phi}_{(1)}\left(t,x,z,k_x\right)\doteqdot\phi_\mu^-\left(t,\langle x,0\rangle,z,\langle k_x,0\rangle\right)\\ &=\theta(z)\overline{\psi}_\mu^-\left(t,x,k_x\right)\doteqdot\theta(z)\psi_\mu^-\left(t,\langle x,0\rangle,\langle k_x,0\rangle\right)\\ &=-\mu\gamma(k)\theta(z)\overline{\chi}_\mu^-\left(t,x,k_x\right)\doteqdot-\mu\gamma(k)\theta(z)\chi_\mu^-\left(t,\langle x,0\rangle,\langle k_x,0\rangle\right)\\ &=-\frac{\mu g}{k\Omega(k)}\theta(z)\tau_\mu\left(\Omega(k)t-k_x x\right)=-\frac{\mu}{k}\sqrt{\frac{2g}{k\sinh 2kd}}\,\tau_\mu\left(\Omega(k)t-k_x x\right),\end{aligned} \tag{8.47}$$

$$
\begin{aligned}
v_{(1)x}(t,\underline{x}) &\doteqdot \bar{v}_{(1)x}(t,x,z,k_x) \doteqdot v_{(1)x}(t,\langle x,0\rangle, z\langle k_x,0\rangle) \\
&= \frac{k_x}{k}\sqrt{\frac{2g}{k\sinh 2kd}}\cosh k(z+d)\tau_{-\mu}(\Omega(k)t-k_x x),
\end{aligned}
\tag{8.48}
$$

$$
\begin{aligned}
v_{(1)z}(t,\underline{x}) &\doteqdot \bar{v}_{(1)z}(t,x,z,k_x) \doteqdot v_{(1)z}(t,\langle x,0\rangle, z\langle k_x,0\rangle) \\
&= -\mu\sqrt{\frac{2g}{k\sinh 2kd}}\sinh k(z+d)\tau_{\mu}(\Omega(k)t-k_x x).
\end{aligned}
\tag{8.49}
$$

the understanding being that $v_{(1)y}(t,\underline{x}) \doteqdot 0$. By (8.30), it follows from (8.48) and (8.49) that

$$
\begin{aligned}
&[v_{(1)x}(t,\underline{x})]^2\left[\sqrt{\frac{2g}{k\sinh 2kd}}\cosh k(z+d)\right]^{-2} + [v_{(1)z}(t,\underline{x})]^2\left[\sqrt{\frac{2g}{k\sinh 2kd}}\sinh k(z+d)\right]^{-2} \\
&= \tau_{-\mu}{}^2(\Omega(k)t-k_x x) + \tau_{-\mu}{}^2(\Omega(k)t-k_x x) = 1.
\end{aligned}
\tag{8.50}
$$

### 8.3.3. A standing plane monochromatic gravity water wave

**Convention 8.2.** For the sake of brevity and without loss of generality, I shall, under definition (8.30), write the non-scaled velocity potential of a *plane standing monochromatic gravity water wave* as

$$
\phi_{(1)}(t,\underline{x}) \doteqdot \phi_{\mu\nu}(t,\underline{x},\underline{k}_2) = \theta(z)\psi_{\mu\nu}(t,\underline{x}_2,\underline{k}_2) = -\mu\gamma(k)\theta(z)\chi_{\mu\nu}(t,\underline{x}_2,\underline{k}_2)
\tag{8.51}
$$

and hence the respective scaled one as:

$$
\Phi_{(1)}(t,\underline{x}) \doteqdot \varepsilon\phi_{\mu\nu}(t,\underline{x},\underline{k}_2) = \varepsilon\theta(z)\psi_{\mu\nu}(t,\underline{x}_2,\underline{k}_2) = -\mu ak\gamma(k)\theta(z)\chi_{\mu\nu}(t,\underline{x}_2,\underline{k}_2),
\tag{8.52}
$$

subject to (8.6) and hence subject to some particular $\mu \in \{1,-1\}$ and $\nu \in \{1,-1\}$.•

**1.** Under definition (8.30) it follows from (8.6) that for each $\mu \in \{1,-1\}$ and each $\nu \in \{1,-1\}$:

$$
\chi_{\mu\nu}(t,\underline{x}_2,\underline{k}_2) = \tau_{\mu}(\Omega(k)t)\tau_{\nu}(k_2\cdot\underline{x}_2),
\tag{8.53}
$$

$$
\frac{\partial\chi_{\mu\nu}(t,\underline{x}_2,\underline{k}_2)}{\partial t} = \frac{\partial\tau_{\mu}(\Omega(k)t)}{\partial t}\tau_{\nu}(k_2\cdot\underline{x}_2) = \mu\Omega(k)\tau_{-\mu}(\Omega(k)t)\tau_{\nu}(k_2\cdot\underline{x}_2),
\tag{8.54}
$$

$$
\begin{gathered}
\frac{\partial\chi_{\mu\nu}(t,\underline{x}_2,\underline{k}_2)}{\partial x_i} = \tau_{\mu}(\Omega(k)t)\frac{\partial\tau_{\nu}(k_2\cdot\underline{x}_2)}{\partial x_i} = -\mu\nu k_i\tau_{\mu}(\Omega(k)t_2)\tau_{-\nu}(k_2\cdot\underline{x}_2) \\
\text{for each } i \in \{1,2\}.
\end{gathered}
\tag{8.55}
$$

Consequently, by (7.32), (8.10), and (8.30), equation (8.50) (e.g.) becomes

$$\begin{aligned}\phi_{(1)}(t,\underline{x}) &\doteqdot \phi_{\mu\nu}(t,\underline{x},\underline{k}_2) = \theta(z)\psi_{\mu\nu}(t,\underline{x}_2,\underline{k}_2) = -\mu\gamma(k)\theta(z)\chi_{\mu\nu}(t,\underline{x}_2,\underline{k}_2) \\ &= -\frac{\mu g}{k\Omega(k)}\theta(z)\tau_\mu(\Omega(k)t)\tau_\nu(k_2\cdot\underline{x}_2) \\ &= -\frac{\mu}{k}\sqrt{\frac{g}{k\tanh kd}}\frac{\cosh k(z+d)}{\cosh kd}\tau_\mu(\Omega(k)t)\tau_\nu(k_2\cdot\underline{x}_2),\end{aligned} \tag{8.56}$$

whence, in view of (8.55),

$$\begin{aligned}v_{(1)i}(t,\underline{x}) &= \nabla_i\phi_{(1)}(t,\underline{x}) = \frac{\partial\phi_{(1)}(t,\underline{x})}{\partial x_i} = -\mu\gamma(k)\theta(z)\tau_\mu(\Omega(k)t)\frac{\partial\tau_\nu(\underline{k}_2\cdot\underline{x}_2)}{\partial x_i} \\ &= -\mu\nu\frac{k_i}{k}\sqrt{\frac{g}{k\tanh kd}}\frac{\cosh k(z+d)}{\cosh kd}\tau_\mu(\Omega(k)t)\tau_{-\nu}(\underline{k}_2\cdot\underline{x}_2)\text{ for each } i\in\{1,2\},\end{aligned} \tag{8.57a}$$

$$\begin{aligned}v_{(1)3}(t,\underline{x}) &= \nabla_3\phi_{(1)}(t,\underline{x}) = \frac{\partial\phi_{(1)}(t,\underline{x})}{\partial x_3} = -\mu\gamma(k)\frac{\partial\theta(z)}{\partial z}\tau_\mu(\Omega(k)t)\tau_\nu(\underline{k}_2\cdot\underline{x}_2) \\ &= -\mu\sqrt{\frac{g}{k\tanh kd}}\frac{\sinh k(z+d)}{\cosh kd}\tau_\mu(\Omega(k)t)\tau_\nu(\underline{k}_2\cdot\underline{x}_2)\text{ for } i=3.\end{aligned} \tag{8.57b}$$

Particularly, equations (8.56), (8.57a), and (8.57b) at $z$=0 become

$$\phi_{(1)}^{(0)}(t,\underline{x}_2) \doteqdot \left[\phi_{(1)}(t,\underline{x})\right]_{z=0} = -\frac{\mu}{k}\sqrt{\frac{g}{k\tanh kd}}\tau_\mu(\Omega(k)t)\tau_\nu(\underline{k}_2\cdot\underline{x}_2), \tag{8.56'}$$

$$v_{(1)i}^{(0)}(t,\underline{x}_2) \doteqdot \left[v_{(1)i}(t,\underline{x})\right]_{z=0} = \mu\nu\frac{k_i}{k}\sqrt{\frac{g}{k\tanh kd}}\tau_\mu(\Omega(k)t)\tau_{-\nu}(\underline{k}_2\cdot\underline{x}_2)\quad\text{for each } i\in\{1,2\}, \tag{8.57a'}$$

$$v_{(1)3}^{(0)}(t,\underline{x}_2) \doteqdot \left[v_{(1)3}(t,\underline{x})\right]_{z=0} = -\mu\sqrt{\frac{g\tanh kd}{k}}\tau_\mu(\Omega(k)t)\tau_\nu(\underline{k}_2\cdot\underline{x}_2)\ \text{ for } i=3. \tag{8.57b'}$$

At the same time, equation (8.24) becomes

$$\zeta_{(1)}(t,\underline{x}_2) \doteqdot \zeta_{\mu,\nu}(t,\underline{x}_2,\underline{k}_2) = \frac{1}{k}\chi_{-\mu,\nu}(t,\underline{x}_2,\underline{k}_2) = \frac{1}{k}\tau_{-\mu}(\Omega(k)t)\tau_\nu(\underline{k}_2\cdot\underline{x}_2). \tag{8.58}$$

**2.** By (8.3), (8.57a), and (8.57b), equation (5.50) at $l$=2 can be developed thus:

$$\begin{aligned}e_{k(2)}(t,\underline{x}) &= \frac{1}{2}\rho_9\sum_{i=1}^{3}\left(v_{(1)i}(t,\underline{x})\right)^2 = \frac{1}{2}\rho_9\left[\sum_{i=1}^{2}\left(v_{(1)i}(t,\underline{x})\right)^2 + \left(v_{(1)3}(t,\underline{x})\right)^2\right] \\ &= \frac{\rho_9 g}{2k\tanh kd\cosh^2 kd}\left[\cosh^2 k(z+d)\tau_{-\nu}{}^2(\underline{k}_2\cdot\underline{x}_2)\right. \\ &\qquad \left.+\sinh^2 k(z+d)\tau_\nu{}^2(\underline{k}_2\cdot\underline{x}_2)\right]\tau_\mu{}^2(\Omega(k)t).\end{aligned} \tag{8.59}$$

By (8.30), the pertinent instances of (8.39) become

$$\tau_{\pm\mu}{}^2(\Omega(k)t) = \frac{1}{2}[1 \mp \mu\cos 2\Omega(k)t] \text{ for each } \mu \in \{1,-1\}, \tag{8.60}$$

$$\tau_{\pm\nu}{}^2(\underline{k}_2 \cdot \underline{x}_2) = \frac{1}{2}[1 \mp \nu\cos 2(\underline{k}_2 \cdot \underline{x}_2)] \text{ for each } \nu \in \{1,-1\}. \tag{8.61}$$

Making use of (8.60) and (8.61) and of the instances of (8.42b) and (8.42c) with $\alpha \doteqdot k(z+d)$ in that order, and also making use of the train of equations (8.44), equation (8.59) can be developed thus:

$$\begin{aligned} e_{\mathrm{k}(2)}(t,\underline{x}) &= \frac{\rho_9 g}{4k\sinh 2kd}\{\cosh^2 k(z+d)[1+\nu\cos 2(\underline{k}_2 \cdot \underline{x}_2)] \\ &\quad + \sinh^2 k(z+d)[1-\nu\cos 2(\underline{k}_2 \cdot \underline{x}_2)]\}.[1-\mu\cos 2\Omega(k)t] \\ &= \frac{\rho_9 g}{4k\sinh 2kd}[\cosh 2k(z+d) + \nu\cos 2(\underline{k}_2 \cdot \underline{x}_2)].[1-\mu\cos 2\Omega(k)t]. \end{aligned} \tag{8.62}$$

Particularly, equation (8.62) at $z$=0 becomes

$$\begin{aligned} e^{(0)}_{\mathrm{k}(2)}(t,\underline{x}_2) &\doteqdot [e_{\mathrm{k}(2)}(t,\underline{x})]_{z=0} \\ &= \frac{\rho_9 g}{4k\sinh 2kd}[\cosh 2kd + \nu\cos 2(\underline{k}_2 \cdot \underline{x}_2)][1-\mu\cos 2\Omega(k)t]. \end{aligned} \tag{8.62'}$$

**3.** If (8.46) hold then. by (8.44), equations (8.56), (8.57a), and (8.57b) reduce to

$$\begin{aligned} \phi_{(1)}(t,\underline{x}) &\doteqdot \overline{\phi}_{\mu\nu}(t,x,z,k_x) \doteqdot \phi_{\mu\nu}(t,\langle x,0\rangle,z,\langle k_x,0\rangle) \\ &= \theta(z)\overline{\psi}_{\mu\nu}(t,x,k_x) \doteqdot \theta(z)\psi_{\mu\nu}(t,\langle x,0\rangle,\langle k_x,0\rangle) \\ &= -\mu\gamma(k)\theta(z)\overline{\chi}_{\mu\nu}(t,x,k_x) \doteqdot -\mu\gamma(k)\theta(z)\chi_{\mu\nu}(t,\langle x,0\rangle,\langle k_x,0\rangle) \\ &= -\frac{\mu g}{k\Omega(k)}\theta(z)\tau_\mu(\Omega(k)t)\tau_\nu(k_x x) = -\frac{\mu}{k}\sqrt{\frac{2g}{k\sinh 2kd}}\theta(z)\tau_\mu(\Omega(k)t)\tau_\nu(k_x x), \end{aligned} \tag{8.63}$$

$$\begin{aligned} v_{(1)x}(t,\underline{x}) &\doteqdot \overline{v}_{(1)x}(t,x,z,k_x) \doteqdot v_{(1)x}(t,\langle x,0\rangle,z\langle k_x,0\rangle) \\ &= -\mu\nu\frac{k_x}{k}\sqrt{\frac{g}{k\tanh kd}}\frac{\cosh k(z+d)}{\cosh kd}\tau_\mu(\Omega(k)t)\tau_{-\nu}(k_x x) \\ &= -\mu\nu\frac{k_x}{k}\sqrt{\frac{2g}{k\sinh 2kd}}\cosh k(z+d)\tau_\mu(\Omega(k)t)\tau_{-\nu}(k_x x), \end{aligned} \tag{8.64}$$

$$\begin{aligned} v_{(1)z}(t,\underline{x}) &\doteqdot \overline{v}_{(1)z}(t,x,z,k_x) \doteqdot v_{(1)z}(t,\langle x,0\rangle,z\langle k_x,0\rangle) \\ &= -\mu\sqrt{\frac{g}{k\tanh kd}}\frac{\sinh k(z+d)}{\cosh kd}\tau_\mu(\Omega(k)t)\tau_\nu(k_x x) \\ &= -\mu\sqrt{\frac{2g}{k\sinh 2kd}}\sinh k(z+d)\tau_\mu(\Omega(k)t)\tau_\nu(k_x x). \end{aligned} \tag{8.65}$$

the understanding being that $v_{(1)y}(t,\underline{x}) \doteqdot 0$ . By (8.30), it follows from (8.64) and (8.65) that

$$\left[v_{(1)x}(t,\underline{x})\right]^2\left[\sqrt{\frac{2g}{k\sinh 2kd}}\cosh k(z+d)\right]^{-2}+\left[v_{(1)z}(t,\underline{x})\right]^2\left[\sqrt{\frac{2g}{k\sinh 2kd}}\sinh k(z+d)\right]^{-2} \quad (8.66))$$
$$=\tau_\mu{}^2(\Omega(k)t)\left[\tau_{-\nu}{}^2(k_x x)+\tau_\nu{}^2(k_x x)\right]=\tau_\mu{}^2(\Omega(k)t).$$

# 9. The cyclic frequency and related scalar characteristic of a progressive plane monochromatic gravity water wave

## 9.1. The second dimensionless parameter and basic scalar characteristics of a progressive plane monochromatic gravity water wave

**1.** Besides the dimensionless strictly positive parameter $\varepsilon \equiv ka > 0$, there is in the recursive asymptotic problem in question another dimensionless strictly positive parameter $\delta \equiv kd > 0$, whose value affects the character of solutions of the problem (cf. (7.78), (7.79), and (8.11)). Various aspects of this parameter are made explicated below in this subsection,

**Definition 9.1.** Given $d$>0,

$$\Sigma(\delta) \overset{\rightarrow}{\equiv} \sqrt{\frac{d}{g}}\,\Omega\!\left(\frac{\delta}{d},d\right)=\sqrt{\delta\tanh\delta}\geq 0\,, \tag{9.1}$$

the understanding being that, given $\underline{k}_2 \in \underline{E}_2$,

$$\delta \overset{\rightarrow}{\equiv} \left|\underline{\delta}_2\right| = \sqrt{\delta_1^2+\delta_2^2} = \left|\underline{k}_2\right| d = kd \geq 0\text{, subject to } \underline{\delta}_2 \overset{\rightarrow}{\equiv} \underline{k}_2 d\,. \tag{9.2}$$ •

**Lemma 9.1 (**and at the same time a definition of ‘ $s_{\mathrm{g}}$ ’, ‘ $s_{\mathrm{p}}$ ’, and ‘ $m_{\pm 1}$ ’**).** For each $\delta \in (0,+\infty)$*:*

$$s_{\mathrm{g}}(\delta) \overset{\rightarrow}{\equiv} \frac{\partial\Sigma(\delta)}{\partial\delta} = m_1(\delta)s_{\mathrm{p}}(\delta) > 0\,, \tag{9.3}$$

where

$$s_{\mathrm{p}}(\delta) \overset{\rightarrow}{\equiv} \frac{\Sigma(\delta)}{\delta} = \sqrt{\frac{\tanh\delta}{\delta}} > 0\,, \tag{9.4}$$

$$m_{\pm 1}(2\delta) \overset{\rightarrow}{\equiv} \frac{1}{2}\left(1 \pm \frac{2\delta}{\sinh 2\delta}\right) > 0\,. \tag{9.5}$$

By (9.3), the values of the functional form ‘ $\Sigma(\delta)$ ’, as defined by (9.1), monotonically increase from $\Sigma(0)=0$ to $\Sigma(+\infty)=+\infty$ as $\delta$ increases from $0$ to $+\infty$.

**Proof:** It follows from (9.1) that

$$\frac{\partial \Sigma(\delta)}{\partial \delta} = \frac{1}{2}\left[\delta^{-1/2}(\tanh\delta)^{1/2} + \delta^{1/2}(\tanh\delta)^{-1/2}(\cosh\delta)^{-2}\right]$$
$$= \frac{(\delta\tanh\delta)^{1/2}}{2\delta}\left(1+\frac{\delta}{\sinh\delta\cosh\delta}\right) = \frac{\Sigma(\delta)}{2\delta}\left(1+\frac{2\delta}{\sinh 2\delta}\right). \quad (9.3_1)$$

QED.•

**Comment 9.1.** Except for the useful relations

$$s_{\mathrm{p}}(\delta) - s_{\mathrm{g}}(\delta) = m_{-1}(2\delta)s_{\mathrm{p}}(\delta), \quad (9.6)$$

$$m_1(2\delta) + m_{-1}(2\delta) = 1, \quad (9.7)$$

which immediately follow from (9.3) and (9.5) respectively, the functional form '$m_{-1}(2\delta)$' is irrelevant to '$s_{\mathrm{g}}(\delta)$' as such. This form appears in some formulae relevant to depth-integrated characteristics of the fluid flow.•

**Lemma 9.2 (**and at the same time a definition of '$\Sigma^{-1}$'**).** For each $\sigma \in [0,+\infty)$: there is exactly one $\Sigma^{-1}(\sigma) \in [0,\infty)$ such that for each $\delta \in [0,+\infty)$:

$$\delta = \Sigma^{-1}(\sigma) \text{ if and only if } \Sigma(\delta) = \sigma. \quad (9.8)$$

**Proof:** Owing to strictly monotonic increase of $\Sigma(\delta)$ with increase of $\sigma$ in the real semi-axis, as stated in Lemma 9.1, given $\sigma \in [0,+\infty)$: the equation $\Sigma(\delta) = \sigma$ has a unique solution with respect to '$\delta$'. This solution is denoted by '$\Sigma^{-1}(\sigma)$', so that $\Sigma^{-1}$ is *the inverse of the bijective function Σ*. QED.•

**Comment 9.2.** For the sake of brevity, I use '$\Omega^2(k)$' (e.g.) interchangeably with '$[\Omega(k)]^2$'. At the same time, '$\Sigma^{-1}(\sigma)$' has by definition nothing to do either with '$[\Sigma(\sigma)]^{-1}$' or with '$1/\Sigma(\sigma)$'.•

**Corollary 9.1 (**and at the same time a definition of '$c_{\mathrm{g}}$' and '$c_{\mathrm{p}}$'**).** Given $d \in (0,\infty)$, for each $k \in [0,+\infty)$:

$$\Omega(k) \doteqdot \Omega(k,d) = \sqrt{\frac{g}{d}}\Sigma(kd) = \frac{\sqrt{gd}}{d}\Sigma(kd) = \frac{c_0}{d}\Sigma(kd), \quad (9.9)$$

$$c_{\mathrm{g}}(k) \doteqdot c_{\mathrm{g}}(k,d) \doteqdot \frac{\partial\Omega(k,d)}{\partial k} = \sqrt{\frac{g}{d}}\left[\frac{d\Sigma(\delta)}{d\delta}\right]_{\delta=kd}\frac{d(kd)}{dk} = \sqrt{gd}\,s_{\mathrm{g}}(kd) = c_0 s_{\mathrm{g}}(kd), \quad (9.10)$$

$$c_{\mathrm{p}}(k) \doteqdot c_{\mathrm{p}}(k,d) \doteqdot \frac{\Omega(k,d)}{k} = \sqrt{\frac{g}{d}}\frac{\Sigma(kd)}{k} = \sqrt{gd}\,\frac{\Sigma(kd)}{kd} = \sqrt{gd}\,s_{\mathrm{p}}(kd) = c_0 s_{\mathrm{p}}(kd), \quad (9.11)$$

the understanding being that

$$c_0 \doteqdot \sqrt{gd} > 0, \tag{9.12}$$

and that, in accordance with (9.6),

$$c_{\mathrm{g}}(k,d) = m_1(2kd)c_{\mathrm{p}}(k,d) > 0, \tag{9.13}$$

It is understood that for each $\delta \in [0,+\infty)$:

$$s_{\mathrm{g}}(\delta) = (gd)^{-1/2}c_{\mathrm{g}}(\delta/d,d) = c_0^{-1}c_{\mathrm{g}}(\delta/d,d), \tag{9.10$_1$}$$

$$s_{\mathrm{p}}(\delta) \doteqdot (gd)^{-1/2}c_{\mathrm{p}}(\delta/d,d) = c_0^{-1}c_{\mathrm{p}}(\delta/d,d), \tag{9.11$_1$}$$

which are converse of (9.10) and (9.11).

**Proof:** The trains of equations (9.9)–(9.11) along with (9.12) and (9.13) immediately follow from (9.1) and (9.3)–(9.5). Equations (9.10$_1$) and (9.11$_1$) immediately follow from (9.10) and (9.11). QED.•

**Comment 9.3.** 1) The dispersion functional form '$\Omega(k,d)$' is a characteristic of the whole recursive asymptotic wave problem in question. Still, the most natural interpretation of '$\Omega(k,d)$' is that, given a *wave vector* $\underline{k}_2 \in \underline{E}_2 - \langle 0,0 \rangle$, $\Omega(k,d)$ is *the cyclic frequency of a progressive plane monochromatic gravity water wave, of the wave number* $k \doteqdot |\underline{k}_2| > 0$ and hence *of the wavelength* $\lambda \doteqdot 2\pi/k$, the understanding being that *the velocity potential of the associated fluid flow* is a particular solution of the problem $\mathsf{T}_{\mathrm{u}(1)}(\phi_{(1)})$, i.e. of the triple of equations (7.1)–(7.3) at $l$=1 subject to (7.4). In this case, $c_{\mathrm{g}}(k,d)$ is the *wave group speed*, whereas $c_{\mathrm{p}}(k,d)$ is the *wave phase speed*. In contrast to the *dimensional quantities* $\Omega(k,d)$, $c_{\mathrm{g}}(k,d)$, and $c_{\mathrm{p}}(k,d)$, the corresponding *dimensionless quantities* $\Sigma(\delta)$, $s_{\mathrm{g}}(\delta)$, and $s_{\mathrm{p}}(\delta)$ are called "the *reduced* wave cyclic frequency", "the *reduced* wave group speed", and "the *reduced* wave phase speed", respectively, – in the sense that these are independent either of '$g$' or of '$d$'.

2) By (7.38), the pertinent straightforward calculations implied in (9.10) yield

$$c_{\mathrm{g}}(k) \risingdotseq c_{\mathrm{g}}(k,d) \risingdotseq \frac{\partial\Omega(k,d)}{\partial k} = \frac{\partial\sqrt{gk\tanh kd}}{\partial k} = \frac{1}{2}\sqrt{\frac{g}{k\tanh kd}}\,\frac{\partial(k\tanh kd)}{\partial k}$$
$$= \frac{1}{2}\sqrt{\frac{g}{k\tanh kd}}\left(\tanh kd + \frac{kd}{\cosh^2 kd}\right) = \frac{1}{4}\sqrt{\frac{gd}{kd\tanh kd}}\left(\frac{\sinh 2kd + 2kd}{\cosh^2 kd}\right) \qquad (9.10')$$
$$= c_0 \frac{\sinh 2\delta + 2\delta}{4\sqrt{\delta\tanh\delta}\cosh^2\delta} == c_0 s_{\mathrm{g}}(\delta)$$

subject to (9.12).•

**Corollary 9.2 (**and at the same time a definition of '*K*'**).** Given $d \in (0,+\infty)$, for each $\omega \in (0,+\infty)$, there is exactly one

$$K(\omega) \risingdotseq K(\omega,d) \risingdotseq \frac{1}{d}\Sigma^{-1}\left(\sqrt{\frac{d}{g}}\omega\right), \qquad (9.14)$$

such that for each $k \in [0,+\infty)$:

$$k = K(\omega,d) \text{ if and only if } \Omega(\kappa,d) = \omega. \qquad (9.15)$$

**Proof:** The corollary follows from Lemma 9.2 by (9.1) and (9.2), with the understanding that

$$\sigma \risingdotseq \sqrt{\frac{d}{g}}\omega \text{ and conversely } \omega \risingdotseq \sqrt{\frac{g}{d}}\sigma. \qquad (9.16)•$$

## 9.2. A long-wave range

**Corollary 9.3.** For each $\delta \in [0,\pi/2)$:

$$\Sigma^2(\delta) \risingdotseq [\Sigma(\delta)]^2 = \delta^2\left[1 - \frac{\delta^2}{3} + \frac{2\delta^4}{15} + o(\delta^4)\right], \qquad (9.17)$$

$$\Sigma(\delta) = \delta\left[1 - \frac{\delta^2}{6} + \frac{19\delta^4}{360} + o(\delta^4)\right], \qquad (9.18)$$

$$s_{\mathrm{g}}(\delta) = 1 - \frac{\delta^2}{2} + \frac{19\delta^4}{72} + o(\delta^4), \qquad (9.19)$$

$$s_{\mathrm{p}}(\delta) \risingdotseq \frac{\Sigma(\delta)}{\delta} = 1 - \frac{\delta^2}{6} + \frac{19\delta^4}{360} + o(\delta^4), \qquad (9.20)$$

$$m_{-1}(2\delta) = \frac{2\delta^2}{3} - \frac{14\delta^4}{45} + o(\delta^4). \qquad (9.21)$$

The Maclaurin series for '$m_1(2\delta)$' is determined by (9.7) subject to (9.21).

**Proof:** 1) The known Maclaurin series for '$\tanh\delta$' (see, e.g., Gradshteyn and Ryzhik [1980, p. 35, art. 1.411, item 6]) can be written for each $\delta \in (-\pi/2, \pi/2)$ as

$$\tanh\delta = \delta\left[1 - \chi_0\left(\delta^2\right)\right], \tag{9.22}$$

where

$$\chi_0\left(\delta^2\right) \doteqdot \frac{\delta^2}{3} - \frac{2\delta^4}{15} + o\left(\delta^4\right). \tag{9.23}$$

Equation (9.17) immediately follows from (9.1) by (9.23).

2) In order to prove (9.18), notice that $\tanh 0 = 0$ and $\tanh\infty = 1$, whereas $\frac{d\tanh\delta}{d\delta} = (\cosh\delta)^{-2} \in (1,0)$ for each $\delta \in (0,\infty)$. Hence,

$$0 < \tanh\delta < \delta \text{ for each } \delta \in (0,\infty). \tag{9.24}$$

By (9.24), it follows from (9.22) that

$$\chi_0(\delta^2) \in [0,1) \text{ for each } \delta \in (-\pi/2, \pi/2). \tag{9.25}$$

By (9.1), (9.22), and (9.25), the following Maclaurin series with respect to '$\chi_0(\delta^2)$' converges for each $\delta \in [0, \pi/2)$:

$$\frac{\Sigma(\delta)}{\delta} = \sqrt{1 - \chi_0\left(\delta^2\right)} = 1 - \frac{1}{2}\chi_0\left(\delta^2\right) - \frac{1}{8}\chi_0^2\left(\delta^2\right) - o\left(\chi_0^2\left(\delta^2\right)\right). \tag{9.26}$$

By (9.23), equation (9.26) yields (9.18). Equation (9.19) immediately follows from (9.3) by (9.18). Lastly, (9.21) immediately follows from (9.5) with the help of the instance of the known Laurent series (see, e.g., Gradshteyn and Ryzhik [1980, p. 35, art. 1.412, item 12])

$$\operatorname{csch} y \doteqdot (\sinh y)^{-1} = \frac{1}{y}\left[1 - \frac{y^2}{6} + \frac{7y^4}{360} + o(y^4)\right] \text{ for each } |y| \in (0,\pi), \tag{9.27}$$

with $y \doteqdot 2\delta = 2kd$. The corollary is established.•

**Corollary 9.4.** Given $d \in (0,\infty)$, for each $kd \in (0, \pi/2)$:

$$0 < 1 - \frac{\Omega^2(k,d)}{c_0^2 k^2} = 1 - \frac{c_{\mathrm{p}}^2(k,d)}{c_0^2} < \frac{(kd)^2}{3}, \tag{9.28}$$

$$0 < 1 - \frac{\Omega(k,d)}{c_0 k} = 1 - \frac{c_{\mathrm{p}}(k,d)}{c_0} < \frac{(kd)^2}{6}, \tag{9.29}$$

$$0 < 1 - \frac{c_{\mathrm{g}}(k,d)}{c_0} < \frac{(kd)^2}{2}, \tag{9.30}$$

the understanding being that

$$\lim_{k\to 0} c_{\mathrm{p}}(k,d) = \lim_{k\to 0} c_{\mathrm{g}}(k,d) = c_0. \tag{9.31}$$

**Proof:** The corollary follows from Corollary 9.3 by (9.2) and (9.9)–(9.12).•

**Definition 9.2.** Given $d \in (0,\infty)$, a progressive wave of a wave number $k \in (0,\infty)$ is said to be *long* in regard to a liquid layer of depth $d$, – or, alternatively, given $k \in (0,\infty)$, a liquid layer of a depth $d \in (0,\infty)$ is said to be *shallow*, or *thin*, in regard to a progressive wave of a wave number $k$, – if and only if

$$kd \le \sqrt{0.6} \cong 0.775, \tag{9.32}$$

which corresponds to $\frac{1}{6}(kd)^2 \le \frac{1}{10}$ in (9.29).•

**Comment 9.4.** 1) Criterion (9.32) can be rewritten as

$$\frac{\lambda}{d} > \frac{2\pi}{\sqrt{0.6}} \cong 8.11, \tag{9.32'}$$

where $\lambda = 2\pi / k$ is the wavelength.

2) There is an empiric fact that a progressive quasi-plane quasi-monochromatic gravity water wave of local amplitude $a$ and of local wave number $k$ is disintegrated (breaks) when it approaches some region, in which

$$\varepsilon \doteqdot ka < \sqrt{0.6}\,. \tag{9.32a}$$

and which often is also qualified as a *shallow water* one. However, criterion (9.32a) is independent of criterion (9.32). Therefore, a water wave region that satisfies criterion (9.32a) should more correctly be qualified as a *high-amplitude* one. It is noteworthy that the effect of gravity wave break in a high-amplitude region due to bottom effects cannot be associated with the velocity potential $\Phi_{(2)}(t,\underline{x})$ of second order with respect to *ka*, because the latter potential turns out to be bounded throughout, as shown in the next section. For avoidance of the wave disintegration, the assumption that the wave amplitude *a* is small as compare to the depth *d* should be made.•

**Definition 9.3.** In analogy with (8.30), it is convenient for our purpose at hand to modify definition (7.32) thus:

$$\theta_{-1}(z) \doteqdot \theta_{-1}(z,k) \doteqdot \theta_{-1}(z,k,d) \doteqdot \frac{\cosh k(z+d)}{\cosh kd} \doteqdot \theta(z,k,d) \doteqdot \theta(z,k) \doteqdot \theta(z) \tag{9.33}$$

and to supplement it with the definition:

$$\theta_1(z) \doteqdot \theta_1(z,k) \doteqdot \theta_1(z,k,d) \doteqdot \frac{1}{k}\frac{\partial\theta_{-1}(z,k,d)}{\partial z} = \frac{\sinh k(z+d)}{\cosh kd}. \tag{9.34}$$

Thus, the functional variables '$\theta_{-1}$' and '$\theta$' are synonyms, any one of which determines the dependence on '$z$' of each one of the functional forms $\phi_{(l)i}(t,\underline{x})$ and $v_{(l)i}(t,\underline{x})$ for each $l \in \omega_1$ and each $i \in \{1,2\}$, whereas the functional variable '$\theta_1$' determines the dependence on '$z$' of the functional form $v_{(l)3}(t,\underline{x})$ for each $l \in \omega_1$.•

**Corollary 9.5.** Given $d \in (0,\infty)$, for each $kd \in [0,\pi/2)$, for each $z \in (-\infty,+\infty)$:

$$\theta_{-1}(z,k,d) = 1 + \frac{1}{2}k^2 z(z+2d) - \frac{1}{24}k^4 z(z+2d)\left(4d^2 - 2dz - z^2\right) + o\left(k^4(z+d)^4\right), \tag{9.35}$$

$$\theta_1(z,k,d) = k(z+d) - \frac{1}{12}k^3(z+d)\left(4d^2 - 6dz - 3z^2\right) + o\left(k^3(z+d)^3\right). \tag{9.36}$$

**Proof:** In accordance with (9.33) and (9.33), multiplication of each one of the known series (see, e.g., Gradshteyn and Ryzhik [1980, p. 35, art. 1.411, items 4 and 2]):

$$\cosh x = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!},\ \sinh x = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}, \text{ for each } x \in (-\infty,+\infty), \tag{9.37}$$

with $x = k(z+d)$ by the known series (*ibid.*, p. 35, art. 1.411, item 10):

$$\operatorname{sech}\delta \doteqdot (\cosh\delta)^{-1} = 1 - \frac{\delta^2}{2!} + \frac{5\delta^4}{4!} + o\left(\delta^4\right) \text{ for each } \delta \in (-\pi/2,\pi/2), \tag{9.38}$$

subject to (9.2), yields (9.35) and (9.36) respectively. In this case, once (9.35) is deduced, equation (9.36) can, alternatively, be obtained by differentiating (9.35) with respect '$z$', in accordance with (9.34). QED.•

**Comment 9.5.** Under Convention 8.1, given $\mu \in \{1,-1\}$,

$$\Phi_{(1)}(t,\underline{x}) = -\mu a k\gamma(k)\theta_{-1}(z,k,d)\tau_\mu\left(\Omega(k)t - k_2\cdot\underline{x}_2\right), \tag{9.39}$$

by (8.29)–(8.31) and (9.33). At the same time, by (9.1), (9.2), and (9.12), equation (8.10) becomes

$$\gamma(k) \doteqdot \frac{g}{k\Omega(k)} = \frac{1}{k}\sqrt{\frac{g}{k\tanh kd}} = \frac{1}{k}\sqrt{\frac{gd}{\delta\tanh\delta}} = \frac{c_0}{k\Sigma(\delta)} > 0. \tag{9.40}$$

If $kd \in (0,\pi/2)$ then it follows from (9.40) by (9.18) that

$$\gamma(k) = \frac{c_0}{k\Sigma(\delta)} = \frac{c_0}{k\delta}\left[1 + \frac{\delta^2}{6} - \frac{\delta^4}{40} + o\left(\delta^4\right)\right]. \tag{9.41}$$

In accordance with (9.35) and (9.40), the functional form $\Phi_{(1)}(t,\underline{x})$ as defined by (9.39) becomes unbounded as $k \to 0$, unless of course a special additional assumption that $a \to 0$ as $k \to 0$ is made. Still, all measurable characteristics of the pertinent fluid flow, which are expressed in terms of partial derivatives of $\Phi_{(1)}(t,\underline{x})$ with respect to ‘ $t$ ’ or ‘ $x_i$ ’, – such characteristics, e.g., as as $V_{(1)i}(t,\underline{x})$ at $i \in \{1,2\}$, $V_{(1)3}(t,\underline{x})$, $P_{d(1)}(t,\underline{x})$, $E_{\mathrm{k}(2)}(t,\underline{x})$, etc, – remain regular as $k \to 0$. All these characteristics can immediately be written in the first non-vanishing approximation with respect to $kd$ with the help of the series obtained above in this subsection.•

## 9.3. A short-wave range

**Corollary 9.6.** For each $q \in (0,\infty)$:

$$\Sigma^2(\delta) = \delta\left[1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-2n\delta}\right]. \tag{9.42}$$

$$\Sigma(\delta) = \sqrt{\delta}\left[1 - e^{-2\delta} + \frac{1}{2}e^{-4\delta} + o\left(e^{-4\delta}\right)\right], \tag{9.43}$$

$$s_{\mathrm{g}}(\delta) = \frac{1}{2\sqrt{\delta}}\left[1 + (4\delta - 1)e^{-2\delta} - \frac{1}{2}(8\delta - 1)e^{-4\delta} + o\left(e^{-4\delta}\right)\right], \tag{9.44}$$

$$s_{\mathrm{p}}(\delta) = \frac{1}{\sqrt{\delta}}\left[1 - e^{-2\delta} + \frac{1}{2}e^{-4\delta} + o\left(e^{-4\delta}\right)\right], \tag{9.45}$$

$$m_{\pm 1}(2\delta) = \frac{1}{2} \pm 2\delta e^{-2\delta}\sum_{n=0}^{\infty} e^{-4n\delta} \tag{9.46}$$

**Proof:** By the conventional definition,

$$\tanh\delta \doteqdot \frac{e^{\delta} - e^{-\delta}}{e^{\delta} + e^{-\delta}} = \frac{1 - e^{-2\delta}}{1 + e^{-2\delta}} \doteqdot 1 - \chi_{\infty}(2\delta) \text{ for each } \delta \in (0,\infty), \tag{9.47}$$

the understanding being that

$$\chi_{\infty}(y) \doteqdot \frac{2e^{-y}}{1 + e^{-y}} = -2\sum_{n=1}^{\infty}(-1)^n e^{-ny} \text{ for each } y \in (0,\infty) \tag{9.48}$$

(see, e.g., Gradshteyn and Ryzhik [1980, p. 23, art. 1.232, item 1]). Equation (9.42) immediately follows from (9.1), by (9.47) and by (9.48) with $y = 2\delta$. It then follows from the first expression for ‘ $\chi_{\infty}(y)$ ’ in (9.48) that

$$\frac{d\chi_\infty(y)}{dy} = -\frac{2e^{-y}}{\left(1+e^{-y}\right)^2} < 0 \text{ for each } y \in (0,\infty)\,. \tag{9.49}$$

Thus, the values of '$\chi_\infty(y)$' monotonically decrease from $\chi_\infty(0) = 1$ to $\chi_\infty(\infty) = 0$ as $y$ increases from $0$ to $\infty$. Therefore,

$$\chi_\infty(y) \in (0,1) \text{ for each } y \in (0,\infty) \tag{9.50}$$

(cf. (9.26). Hence, by (9.47) and by (9.50) with $y = 2\delta$, it follows from (9.1) that the following Maclaurin series with respect to '$\chi_\infty(2\delta)$' converges for each $\delta \in (0,\infty)$:

$$\delta^{-1/2}\Sigma(\delta) = \sqrt{\tanh\delta} = \sqrt{1-\chi_\infty(2\delta)} = 1 - \frac{1}{2}\chi_\infty(2\delta) - \frac{1}{8}\chi_\infty^2(2\delta) - o\left(\chi_\infty^2(2\delta)\right) \tag{9.51}$$

(cf. (9.26)). By the last expression for '$\chi_\infty(y)$' in (9.48) with $y \doteq 2\delta$, equation (9.51) reduces to (9.43), whereas. equations (9.44) and (9.45) immediately follow from (9.3) by (9.43). Lastly, (9.46) immediately follows from (9.5) with the help of this self-evident equation

$$\operatorname{csch} y \doteq (\operatorname{sh} y)^{-1} = \frac{2}{e^y - e^{-y}} = \frac{2e^{-y}}{1-e^{-2y}} = 2e^{-y}\sum_{n=0}^{\infty} e^{-2ny}\,,\ y \in (0,\infty) \tag{9.52}$$

(see, e.g., Gradshteyn and Ryzhik [1980, p. 23, art. 1.232, item 2]) at $y = 2\delta$. The corollary is established.•

**Comment 9.6.** It follow from (9.1)–(9.5) and (9.9)–(9.12) by (9.43)–(9.46) that, given $k \in (0,\infty)$:

$$\Omega(k,\infty) \doteq \lim_{d\to\infty}\Omega(k,d) = \sqrt{gk}\,, \tag{9.53}$$

$$c_{\mathrm{p}}(k,\infty) \doteq \lim_{d\to\infty} c_{\mathrm{p}}(k,d) = \frac{\Omega(k,\infty)}{k} = \sqrt{\frac{g}{k}}\,, \tag{9.54}$$

$$c_{\mathrm{g}}(k,\infty) \doteq \lim_{d\to\infty} c_{\mathrm{g}}(k,d) = \frac{d\Omega(k,\infty)}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}c_{\mathrm{p}}(k,\infty)\,, \tag{9.53}$$

$$m_{\pm 1}(\infty) = \lim_{d\to\infty} m_{\pm 1}(2kd) = \frac{1}{2}\,. \tag{9.56}$$

By (9.11), (9.53), and (9.54), it follows from (9.42) and (9.43) that given $d \in (0,\infty)$, for each $k \in (0,\infty)$:

$$0 < 1 - \frac{\Omega^2(k,d)}{\Omega^2(k,\infty)} = 1 - \frac{c_{\mathrm{p}}^2(k,d)}{c_{\mathrm{p}}^2(k,\infty)} < 2e^{-2kd}\,, \tag{9.57}$$

$$0 < 1 - \frac{\Omega(k,d)}{\Omega(k,\infty)} = 1 - \frac{c_{\mathrm{p}}(k,d)}{c_{\mathrm{p}}(k,\infty)} < e^{-2kd}\,. \qquad (9.59)\bullet$$

**Definition 9.4.** Given $d \in (0,\infty)$, a progressive mode of a wave number $k \in (0,\infty)$ is said either to be *intermediate* or to be *short* inregard to a liquid layer of depth $d$, – or, alternatively, given $k \in (0,\infty)$: a liquid layer of a depth $d \in (0,\infty)$ is said either to be *transitional* or to be *deep* (or *thick*), in regard to a progressive of wave number $k$, – depending on whether

$$\sqrt{0.6} \le kd \le 0.5\ln 10 \qquad (9.59)$$

or whether

$$kd > 0.5\ln 10 \cong 1.151\,, \qquad (9.60)$$

respectively. In this case, the criterion $kd = 0.5\ln 10$ is equivalent to $e^{-2kd} = 0.1$ (cf. (9.58)).•

$$\frac{\lambda}{d} > \frac{2\pi}{\sqrt{0.6}} \cong 8.11\,, \qquad (9.32')$$

**Comment 9.7.** Criteria (9.59) and (9.60), can be rewritten as

$$\frac{4\pi}{\ln 10} \le \frac{\lambda}{d} \le \frac{2\pi}{\sqrt{0.6}} \qquad (9.59')$$

and as `

$$\frac{\lambda}{d} < \frac{4\pi}{\ln 10} \cong 5.46 \qquad (9.60')$$

respectively, where $\lambda = 2\pi / k$. In the case of a liquid layer with a mildly varying depth $d = h(\underline{x}_2)$, condition (3.46) does not suffice for the refraction of a wave of a local wave number $k(\underline{x}_2)$ to cease. Therefore, the term '*deep layer*', or '*thick layer*', must be redefined if one wants it to connote a certain criterion of the absence of wave refraction.•

**Corollary 9.7.** Given $d \in (0,\infty)$, given $k \in (0,\infty)$, for each $z \in [0,-d]$:

$$\theta_{\mp 1}(z,k,d) = [e^{kz} \pm e^{-k(z+2d)}]\sum_{n=0}^{\infty}(-1)^n e^{-2nkd}\,. \qquad (9.61)$$

Hence, given $k \in (0,\infty)$: given $z \in [0,-\infty)$:

$$\theta_{\mp 1}(z,k,\infty) \rightleftarrows \lim_{d\to\infty}\theta_{\mp 1}(z,k,d) = e^{kz} = e^{-k|z|}\,. \qquad (9.62)$$

**Proof:** In accordance with (9.35) and (9.36), multiplication of the instance at $x = k(z+d)$ of each one of the evident equations

$$\cosh x = \frac{1}{2}e^{x}(1+e^{-2x}), \sinh x = \frac{1}{2}e^{x}(1+e^{-2x}), \tag{9.63}$$

which are valid for all $x \in (-\infty,+\infty)$, by the evident equation

$$\operatorname{sech}\delta \doteqdot (\cosh \delta)^{-1} = \frac{2}{e^{\delta}+e^{-\delta}} = \frac{2e^{-\delta}}{1+e^{-2\delta}} = 2e^{-\delta}\sum_{n=0}^{\infty}(-1)^{n}e^{-2n\delta}, \ \delta \in (0,+\infty) \tag{9.64}$$

(cf. Gradshteyn and Ryzhik [1980, p. 23, art. 1.232, item 3]), subject to (9.2), yields (9.61).•

**Comment 9.8.** 1) By (9.2), (9.43) and by (9.52) with $y \doteqdot 2\delta = 2kd$, equation (8.41) can be developed thus

$$\gamma(k) = \frac{c_0}{k\Sigma(\delta)} = \frac{\sqrt{g}}{k^{3/2}}\left[1-e^{-2\delta}+\frac{1}{2}e^{-4\delta}+o\left(e^{-4\delta}\right)\right] > 0. \tag{9.65}$$

Consequently, under Convention 8.1, given $\mu \in \{1,-1\}$, $\Phi_{(1)}(t,\underline{x})$ is defined by (9.39) subject to (9.61) and (9,65), so that $\Phi_{(1)}(t,\underline{x})$ is proportional to $k^{-1/2}$. That is, just as in the case of a shallow liquid layer (see Comment 9.5), $\Phi_{(1)}(t,\underline{x})$ of the pertinent fluid flow in the case of an intermediate or deep liquid layer becomes unbounded as $k \to 0$ Still, as before, all measurable characteristics of the fluid flow, which are expressed in terms of partial derivatives of $\Phi_{(1)}(t,\underline{x})$ with respect to '$t$' or '$x_i$', turns out to be bounded as $k \to 0$.

2) Although all series which have been deduced above this subsection converge for each $\delta \doteqdot kd \in [0,\infty)$, the sum of several first terms (in particular, the first term alone) of a series can be used as an approximation to the corresponding prototype functional form only in the case of intermediate and short waves or, equivalently, in the case of transitional and deep (thick) liquid layers. If particularly $\delta \doteqdot kd \in [1{,}151/2,\infty)$ then the given plane monochromatic wave is short, or equivalently, the given liquid layer is deep, – in accordance with Definition 9.4. If $k$ becomes large enough then surface tension must be taken into account.•

# 10. The second-order asymptotic approximations to the velocity potentials of progressive and standing plane monochromatic gravity water waves

## 10.1. The inhomogeneous terms in the boundary conditions at $z=0$ in the presence of a progressive wave

**Lemma 10.1.** Under Convention 8.1, given $\mu \in \{1,-1\}$, given $\underline{k}_2 \in \underline{E}_2 - \langle 0,0 \rangle$, for each $\langle t,\underline{x}_2 \rangle \in R \times \underline{E}_2$:

$$\alpha_{\mathrm{d}(2)}(t,\underline{x}_2) = -\frac{1}{2k\sinh 2kd}\left[1+\mu(2-\cosh 2kd)\cos 2(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2)\right], \tag{10.1}$$

$$\alpha_{\mathrm{k}(2)}(t,\underline{x}_2) = -\mu\sqrt{\frac{g}{k\tanh kd}}\sin 2(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2), \tag{10.2}$$

$$\begin{aligned}\alpha_{(2)}(t,\underline{x}_2) &= \frac{3\mu}{\sinh 2kd}\sqrt{\frac{g\tanh kd}{k}}\sin 2(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2)\\ &= \frac{3\mu\Omega(k)}{k\sinh 2kd}\sin 2(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2).\end{aligned} \tag{10.3}$$

**Proof:** I proceed from equations (6.35) and (6.38) in their before-last forms and also from equation (6.51) at *l*=2, namely

$$\alpha_{\mathrm{d}(2)}(t,\underline{x}_2) = -\frac{1}{\rho_0 g}\left[e^{(0)}_{\mathrm{k}(2)}(t,\underline{x}_2)+\rho_0\frac{\partial v^{(0)}_{(1)3}(t,\underline{x}_2)}{\partial t}\zeta_{(1)}(t,\underline{x}_2)\right], \tag{10.1$_0$}$$

$$\alpha_{\mathrm{k}(2)}(t,\underline{x}_2) = -\sum_{i=1}^{2}\nabla_i\left[v^{(0)}_{(1)i}(t,\underline{x}_2)\zeta_{(1)}(t,\underline{x}_2)\right], \tag{10.2$_0$}$$

$$\alpha_{(2)}(t,\underline{x}_2) \doteq \frac{\partial\alpha_{\mathrm{d}(2)}(t,\underline{x}_2)}{\partial t}-\alpha_{\mathrm{k}(2)}(t,\underline{x}_2). \tag{10.3$_0$}$$

1) The first summand in the square brackets on the right-hand side of equation (10.1$_0$) is given by (8.45'), whereas the second one is determined by (8.35b') and (8.36) as follows:

$$\begin{aligned}\frac{\partial v^{(0)}_{(1)3}(t,\underline{x}_2)}{\partial t}\zeta_{(1)}(t,\underline{x}_2) &= -\frac{\mu}{k}\sqrt{\frac{g\tanh kd}{k}}\frac{\partial\tau_\mu(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2)}{\partial t}\tau_{-\mu}(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2)\\ &= -\frac{\mu^2 g\tanh kd}{k}\tau_{-\mu}{}^2(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2) = -\frac{g\tanh kd}{2k}\left[1+\mu\cos 2(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2)\right],\end{aligned} \tag{10.1$_1$}$$

where use of the pertinent instance of (8.40) has been made. By (8.45') and (10.1$_1$), equation (10.1$_0$) can be developed thus:

$$\alpha_{\mathrm{d}(2)}(t,\underline{x}_2) = -\frac{1}{\rho_0 g}\left\{\frac{\rho_9 g}{2k\sinh 2kd}\left[\cosh 2kd + \mu\cos 2\left(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2\right)\right]\right.$$

$$\left. - \frac{\rho_0 g \tanh kd}{2k}\left[1 + \mu\cos 2\left(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2\right)\right]\right\} \tag{10.1$_2$}$$

$$= -\frac{1}{2k\sinh 2kd}\left[\cosh 2kd - \tanh kd \sinh 2kd\right.$$

$$\left. + \mu\left(1 - \tanh kd\sinh 2kd\right)\cos 2\left(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2\right)\right].$$

By the pertinent instances of (8.42,b,c), it follows that

$$\cosh 2kd - \tanh kd \sinh 2kd = \cosh^2 kd + \sinh^2 kd - \frac{2\sinh^2 kd \cosh kd}{\cosh kd}$$

$$= \cosh^2 kd - \sinh^2 kd = 1, \tag{10.1$_3$}$$

$$1 - \tanh kd \sinh 2kd = \cosh^2 kd - \sinh^2 kd - \frac{2\sinh^2 kd \cosh kd}{\cosh kd}$$

$$= \cosh^2 kd - 3\sinh^2 kd = 1 - 2\sinh^2 kd = 1 - 2\sinh^2 kd = 2 - \cosh 2kd, \tag{10.1$_4$}$$

By (10.1$_3$) and (10.1$_4$), equation (10.1$_2$) immediately turns into (10.1).

2) By (8.35a') and (8.36), and also by the pertinent instance of (8.40), equation (10.2$_0$) can be developed thus:

$$\alpha_{\mathrm{k}(2)}(t,\underline{x}_2) = -\frac{1}{k^2}\sqrt{\frac{g}{\mathrm{k}\tanh kd}}\sum_{i=1}^{2} k_i \nabla_i {\tau_{-\mu}}^2\left(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2\right)$$

$$= -\frac{1}{2k^2}\sqrt{\frac{g}{\mathrm{k}\tanh kd}}\sum_{i=1}^{2} k_i \nabla_i \left[1 + \mu\cos 2\left(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2\right)\right]$$

$$= -\frac{\mu}{2k^2}\sqrt{\frac{g}{\mathrm{k}\tanh kd}}\left(\sum_{i=1}^{2} k_i k_i\right)\sin 2\left(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2\right) \tag{10.2$_1$}$$

$$= -\mu\sqrt{\frac{g}{\mathrm{k}\tanh kd}}\sin 2\left(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2\right),$$

which proves (10.2).

3) Differentiation of both sides of equation (10.1) with respect to ‘$t$’ yields

$$\frac{\partial \alpha_{\mathrm{d}(2)}(t,\underline{x}_2)}{\partial t} = \frac{\mu\Omega(k)\left(2 - \cosh 2kd\right)}{k\sinh 2kd}\sin 2\left(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2\right)$$

$$= \mu\sqrt{\frac{g\tanh kd}{k}}\,\frac{\left(2 - \cosh 2kd\right)}{\sinh 2kd}\sin 2\left(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2\right). \tag{10.3$_1$}$$

where use of (7.38) has been made in developing the final expression. Substitution of (10.3$_1$) and (10.2) into (10.3$_0$) yields

$$\alpha_{(2)}(t,\underline{x}_2)=\mu\sqrt{\frac{g\tanh kd}{k}}\left(\frac{2-\cosh 2kd}{\sinh 2kd}+\frac{1}{\tanh kd}\right)\sin 2\left(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\right)$$
$$=\frac{3\mu}{\sinh 2kd}\sqrt{\frac{g\tanh kd}{k}}\sin 2\left(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\right), \tag{10.3$_2$}$$

because

$$\frac{2-\cosh 2kd}{\sinh 2kd}+\frac{1}{\tanh kd}=\frac{2-\cosh 2kd+2\cosh^2 kd}{2\sinh kd\cosh kd}$$
$$=\frac{2-\cosh 2kd+(\cosh 2kd+1)}{\sinh 2kd}=\frac{3}{\sinh 2kd}, \tag{10.3$_3$}$$

where use of the instance of (8.43b) with $\alpha=kd$ has been made. The train of equations (10.3$_2$) proves (10.3). The lemma is established.•

## 10.2. The second-order approximation of the velocity potential in the presence of a progressive wave

In accordance with (7.10) at $l$=2 subject to (7.32),

$$\phi_{(2)}(t,\underline{x})\doteqdot\theta(z)\psi_{(2)}(t,\underline{x}_2)\doteqdot\frac{\cosh k(z+d)}{\cosh kd}\psi_{(2)}(t,\underline{x}_2), \tag{10.4}$$

while for each $\langle t,\underline{x}_2\rangle\in R\times\underline{E}_2$, the functional form $\psi_{(2)}(t,\underline{x}_2)$, defined by (7.48) at $l$=2 as:

$$\psi_{(2)}(t,\underline{x}_2)\doteqdot\psi_{(2)*}(t,\underline{x}_2,0)\doteqdot\frac{g}{\Omega(k)}\int_0^t\alpha_{(2)}(t',\underline{x}_2)\sin[\Omega(k)(t-t')]\,dt', \tag{10.5}$$

is the only pertinent *full solution* of equation (7.34), – the solution that automatically satisfies its *identifying conditions* (7.49) at $l$=2:

$$\psi_{(2)}(0,\underline{x}_2)=0 \text{ (a) and } \left[\frac{\partial\psi_{(2)}(t,\underline{x}_2)}{\partial t}\right]_{t=0}=0 \text{ (b)}, \tag{10.6}$$

so that, by (7.48) at $l$=2,

$$\psi_{(2)}(t,\underline{x}_2)\doteqdot c_{(2)}\psi_{(1)}(t,\underline{x}_2)+\psi_{(2)*}(t,\underline{x}_2,0) \text{ subject to } c_{(2)}\doteqdot 0. \tag{10.7}$$

Substitution of (10.3) into (10.5) yields

$$\psi_{(2)}(t,\underline{x}_2)=\frac{3\mu g}{k\sinh 2kd}I_{\mathrm{p}(2)}(t,\underline{x}_2), \tag{10.8}$$

where

$$I_{\mathrm{p}(2)}(t,\underline{x}_2) \doteq \int_0^t \sin 2\left[\Omega(k)t' - \underline{k}_2 \cdot \underline{x}_2\right]\sin\left[\Omega(k)(t-t')\right]dt' ; \tag{10.9}$$

the subscript ‘p’ is the first letter of ‘progresive’.

It follows from (7.66,b) that

$$\sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right] . \tag{10.10}$$

Hence, the integrand of (10.9) can be developed as the following instance of (10.10):

$$\begin{aligned} &\sin\left[\Omega(k)(t-t')\right]\sin 2\left[\Omega(k)t' - \underline{k}_2 \cdot \underline{x}_2\right] \\ &= \frac{1}{2}\left\{\cos\left[\Omega(k)(t-3t') + \underline{2k}_2 \cdot \underline{x}_2\right] - \cos\left[\Omega(k)(t+t') - \underline{2k}_2 \cdot \underline{x}_2\right]\right\} \\ &= \frac{1}{2}\left\{\cos\left[\Omega(k)(3t'-t) - \underline{2k}_2 \cdot \underline{x}_2\right] - \cos\left[\Omega(k)(t'+t) - \underline{2k}_2 \cdot \underline{x}_2\right]\right\}. \end{aligned} \tag{10.9$_1$}$$

Consequently, he integral (10.9) subject to (10.9$_1$) is calculated thus:

$$\begin{aligned} I_{\mathrm{p}(2)}(t,\underline{x}_2) &= \frac{1}{2}\int_0^t \left\{\cos\left[\Omega(k)(3t'-t) - \underline{2k}_2 \cdot \underline{x}_2\right] - \cos\left[\Omega(k)(t'+t) - \underline{2k}_2 \cdot \underline{x}_2\right]\right\}dt' \\ &= \frac{1}{2\Omega(k)}\left\{\frac{1}{3}\sin\left[\Omega(k)(3t'-t) - \underline{2k}_2 \cdot \underline{x}_2\right] - \sin\left[\Omega(k)(t'+t) - \underline{2k}_2 \cdot \underline{x}_2\right]\right\}_{t'=0}^{t'=t} \\ &= \frac{1}{2\Omega(k)}\left\{-\frac{2}{3}\sin 2\left[\Omega(k)t - \underline{k}_2 \cdot \underline{x}_2\right] + \frac{1}{3}\sin\left[\Omega(k)t + \underline{2k}_2 \cdot \underline{x}_2\right] + \sin\left[\Omega(k)t - \underline{2k}_2 \cdot \underline{x}_2\right]\right\}. \end{aligned} \tag{10.9$_2$}$$

Thus, $\psi_{(2)}(t,\underline{x}_2)$ is given by (10.8) subject to (10.9$_2$), while $\phi_{(2)}(t,\underline{x})$ is expressed in terms of $\psi_{(2)}(t,\underline{x}_2)$ by (10.4). Hence finally, by (5.14),

$$\begin{aligned} &\Phi_{(2)}(t,\underline{x}) \doteq \Phi_{(2)}(t;\underline{x},\varepsilon) = \varepsilon^2\phi_{(2)}(t,\underline{x}) = (ak)^2\phi_{(2)}(t,\underline{x}) \\ &= \frac{(ak)^2\cosh k(z+d)}{\cosh kd}\psi_{(2)}(t,\underline{x}_2) = \frac{3\mu a^2 gk\cosh k(z+d)}{\cosh kd\sinh 2kd}I_{\mathrm{p}(2)}(t,\underline{x}_2) \\ &= \frac{\mu a^2\Omega(k)}{2\sinh kd\sinh 2kd}\cosh k(z+d) \\ &\cdot\left\{-2\sin 2\left[\Omega(k)t - \underline{k}_2 \cdot \underline{x}_2\right] + \sin\left[\Omega(k)t + 2\underline{k}_2 \cdot \underline{x}_2\right] + 3\sin\left[\Omega(k)t - 2\underline{k}_2 \cdot \underline{x}_2\right]\right\}, \end{aligned} \tag{10.11}$$

because

$$\frac{gk}{\cosh kd} = \frac{\Omega^2(k)}{\tanh kd\cosh kd} = \frac{\Omega^2(k)}{\sinh kd}. \tag{10.11$_1$}$$

### 10.3. The inhomogeneous terms in the boundary conditions at $z = 0$ in the presence of a standing wave

**Lemma 10.2.** Under Convention 8.2, given $\mu \in \{1,-1\}$, given $\nu \in \{1,-1\}$, given $\underline{k}_2 \in \underline{E}_2 - \langle 0,0\rangle$, for each $\langle t, \underline{x}_2\rangle \in R \times \underline{E}_2$:

$$\alpha_{\mathrm{d}(2)}(t,\underline{x}_2) = \frac{1}{4k\sinh 2kd}\{-[1+\nu\cosh 2kd\cos 2(\underline{k}_2\cdot\underline{x}_2)] \\ +\mu[2\cosh 2kd - 1 + \nu(2-\cosh 2kd)\cos 2(\underline{k}_2\cdot\underline{x}_2)]\cos 2\Omega(k)t\}, \tag{10.12}$$

$$\alpha_{\mathrm{k}(2)}(t,\underline{x}_2) = -\frac{\mu\nu}{2}\sqrt{\frac{g}{k\tanh kd}}\sin 2\Omega(k)t\cos 2(\underline{k}_2\cdot\underline{x}_2), \tag{10.13}$$

$$\alpha_{(2)}(t,\underline{x}_2) \doteqdot \frac{\partial\alpha_{\mathrm{d}(2)}(t,\underline{x}_2)}{\partial t} - \alpha_{\mathrm{k}(2)}(t,\underline{x}_2) \\ = -\mu\sqrt{\frac{g\tanh kd}{k}}\frac{(2\cosh 2kd - 1)}{2\sinh 2kd}[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)]\sin 2\Omega(k)t \\ = -\frac{\mu(2\cosh 2kd-1)\Omega(k)}{2k\sinh 2kd}[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)]\sin 2\Omega(k)t \tag{10.14}$$

**Proof:** Just as in the case of a progressive wave, I proceed from the general equations ($10.1_0$)–($10.3_0$).

1) The first summand in the square brackets on the right-hand side of equation ($10.1_0$) is given by (8.62'), whereas he second one is determined by (8.57b) and (8.58) as follows:

$$\frac{\partial v^{(0)}_{(1)3}(t,\underline{x}_2)}{\partial t}\zeta_{(1)}(t,\underline{x}_2) = -\frac{\mu}{k}\sqrt{\frac{g\tanh kd}{k}}\frac{\partial\tau_\mu(\Omega(k)t)}{\partial t}\tau_{-\mu}(\Omega(k)t)\tau_\nu{}^2(\underline{k}_2\cdot\underline{x}_2) \\ = -\frac{\mu^2 g\tanh kd}{k}\tau_{-\mu}{}^2(\Omega(k)t)\tau_\nu{}^2(\underline{k}_2\cdot\underline{x}_2) \\ = -\frac{g\tanh kd}{4k}[1+\mu\cos 2\Omega(k)t][1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)], \tag{$10.12_1$}$$

where use of the pertinent instances of (8.60) and (8.61) has been made. By (8.62') and ($10.12_1$), equation ($10.1_0$) becomes:

$$\alpha_{\mathrm{d}(2)}(t,\underline{x}_2) = \frac{1}{4k}\left\{-\frac{1}{\sinh 2kd}[\cosh 2kd + \nu\cos 2(\underline{k}_2\cdot\underline{x}_2)][1-\mu\cos 2\Omega(k)t]\right. \\ \left. + \tanh kd[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)][1+\mu\cos 2\Omega(k)t]\right\}. \tag{$10.12_2$}$$

This can conveniently be written as`

$$\alpha_{\mathrm{d}(2)}(t,\underline{x}_2) = \frac{1}{4k}\cdot[c_0(\underline{x}_2) + \mu c_2(\underline{x}_2)\cos 2\Omega(k)t], \tag{$10.12_3$}$$

where

$$c_0(\underline{x}_2) \cong -\frac{1}{\sinh 2kd}\left[\cosh 2kd + \nu\cos 2(\underline{k}_2\cdot\underline{x}_2)\right] + \tanh kd\left[1 - \nu\cos 2(\underline{k}_2\cdot\underline{x}_2)\right]$$
$$= -\frac{\cosh 2kd}{\sinh 2kd} + \tanh kd - \nu\left(\frac{1}{\sinh 2kd} + \tanh kd\right)\cos 2(\underline{k}_2\cdot\underline{x}_2) \qquad (10.12_4)$$
$$= -\frac{1}{\sinh 2kd}\left[1 + \nu\cosh 2kd\cos 2(\underline{k}_2\cdot\underline{x}_2)\right],$$

$$c_2(\underline{x}_2) \cong \frac{1}{\sinh 2kd}\left[\cosh 2kd + \nu\cos 2(\underline{k}_2\cdot\underline{x}_2)\right] + \tanh kd\left[1 - \nu\cos 2(\underline{k}_2\cdot\underline{x}_2)\right]$$
$$= \frac{\cosh 2kd}{\sinh 2kd} + \tanh kd + \nu\left(\frac{1}{\sinh 2kd} - \tanh kd\right)\cos 2(\underline{k}_2\cdot\underline{x}_2) \qquad (10.12_5)$$
$$= \frac{1}{\sinh 2kd}\left[2\cosh 2kd - 1 + \nu(2 - \cosh 2kd)\cos 2(\underline{k}_2\cdot\underline{x}_2)\right],$$

because

$$-\frac{\cosh 2kd}{\sinh 2kd} + \tanh kd = \frac{-\cosh 2kd}{2\sinh kd\cosh kd} + \frac{\sinh kd}{\cosh kd}$$
$$= \frac{-\cosh 2kd + 2\sinh^2 kd}{2\sinh kd\cosh kd} = \frac{-\cosh 2kd + \cosh 2kd - 1}{\sinh 2kd} = -\frac{1}{\sinh 2kd}, \qquad (10.12_6)$$

$$\frac{1}{\sinh 2kd} + \tanh kd = \frac{1 + 2\sinh^2 kd}{\sinh 2kd} = \frac{\cosh 2kd}{\sinh 2kd} = \coth 2kd\,, \qquad (10.12_7)$$

$$\frac{\cosh 2kd}{\sinh 2kd} + \tanh kd = \frac{\cosh 2kd}{2\sinh kd\cosh kd} + \frac{\sinh kd}{\cosh kd} = \frac{\cosh 2kd + 2\sinh^2 kd}{2\sinh kd\cosh kd}$$
$$= \frac{\cosh 2kd + \cosh 2kd - 1}{\sinh 2kd} = \frac{2\cosh 2kd - 1}{\sinh 2kd}, \qquad (10.12_8)$$

$$\frac{1}{\sinh 2kd} - \tanh kd = \frac{1 - 2\sinh^2 kd}{\sinh 2kd} = \frac{2 - \cosh 2kd}{\sinh 2kd}. \qquad (10.12_9)$$

By the final expression in ($10.12_4$) and ($10.12_5$), equation ($10.12_3$) turns into (10.12).

2) By (8.57a') and (8.58), equation ($10.2_0$) can be developed thus`

$$
\begin{aligned}
\alpha_{\mathrm{k}(2)}(t,\underline{x}_2) &= -\frac{\mu\nu}{k^2}\sqrt{\frac{g}{k\tanh kd}}\,\tau_\mu(\Omega(k)t)\tau_{-\mu}(\Omega(k)t) \\
&\quad\cdot\sum_{i=1}^{2} k_i \nabla_i \left[\tau_\nu(\underline{k}_2\cdot\underline{x}_2)\tau_{-\nu}(\underline{k}_2\cdot\underline{x}_2)\right] \\
&= -\frac{\mu\nu}{4k^2}\sqrt{\frac{g}{k\tanh kd}}\sin 2\Omega(k)t\sum_{i=1}^{2} k_i\nabla_i \sin 2(\underline{k}_2\cdot\underline{x}_2) \\
&= -\frac{\mu\nu}{2}\sqrt{\frac{g}{k\tanh kd}}\sin 2\Omega(k)t\cos 2(\underline{k}_2\cdot\underline{x}_2),
\end{aligned}
\tag{10.13$_1$}
$$

because

$$
\sum_{i=1}^{2} k_i\nabla_i \sin 2(\underline{k}_2\cdot\underline{x}_2) = 2\cos 2(\underline{k}_2\cdot\underline{x}_2)\sum_{i=1}^{2}k_i k_i = 2k^2\cos 2(\underline{k}_2\cdot\underline{x}_2). \tag{10.13$_2$}
$$

The train of equations (10.13$_1$) proves (10.13).

3) Differentiation of both sides of equation (10.12) with respect to '$t$' yields

$$
\begin{aligned}
&\frac{\partial\alpha_{\mathrm{d}(2)}(t,\underline{x}_2)}{\partial t} \\
&= -\frac{\mu\Omega(k)}{2k\sinh 2kd}\left[2\cosh 2kd - 1 + \nu(2-\cosh 2kd)\cos 2(\underline{k}_2\cdot\underline{x}_2)\right]\sin 2\Omega(k)t \qquad (10.14_1)\\
&= -\frac{\mu\sqrt{g\tanh kd}}{2\sqrt{k}\sinh 2kd}\left[2\cosh 2kd - 1 + \nu(2-\cosh 2kd)\cos 2(\underline{k}_2\cdot\underline{x}_2)\right]\sin 2\Omega(k)t,
\end{aligned}
$$

where use of (7.38) has been made in developing the final expression. Substitution of (10.14$_1$) and (10.13) into (10.3$_0$) yields

$$
\begin{aligned}
\alpha_{(2)}(t,\underline{x}_2) &\doteqdot \frac{\partial\alpha_{\mathrm{d}(2)}(t,\underline{x}_2)}{\partial t} - \alpha_{\mathrm{k}(2)}(t,\underline{x}_2) = -\frac{\mu\sqrt{g\tanh kd}}{2\sqrt{k}\sinh 2kd}\Bigg[2\cosh 2kd - 1 \\
&\quad + \nu\left(2-\cosh 2kd - \frac{\sinh 2kd}{\tanh kd}\right)\cos 2(\underline{k}_2\cdot\underline{x}_2)\Bigg]\sin 2\Omega(k)t \qquad (10.14_2)\\
&= -\mu\sqrt{\frac{g\tanh kd}{k}}\,\frac{(2\cosh 2kd - 1)}{2\sinh 2kd}\left[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)\right]\sin 2\Omega(k)t,
\end{aligned}
$$

because

$$
\begin{aligned}
2-\cosh 2kd - \frac{\sinh 2kd}{\tanh kd} &= 2-\cosh 2kd - \frac{2\sinh kd\cosh^2 kd}{\sinh kd} \\
&= 2-\cosh 2kd - (\cosh 2kd + 1) = 1 - 2\cosh 2kd,
\end{aligned}
\tag{10.14$_3$}
$$

where use of the instance of (8.43b) with $\alpha \doteqdot kd$ has been made. The train of equations (10.14$_2$) proves (10.14). The lemma is established.•

## 10.4. The second-order approximation of the velocity potential in the presence of a standing wave

In the case of a standing wave, equations (10.4)–(10.7) retain. In this case, substitution of (10.14) into (10.5) yields

$$\psi_{(2)}(t,\underline{x}_2) = -\frac{\mu g(2\cosh 2kd - 1)}{2k\sinh 2kd}\left[1 - \nu\cos 2(\underline{k}_2\cdot\underline{x}_2)\right]I_{\mathrm{s}(2)}(t,\underline{x}_2), \tag{10.15}$$

where

$$I_{\mathrm{s}(2)}(t,\underline{x}_2) \doteq \int_0^t \sin 2\Omega(k)t'\sin[\Omega(k)(t-t')]\,dt'\,; \tag{10.16}$$

the subscript 's' is the first letter of 'standing'. The integrand of (10.6) can be developed as the following instance of (10.10):

$$\begin{aligned}\sin[\Omega(k)(t-t')]\sin 2\Omega(k)t' &= \frac{1}{2}\{\cos[\Omega(k)(t-3t')] - \cos[\Omega(k)(t+t')]\}\\ &= \frac{1}{2}\{\cos[\Omega(k)(3t'-t)] - \cos[\Omega(k)(t'+t)]\}.\end{aligned} \tag{10.16$_1$}$$

Consequently, the integral (10.16) subject to (10.16$_1$) is calculated thus:

$$\begin{aligned}I_{\mathrm{s}(2)}(t,\underline{x}_2) &= \frac{1}{2}\int_0^t\{\cos[\Omega(k)(3t'-t)] - \cos[\Omega(k)(t'+t)]\}dt'\\ &= \frac{1}{2\Omega(k)}\left\{\frac{1}{3}\sin[\Omega(k)(3t'-t)] - \sin[\Omega(k)(t'+t)]\right\}_{t'=0}^{t'=t}\\ &= \frac{1}{2\Omega(k)}\left[-\frac{2}{3}\sin 2\Omega(k)t + \frac{4}{3}\sin\Omega(k)t\right] = \frac{1}{3\Omega(k)}[2\sin\Omega(k)t - \sin 2\Omega(k)t].\end{aligned} \tag{10.16$_2$}$$

Thus, $\psi_{(2)}(t,\underline{x}_2)$ is given by (10.15) subject to (10.16$_2$), while $\phi_{(2)}(t,\underline{x})$ is expressed in terms of $\psi_{(2)}(t,\underline{x}_2)$ by (10.4). Hence finally, by (5.14) and (10.11$_1$),

$$\Phi_{(2)}(t,\underline{x}) \stackrel{\rightarrow}{=} \Phi_{(2)}(t;\underline{x},\varepsilon) = \varepsilon^2 \phi_{(2)}(t,\underline{x}) = (ak)^2 \phi_{(2)}(t,\underline{x})$$
$$= \frac{(ak)^2 \cosh k(z+d)}{\cosh kd} \psi_{(2)}(t,\underline{x}_2)$$
$$= -\frac{\mu a^2 gk(2\cosh 2kd - 1)}{2\cosh kd \sinh 2kd} \cosh k(z+d)[1 - \nu \cos 2(\underline{k}_2 \cdot \underline{x}_2)] I_{s(2)}(t,\underline{x}_2)$$
$$= -\frac{\mu a^2 (2\cosh 2kd - 1)}{6\cosh kd \sinh 2kd} \sqrt{\frac{gk}{\tanh kd}} \tag{10.17}$$
$$\cdot \cosh k(z+d)[1 - \nu \cos 2(\underline{k}_2 \cdot \underline{x}_2)][2\sin \Omega(k)t - \sin 2\Omega(k)t].$$
$$= -\frac{\mu a^2 \Omega(k)(2\cosh 2kd - 1)}{6\sinh kd \sinh 2kd}$$
$$\cdot \cosh k(z+d)[1 - \nu \cos 2(\underline{k}_2 \cdot \underline{x}_2)][2\sin \Omega(k)t - \sin 2\Omega(k)t].$$

because

$$\frac{gk}{\cosh kd} = \frac{gk \tanh kd}{\cosh kd \tanh kd} = \frac{\Omega^2(k)}{\sinh kd}. \tag{10.17$_1$}$$

## 10.5. Main general and concrete results and their implications

### 10.5.1. Preliminary remarks

In accordance with (5.14) (see also (7.72) and (7.73)), given an initial wave number $k>0$, given an initial amplitude $a>0$, the *scaled* partial velocity potential $\Phi_{(l)}(t;\underline{x},\varepsilon)$ of the $l$th asymptotic approximation, subject to $l \in \omega_1$, with respect to the dimensionless parameter '$\varepsilon$', such as $\varepsilon \stackrel{\rightarrow}{=} ka$, to the given *scaled* partial velocity potential $\Phi_{(1)}(t;\underline{x},\varepsilon)$ of a *priming* (*primary*) *progressive*, or *standing*, *plane monochromatic gravity water wave* (briefly *PPPMGWW* or *PSPMGWW* respectively) on a uniform water layer of a depth -$d$ from the equilibrium free surface $z$=0, – the wave, which serves as the first non-vanishing approximation, – is defined as:

$$\Phi_{(l)}(t,\underline{x}) \stackrel{\rightarrow}{=} \Phi_{(l)}(t;\underline{x},\varepsilon) \stackrel{\rightarrow}{=} \varepsilon^l \phi_{(l)}(t,\underline{x}) = (ka)^l \phi_{(l)}(t,\underline{x}) \text{ for each } l \in \omega_1, \tag{10.18}$$

where $\phi_{(l)}(t,\underline{x})$ is the respective $l$th *secondary non-scaled* partial velocity potential; $t$ is a time point, while $\underline{x} \stackrel{\rightarrow}{=} \underline{x}_3 \stackrel{\rightarrow}{=} \langle x_1,x_2,x_3\rangle = \langle x,y,z\rangle$, in accordance with item 4 of subsection 1.2. Given $l \in \omega_1$, the functional form $\Phi_{(l)}(t,\underline{x})$ allows in principle calculating *all* characteristics of the pertinent wave-related fluid flow in the liquid layer in the $l$th asymptotic approximation with respect to '$\varepsilon$', – such characteristics particularly as the liquid velocity components $V_{i(l)}(t,\underline{x})$ for each $i \in \omega_{1,3}$, the dynamic pressure $P_{\mathrm{d}(l)}(t,\underline{x})$, and the vertical displacement $\mathrm{Z}_{(l)}(t,\underline{x}_2)$ of the disturbed free surface from the equilibrium plane z=0, – and it also allows *immediately* (*not mediately via* $\Phi_{(l+1)}(t,\underline{x})$) calculating *strictly some*, i.e. *some but not all*, characteristics of the ed fluid flow in the ($l$+1)th asymptotic approximation with respect to '$\varepsilon$', – such characteristics particularly as the volumetric kinetic energy density $E_{\mathrm{k}(l)}(t,\underline{x})$ and the energy flux density vector (the Poynting vector) components $Q_{i(l)}(t,\underline{x})$ for each $i \in \omega_{1,3}$. The former characteristics will be called *characteristics of first kind* and the latter *characteristics of second kind, with respect to* $\Phi_{(l)}(t,\underline{x})$.

In what follows, I shall summarize various sets of the following interrelated concrete scaled functional forms in the case of a PPPMGWW or PSPMGWW as the first non-vanishing approximation; a properly specified $\Phi_{(1)}(t,\underline{x})$, the related $\Phi_{(2)}(t,\underline{x})$, the displacements of the free surface points from the equilibrium plane $z$=0 in the respective approximations, $\mathrm{Z}_{(1)}(t,\underline{x}_2)$ and $\mathrm{Z}_{(2)}(t,\underline{x}_2)$, and also temporal partial derivatives of the latter, $\partial \mathrm{Z}_{(1)}(t,\underline{x}_2)/\partial t$ and $\partial \mathrm{Z}_{(2)}(t,\underline{x}_2)/\partial t$. All formulas displayed below are subject to definitions (7.38), (8.2)–(8.4), and (8.30), i.e. subject to

$$\Omega(k) \doteqdot \Omega(k,d) \doteqdot \sqrt{gk\tanh kd} > 0\,, \tag{10.19}$$

$$\underline{k}_2 \doteqdot \langle k_1, k_2 \rangle \doteqdot \langle k_x, k_y \rangle,\ |\underline{k}_2|^2 \doteqdot \underline{k}_2{}^2 \equiv k_1{}^2 + k_2{}^2 = k^2\,,\ \underline{x}_2 \doteqdot \langle x_1, x_2 \rangle \doteqdot \langle x, y \rangle\,, \tag{10.20}$$

$$\tau_1 \doteqdot \sin,\ \tau_{-1} \doteqdot \cos\,, \tag{10.21}$$

respectively.

### 10.5.2. The case of a progressive wave

Given $\mu \in \{1,-1\}$:

$$\begin{aligned}\Phi_{(1)}(t,\underline{x}) &\doteqdot \Phi_{(1)}(t,\underline{x},ka) = \Phi_{\mu}^{-}(t;\underline{x},\underline{k}_2,a)\\ &\doteqdot -\mu a\sqrt{\frac{g}{k\tanh kd}}\,\frac{\cosh k(z+d)}{\cosh kd}\,\tau_{\mu}\big(\Omega(k)t - \underline{k}_2\cdot\underline{x}_2\big),\end{aligned} \tag{10.22}$$

– in accordance with (8.34);

$$\begin{aligned}\Phi_{(2)}(t,\underline{x}) \doteqdot \Phi_{(2)}(t,\underline{x},ka) = \frac{\mu a^2}{2\cosh kd\sinh 2kd}\sqrt{\frac{gk}{\tanh kd}}\cosh k(z+d)\\ \cdot\{-2\sin 2[\Omega(k)t - \underline{k}_2\cdot\underline{x}_2] + \sin[\Omega(k)t + 2\underline{k}_2\cdot\underline{x}_2] + 3\sin[\Omega(k)t - 2\underline{k}_2\cdot\underline{x}_2]\}.\end{aligned} \tag{10.23}$$

– in accordance with (10.11);

$$Z_{(1)}(t,\underline{x}_2)=\mathrm{A}_{\mathrm{d}(1)}(t,\underline{x}_2)-\frac{1}{g}\frac{\partial\Phi_{(1)}^{(0)}(t,\underline{x}_2)}{\partial t}=-\frac{1}{g}\left[\frac{\partial\Phi_{(1)}(t,\underline{x})}{\partial t}\right]_{z=0} \quad (10.24)$$
$$=a\tau_{-\mu}(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2),$$

$$Z_{(2)}(t,\underline{x}_2)=\mathrm{A}_{\mathrm{d}(2)}(t,\underline{x}_2)-\frac{1}{g}\frac{\partial\Phi_{(2)}^{(0)}(t,\underline{x}_2)}{\partial t}=\mathrm{A}_{\mathrm{d}(2)}(t,\underline{x}_2)-\frac{1}{g}\left[\frac{\partial\Phi_{(2)}(t,\underline{x})}{\partial t}\right]_{z=0}$$
$$=-\frac{a^2k}{2\sinh 2kd}\{1+\mu[-(2+\cosh 2kd)\cos 2(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2) \quad (10.25)$$
$$+\cos(\Omega(k)t+2\underline{k}_2\cdot\underline{x}_2)+3\cos(\Omega(k)t-2\underline{k}_2\cdot\underline{x}_2)]\},$$

– in accordance with (5.7), (6.34), (6.52), (6.55), (10.1), (10.22), and (10.23), because particularly

$$\mathrm{A}_{\mathrm{d}(2)}(t,\underline{x}_2)=-\frac{a^2k}{2\sinh 2kd}[1+\mu(2-\cosh 2kd)\cos 2(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2)], \quad (10.25_1)$$

by (6.34) and (10.1), and also because

$$-\frac{1}{g}\left[\frac{\partial\Phi_{(2)}(t,\underline{x})}{\partial t}\right]_{z=0}=-\frac{\mu a^2k}{2\sinh 2kd} \quad (10.25_2)$$
$$\cdot[-4\cos 2(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2)+\cos(\Omega(k)t+\underline{2k}_2\cdot\underline{x}_2)+3\cos(\Omega(k)t-\underline{2k}_2\cdot\underline{x}_2)],$$

by (10.23);

$$\frac{\partial Z_{(1)}(t,\underline{x}_2)}{\partial t}=\mathrm{A}_{\mathrm{k}(1)}(t,\underline{x}_2)+\Phi_{(l)}^{(1)}(t,\underline{x}_2)=\left[\frac{\partial\Phi_{(1)}(t,\underline{x})}{\partial z}\right]_{z=0}$$
$$=-\mu ak\sqrt{\frac{g}{k\tanh kd}}\frac{\sinh kd}{\cosh kd}\tau_{\mu}(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2) \quad (10.26)$$
$$=-\mu a\sqrt{gk\tanh kd}\,\tau_{\mu}(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2)=-\mu a\Omega(k)\tau_{\mu}(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2),$$

$$\frac{\partial Z_{(2)}(t,\underline{x}_2)}{\partial t}=\mathrm{A}_{\mathrm{k}(2)}(t,\underline{x}_2)+\Phi_{(2)}^{(1)}(t,\underline{x}_2)=\mathrm{A}_{\mathrm{k}(2)}(t,\underline{x}_2)+\left[\frac{\partial\Phi_{(2)}(t,\underline{x})}{\partial z}\right]_{z=0}$$
$$=\frac{\mu a^2k}{4\cosh^2 kd}\sqrt{\frac{gk}{\tanh kd}}[-2(1+2\cosh^2 kd)\sin 2(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2)$$
$$+\sin(\Omega(k)t+2\underline{k}_2\cdot\underline{x}_2)+3\sin(\Omega(k)t-2\underline{k}_2\cdot\underline{x}_2)] \quad (10.27)$$
$$=\frac{\mu a^2k\Omega(k)}{2\sinh 2kd}[-2(1+2\cosh^2 kd)\sin 2(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2)$$
$$+\sin(\Omega(k)t+2\underline{k}_2\cdot\underline{x}_2)+3\sin(\Omega(k)t-2\underline{k}_2\cdot\underline{x}_2)],$$

– in accordance with (5.7), (6.37), (6.53), (6.55), (10.2), (10.22), and (10.23), because particularly

$$\mathrm{A}_{\mathrm{k}(2)}\left(t,\underline{x}_2\right)=-\mu a^2 k\sqrt{\frac{gk}{\tanh kd}}\sin 2\left(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\right), \qquad (10.27_1)$$

by (6.37) and (10.2), and also because

$$\left[\frac{\partial\Phi_{(2)}\left(t,\underline{x}\right)}{\partial z}\right]_{z=0}=\frac{\mu a^2 k}{4\cosh^2 kd}\sqrt{\frac{gk}{\tanh kd}} \qquad (10.27_2)$$
$$\cdot\left[-2\sin 2\left(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\right)+\sin\left(\Omega(k)t+2\underline{k}_2\cdot\underline{x}_2\right)+3\sin\left(\Omega(k)t-2\underline{k}_2\cdot\underline{x}_2\right)\right],$$

by (10.23), whereas

$$\frac{1}{2\cosh^2 kd}\sqrt{\frac{gk}{\tanh kd}}=\frac{\sqrt{gk\tanh kd}}{2\cosh^2 kd\tanh kd}=\frac{\Omega(k)}{\sinh 2kd}, \qquad (10.27_3)$$

$$1+2\cosh^2 kd=2+\cosh 2kd, \qquad (10.27_4)$$

by (7.38), (8.42a), and (8.43b).

In order to be doubly sure in self-consistency of the above results, here follow straightforward calculations of $\partial\mathrm{Z}_{(1)}\left(t,\underline{x}_2\right)/\partial t$ and $\partial\mathrm{Z}_{(2)}\left(t,\underline{x}_2\right)/\partial t$ by differentiating both sides of (10.24) and both sides of (10.25) with respect to '$t$':

$$\frac{\partial\mathrm{Z}_{(1)}\left(t,\underline{x}_2\right)}{\partial t}=a\frac{\partial\tau_{-\mu}\left(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\right)}{\partial t}=-\mu a\Omega(k)\tau_{\mu}\left(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\right), \qquad (10.26a)$$

$$\frac{\partial\mathrm{Z}_{(2)}\left(t,\underline{x}_2\right)}{\partial t}=\frac{\partial\mathrm{A}_{\mathrm{d}(2)}\left(t,\underline{x}_2\right)}{\partial t}-\frac{1}{g}\left[\frac{\partial^2\Phi_{(2)}\left(t,\underline{x}\right)}{\partial t^2}\right]_{z=0}$$
$$=\frac{\mu a^2 k\Omega(k)}{2\sinh 2kd}\left[-2\left(2+\cosh 2kd\right)\sin 2\left(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\right)\right. \qquad (10.27a)$$
$$\left.+\sin\left(\Omega(k)t+2\underline{k}_2\cdot\underline{x}_2\right)+3\sin\left(\Omega(k)t-2\underline{k}_2\cdot\underline{x}_2\right)\right],$$

which coincide with (10.26) and (10.27) respectively, as expected.

Under the general definition

$$\overline{f(t)}^{t}\doteq\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}f(t)dt, \qquad (10.28)$$

it follows from (10.24)–(10.27) that

$$\overline{\mathrm{Z}}_{(1)}\left(\underline{x}_2\right)\doteq\overline{\mathrm{Z}_{(1)}\left(t,\underline{x}_2\right)}^{t}=a\overline{\tau_{-\mu}\left(\Omega(k)t-\underline{k}_2\cdot\underline{x}_2\right)}^{t}=0, \qquad (10.29)$$

$$\overline{\mathrm{Z}}_{(2)}\left(\underline{x}_2\right)\doteq\overline{\mathrm{Z}_{(2)}\left(t,\underline{x}_2\right)}^{t}=-\frac{a^2 k}{2\sinh 2kd}, \qquad (10.30)$$

$$\overline{\frac{\partial\mathrm{Z}_{(1)}\left(t,\underline{x}_2\right)}{\partial t}}^{t}=\overline{\frac{\partial\mathrm{Z}_{(2)}\left(t,\underline{x}_2\right)}{\partial t}}^{t}=0. \qquad (10.31)$$

Consequently, given $a \in (0,\infty)$, given $k \in (0,\infty)$, it follows from (10.30) that

$$\lim_{d\to\infty} \overline{Z_{(2)}(t,\underline{x}_2)}^t = -\lim_{d\to\infty} \frac{a^2 k}{2\sinh 2kd} = -0\,. \tag{10.31$_1$}$$

Also, equation (10.30) coincides with equation (4.12) in Longuet-Higgins and Stewart [1962], which was deduced there from intuitive considerations.

### 10.5.3. The case of a standing wave

Given $\mu \in \{1,-1\}$, given $\nu \in \{1,-1\}$:

$$\begin{aligned}\Phi_{(1)}(t,\underline{x}) &\equiv \Phi_{(1)}(t,\underline{x},ka) \equiv \Phi_{\mu\nu}(t;\underline{x},\underline{k}_2,a)\\ &= -\mu a\sqrt{\frac{g}{k\tanh kd}}\,\frac{\cosh k(z+d)}{\cosh kd}\,\tau_\mu(\Omega(k)t)\tau_\nu(k_2\cdot \underline{x}_2),\end{aligned} \tag{10.32}$$

– in accordance with (8.56);

$$\begin{aligned}\Phi_{(2)}(t,\underline{x}) \equiv \Phi_{(2)}(t,\underline{x},ka) = -\frac{\mu a^2(2\cosh 2kd-1)}{6\cosh kd\sinh 2kd}\sqrt{\frac{gk}{\tanh kd}}\\ \cdot\cosh k(z+d)\left[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)\right]\left[2\sin\Omega(k)t-\sin 2\Omega(k)t\right].\end{aligned} \tag{10.33}$$

– in accordance with (10.17);

$$\begin{aligned}Z_{(1)}(t,\underline{x}_2) &= \mathrm{A}_{\mathrm{d}(1)}(t,\underline{x}_2) - \frac{1}{g}\frac{\partial\Phi^{(0)}_{(1)}(t,\underline{x}_2)}{\partial t} = -\frac{1}{g}\left[\frac{\partial\Phi_{(1)}(t,\underline{x})}{\partial t}\right]_{z=0}\\ &= \frac{\mu a}{\sqrt{gk\tanh kd}}\frac{\partial\tau_\mu(\Omega(k)t)}{\partial t}\tau_\nu(k_2\cdot\underline{x}_2) = \frac{\mu^2 a\Omega(k)}{\sqrt{gk\tanh kd}}\tau_{-\mu}(\Omega(k)t)\tau_\nu(k_2\cdot\underline{x}_2)\\ &= a\,\tau_{-\mu}(\Omega(k)t)\tau_\nu(k_2\cdot\underline{x}_2)\end{aligned} \tag{10.34}$$

and

$$\begin{aligned}Z_{(2)}(t,\underline{x}_2) &= \mathrm{A}_{\mathrm{d}(2)}(t,\underline{x}_2) - \frac{1}{g}\frac{\partial\Phi^{(0)}_{(2)}(t,\underline{x}_2)}{\partial t} = \mathrm{A}_{\mathrm{d}(2)}(t,\underline{x}_2) - \frac{1}{g}\left[\frac{\partial\Phi_{(2)}(t,\underline{x}_2)}{\partial t}\right]_{z=0}\\ &= \frac{a^2 k}{4\sinh 2kd}\{-\left[1+\nu\cosh 2kd\cos 2(\underline{k}_2\cdot\underline{x}_2)\right]\\ &\quad+\mu\left[2\cosh 2kd-1+\nu(2-\cosh 2kd)\cos 2(\underline{k}_2\cdot\underline{x}_2)\right]\cos 2\Omega(k)t\}\\ &\quad+\frac{\mu a^2 k(2\cosh 2kd-1)}{3\sinh 2kd}\left[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)\right]\left[\cos\Omega(k)t-\cos 2\Omega(k)t\right]\end{aligned} \tag{10.35$_0$}$$

(to be reduced below), – in accordance with (5.7), (6.34), (6.52), (6.55), (10.12), (10.32), and (10.33), because particularly

$$\mathrm{A}_{\mathrm{d}(2)}(t,\underline{x}_2)=\frac{a^2k}{4\sinh 2kd}\{-[1+\nu\cosh 2kd\cos 2(\underline{k}_2\cdot\underline{x}_2)]$$
$$+\mu[2\cosh 2kd-1+\nu(2-\cosh 2kd)\cos 2(\underline{k}_2\cdot\underline{x}_2)]\cos 2\Omega(k)t\}, \tag{10.35$_1$}$$

by (6.34) and (10.12), and also because

$$-\frac{1}{g}\left[\frac{\partial\Phi_{(2)}(t,\underline{x}_2)}{\partial t}\right]_{z=0}=\frac{\mu a^2(2\cosh 2kd-1)\Omega(k)}{3g\sinh 2kd}\sqrt{\frac{gk}{\tanh kd}}$$
$$\cdot[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)][\sin\Omega(k)t-\sin 2\Omega(k)t] \tag{10.35$_2$}$$
$$=\frac{\mu a^2k(2\cosh 2kd-1)}{3\sinh 2kd}[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)][\sin\Omega(k)t-\sin 2\Omega(k)t],$$

by (10.33);

$$\frac{\partial Z_{(1)}(t,\underline{x}_2)}{\partial t}=\mathrm{A}_{\mathrm{k}(1)}(t,\underline{x}_2)+\Phi_{(l)}^{(1)}(t,\underline{x}_2)=\left[\frac{\partial\Phi_{(1)}(t,\underline{x})}{\partial z}\right]_{z=0} \tag{10.36}$$
$$=-\mu a\sqrt{gk\tanh kd}\,\tau_\mu(\Omega(k)t)\tau_\nu(k_2\cdot\underline{x}_2)=-\mu a\Omega(k)\tau_\mu(\Omega(k)t)\tau_\nu(k_2\cdot\underline{x}_2)$$

and

$$\frac{\partial Z_{(2)}(t,\underline{x}_2)}{\partial t}=\mathrm{A}_{\mathrm{k}(2)}(t,\underline{x}_2)+\Phi_{(2)}^{(1)}(t,\underline{x}_2)=\mathrm{A}_{\mathrm{k}(2)}(t,\underline{x}_2)+\left[\frac{\partial\Phi_{(2)}(t,\underline{x})}{\partial z}\right]_{z=0}$$
$$-\frac{\mu\nu a^2k(\cosh 2kd+1)\Omega(k)}{2\sinh 2kd}\sin 2\Omega(k)t\cos 2(\underline{k}_2\cdot\underline{x}_2),$$
$$=-\frac{\mu\nu a^2k(\cosh 2kd+1)\Omega(k)}{2\sinh 2kd}\sin 2\Omega(k)t\cos 2(\underline{k}_2\cdot\underline{x}_2) \tag{10.37$_0$}$$
$$-\frac{\mu a^2k(2\cosh 2kd-1)\Omega(k)}{6\sinh 2kd}[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)][2\sin\Omega(k)t-\sin 2\Omega(k)t]$$
$$=-\frac{\mu a^2k\Omega(k)}{6\sinh 2kd}\{2(2\cosh 2kd-1)[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)]\sin\Omega(k)t$$
$$+[3\nu(\cosh 2kd+1)-(2\cosh 2kd-1)[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)]]\sin 2\Omega(k)t\}$$

(to be reduced further below),– in accordance with (5.7), (6.37), (6.53), (6.55), (10.13), (10.32), and (10.33), because particularly

$$\begin{aligned}\mathrm{A}_{\mathrm{k}(2)}(t,\underline{x}_2) &= -\frac{\mu\nu a^2 k}{2}\sqrt{\frac{gk}{\tanh kd}}\sin 2\Omega(k)t\cos 2(\underline{k}_2\cdot\underline{x}_2)\\ &= -\frac{\mu\nu a^2 k\cosh kd\,\Omega(k)}{2\sinh kd}\sin 2\Omega(k)t\cos 2(\underline{k}_2\cdot\underline{x}_2)\\ &= -\frac{\mu\nu a^2 k\cosh^2 kd\,\Omega(k)}{\sinh 2kd}\sin 2\Omega(k)t\cos 2(\underline{k}_2\cdot\underline{x}_2)\\ &= -\frac{\mu\nu a^2 k(\cosh 2kd+1)\Omega(k)}{2\sinh 2kd}\sin 2\Omega(k)t\cos 2(\underline{k}_2\cdot\underline{x}_2),\end{aligned} \tag{10.37$_1$}$$

by (6.37) and (10.13), and also because

$$\begin{aligned}\left[\frac{\partial\Phi_{(2)}(t,\underline{x})}{\partial z}\right]_{z=0} &= -\frac{\mu a^2 k\sinh kd(2\cosh 2kd-1)}{6\cosh kd\sinh 2kd}\sqrt{\frac{gk}{\tanh kd}}\\ &\quad\cdot[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)][2\sin\Omega(k)t-\sin 2\Omega(k)t]\\ &= -\frac{\mu a^2 k(2\cosh 2kd-1)\Omega(k)}{6\sinh 2kd}[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)][2\sin\Omega(k)t-\sin 2\Omega(k)t],\end{aligned} \tag{10.37$_2$}$$

by (10.33) and (10.27$_3$). Equations (10.35$_0$) and (10.37$_0$) are reduced in what follows.

Equations (10.35$_0$) can conveniently be written in the form

$$Z_{(2)}(t,\underline{x}_2) = \frac{a^2 k}{12\sinh 2kd}[C_0(\underline{x}_2)+C_1(\underline{x}_2)\cos\Omega(k)t+C_2(\underline{x}_2)\cos 2\Omega(k)t], \tag{10.35$_3$}$$

where

$$C_0(\underline{x}_2) \doteq -3[1+\nu\cosh 2kd\cos 2(\underline{k}_2\cdot\underline{x}_2)], \tag{10.35$_4$}$$

$$C_1(\underline{x}_2) \doteq 4\mu(2\cosh 2kd-1)[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)], \tag{10.35$_5$}$$

$$\begin{aligned}C_2(\underline{x}_2) &\doteq \mu\{3[2\cosh 2kd-1+\nu(2-\cosh 2kd)\cos 2(\underline{k}_2\cdot\underline{x}_2)]\\ &\quad -4(2\cosh 2kd-1)[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)]\}\\ &= -\mu(2\cosh 2kd-1)+\mu\nu[3(2-\cosh 2kd)+4(2\cosh 2kd-1)]\cos 2(\underline{k}_2\cdot\underline{x}_2)\\ &= -\mu(2\cosh 2kd-1)+\mu\nu(5\cosh 2kd+2)\cos 2(\underline{k}_2\cdot\underline{x}_2).\end{aligned} \tag{10.35$_6$}$$

Thus, equation (10.35$_3$) subject to (10.35$_4$)–(10.35$_6$) can be written as the following single whole reduced equation:

$$\begin{aligned}Z_{(2)}(t,\underline{x}_2) &= \frac{a^2 k}{12\sinh 2kd}\{-3[1+\nu\cosh 2kd\cos 2(\underline{k}_2\cdot\underline{x}_2)]\\ &\quad +4\mu(2\cosh 2kd-1)[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)]\cos\Omega(k)t\\ &\quad +\mu[1-2\cosh 2kd+\nu(5\cosh 2kd+2)\cos 2(\underline{k}_2\cdot\underline{x}_2)]\cos 2\Omega(k)t.\end{aligned} \tag{10.35}$$

In analogy with (10.35$_3$) subject to (10.35$_r$)–(10.35$_6$), equation (10.37$_0$) can conveniently be written in the form

$$\frac{\partial Z_{(2)}(t,\underline{x}_2)}{\partial t} = -\frac{\mu a^2 k\Omega(k)}{6\sinh 2kd}\left[D_1(\underline{x}_2)\sin\Omega(k)t + D_2(\underline{x}_2)\sin 2\Omega(k)t\right], \qquad (10.37_3)$$

where

$$D_1(\underline{x}_2) \stackrel{\rightharpoonup}{=} 2(2\cosh 2kd - 1)[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)], \qquad (10.37_4)$$

$$\begin{aligned} D_2(\underline{x}_2) &\stackrel{\rightharpoonup}{=} 3\nu(\cosh 2kd + 1) - (2\cosh 2kd - 1)[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)] \\ &= -(2\cosh 2kd - 1) + \nu[3(\cosh 2kd + 1) + 2\cosh 2kd - 1]\cos 2(\underline{k}_2\cdot\underline{x}_2) \\ &= 1 - 2\cosh 2kd + \nu(5\cosh 2kd + 2)\cos 2(\underline{k}_2\cdot\underline{x}_2).. \end{aligned} \qquad (10.37_5)$$

Thus, equation ($10.37_3$) subject to ($10.37_4$) and ($10.35_5$) can be written as the following single whole reduced equation:

$$\begin{aligned} \frac{\partial Z_{(2)}(t,\underline{x}_2)}{\partial t} = -\frac{\mu a^2 k\Omega(k)}{6\sinh 2kd}\{&2(2\cosh 2kd - 1)[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)]\sin\Omega(k)t \\ &+ [1 - 2\cosh 2kd + \nu(5\cosh 2kd + 2)\cos 2(\underline{k}_2\cdot\underline{x}_2)]\sin 2\Omega(k)t\}. \end{aligned} \qquad (10.37)$$

Just as in the case of (10.26a) and (10.27a), in order to be doubly` sure in self-consistency of the above results, here follow straightforward calculations of $\partial Z_{(1)}(t,\underline{x}_2)/\partial t$ and $\partial Z_{(2)}(t,\underline{x}_2)/\partial t$ by differentiating both sides of (10.34) and both sides of (10.35) with respect to '$t$':

$$\frac{\partial Z_{(1)}(t,\underline{x}_2)}{\partial t} = a\frac{\partial \tau_{-\mu}(\Omega(k)t)}{\partial t}\tau_\nu(k_2\cdot\underline{x}_2) = -\mu a\Omega(k)\tau_\mu(\Omega(k)t)\tau_\nu(k_2\cdot\underline{x}_2), \qquad (10.36a)$$

$$\begin{aligned} \frac{\partial Z_{(2)}(t,\underline{x}_2)}{\partial t} &= \frac{\partial A_{d(2)}(t,\underline{x}_2)}{\partial t} - \frac{1}{g}\left[\frac{\partial^2\Phi_{(2)}(t,\underline{x}_2)}{\partial t^2}\right]_{z=0} \\ &= \frac{a^2 k}{12\sinh 2kd}\frac{\partial}{\partial t}\left[C_0(\underline{x}_2) + C_1(\underline{x}_2)\cos\Omega(k)t + C_2(\underline{x}_2)\cos 2\Omega(k)t\right] \\ &= -\frac{a^2 k\Omega(k)}{12\sinh 2kd}\left[C_1(\underline{x}_2)\sin\Omega(k)t + 2C_2(\underline{x}_2)\sin 2\Omega(k)t\right] \\ &= -\frac{\mu a^2 k\Omega(k)}{6\sinh 2kd}\{2(2\cosh 2kd - 1)[1-\nu\cos 2(\underline{k}_2\cdot\underline{x}_2)]\cos\Omega(k)t \\ &\quad + [1 - 2\cosh 2kd + \nu(5\cosh 2kd + 2)\cos 2(\underline{k}_2\cdot\underline{x}_2)]\cos 2\Omega(k)t\}, \end{aligned} \qquad (10.37a)$$

which coincide with (10.36) and (10.37) respectively, as expected.

Under the general definition (10.28), it follows from (10.34)–(10.37) that

$$\overline{Z}_{(1)}(\underline{x}_2) \stackrel{\rightharpoonup}{=} \overline{Z_{(1)}(t,\underline{x}_2)}^t = a\overline{\tau_{-\mu}(\Omega(k)t)}^t\tau_\nu(k_2\cdot\underline{x}_2) = 0, \qquad (10.38)$$

$$\overline{Z}_{(2)}(\underline{x}_2) \stackrel{\rightharpoonup}{=} \overline{Z_{(2)}(t,\underline{x}_2)}^t = -\frac{a^2 k}{4\sinh 2kd}[1+\nu\cosh 2kd\cos 2(\underline{k}_2\cdot\underline{x}_2)], \qquad (10.39)$$

$$\overline{\frac{\partial Z_{(1)}(t,\underline{x}_2)}{\partial t}}^{t} = \overline{\frac{\partial Z_{(2)}(t,\underline{x}_2)}{\partial t}}^{t} = 0\,. \tag{10.40}$$

Equation (10.39) comes now instead of (10.30). By definitions (8.2)–(8.4), it follows that

$$\cos 2(\underline{k}_2 \cdot \underline{x}_2) = \cos 2(k_1 x_1 + k_2 x_2) = \cos 4\pi\left(\frac{x_1}{\lambda_1} + \frac{x_2}{\lambda_2}\right), \tag{10.41}$$

so that ‘$\cos 2(\underline{k}_2 \cdot \underline{x}_2)$’ is a doubly periodic functional form of the spatial variables ‘$x_1$’ and ‘$x_2$’ with periods $\lambda_1/2 \doteqdot \pi/k_1$ and $\lambda_2/2 \doteqdot \pi/k_2$ respectively, – a form that takes on values in the interval [–1,1]. At the same time, given $d$>0, given $\nu \in \{1,-1\}$, the functional form ‘$1+\nu\cosh 2kd$’ takes on values in the interval $(-2\sinh^2 kd, 2\cosh^2 kd)$, by (8.43,a.b). Therefore, values of the functional form $\overline{Z_{(2)}(t,\underline{x}_2)}^{t}$, defined by (10.39), satisfy the relation:

$$\overline{Z_{(2)}(t,\underline{x}_2)}^{t} \in \frac{a^2 k}{2\sinh 2kd}\left(-\cosh^2 kd, \sinh^2 kd\right). \tag{10.39$_1$}$$

Under the general definition

$$\overline{F(\underline{x}_2)}^{\underline{x}_2} \doteqdot \lim_{X_1\to\infty}\lim_{X_2\to\infty}\frac{1}{X_1 X_2}\int_{-X_1/2}^{X_1/2}\int_{-X_2/2}^{X_2/2} F(\underline{x}_2)\,dx_2\,dx_1\,, \tag{10.42}$$

it follows from (10.39) that

$$\overline{\overline{Z_{(2)}(t,\underline{x}_2)}^{t}}^{\underline{x}_2} = -\frac{a^2 k}{4\sinh 2kd} \tag{10.43}$$

(cf. (10.30)). Consequently, given $a \in (0,\infty)$, given $k \in (0,\infty)$, it follows from (10.43) that

$$\lim_{d\to\infty}\overline{\overline{Z_{(2)}(t,\underline{x}_2)}^{t}}^{\underline{x}_2} = -\lim_{d\to\infty}\frac{a^2 k}{4\sinh 2kd} = -0 \tag{10.31$_1$}$$

(cf. (10.30$_1$)).

### 10.5.4. Concluding remarks

1) Equation (8.29) subject to (8.34) and equation (8.52) subject to (8.56) are particular cases of (10.18) at $l \doteqdot 1$, while equations (10.11) and (10.17) are particular cases of (10.18) at $l \doteqdot 2$. Accordingly, unless stated otherwise, by ‘$\Phi_{(1)}(t,\underline{x})$’, I shall henceforth mean either the functional form defined by (8.29) subject to (8.34) or that defined by (8.52) subject to (8.56); and likewise, by ‘$\Phi_{(2)}(t,\underline{x})$’, I shall henceforth mean either the functional form defined by (10.11) or that defined by

(10.17). The theory of gravity water waves will be called a *linear* one if it is based on $\Phi_{(1)}(t,\underline{x})$, and a *bilinear* one if it is based on $\Phi_{(1)}(t,\underline{x})+\Phi_{(2)}(t,\underline{x})$. Both $\Omega(k)$ (defined by (7.38)) and $\sinh 2kd$ are of the order of $k$ as $k\to 0$. Therefore, it follows from (10.11) or (10.17) that $\Phi_{(2)}(t,\underline{x})\to 0$ as $k\to 0$ and it also follows that, in either case, values of the functional form $\Phi_{(2)}(t,\underline{x})$ along with values of all its partial derivatives of any order are periodic in *t* and are bounded as $t\to\pm\infty$.

2) In accordance with (5.8), the entire asymptotic expansion of the scaled velocity potential $\Phi(t,\underline{x})$ of a fluid flow in the water layer of a uniform depth *d* in powers of *ka* subject to a *priming progressive*, or *standing*, *plane monochromatic gravity water wave* (briefly *PPPMGWW* or *PSPMGWW* respectively) of a wave number *k* and of a surface amplitude *a* is written as

$$\Phi(t,\underline{x}) \doteqdot \Phi(t;\underline{x},\varepsilon) \sim \Phi_{[\infty,1]}(t;\underline{x},\varepsilon) \doteqdot \sum_{l=1}^{\infty}(ka)^l \phi_{(l)}(t,\underline{x}), \tag{10.44}$$

subject to a well-established an algorithm for successively calculating the non-scaled velocity potentials $\phi_{(l)}(t,\underline{x})$ for all $l\in\omega_2$. At the same time, there are known in mathematics several different kinds of convergence of infinite functional sequences in general and of infinite series and improper integrals in particular, – such kinds of convergence as absolute, conditional, uniform, and mean square ones (see, e.g., Apostol [1963, pp. 353, 359, 360, 390–396, 407,408] and Budak and Fomin [1978, pp. 319–331, 366–374]). Uniform convergence is a quite universal kind of convergence, for which there exist convenient tests as Caushy's, Weierstass', and Abel's ones (see, e.g., Apostol [1963, pp. 395–396] and Budak and Fomin [1978, pp. 328–331]). In this case, uniform convergence implies convergence in the mean square, but not vice versa (Budak and Fomin [1978, pp. 272–274]). It is understood that in order to prove or disprove that a given functional series (as an asymptotic one) converges in a certain sense, one should employ only well-established tests for convergence of the given kind, and not to rely on the intuition. Unfortunately, none of the existing convergence criteria is applicable to an asymptotic series (10.44) for the following reason, because in contrast to '$\Phi_{(2)}(t,\underline{x})$' calculations of '$\Phi_{(3)}(t,\underline{x})$' turn out to be intolerably prolix. Therefore, the question whether or not the asymptotic series (10.44) converges remains unanswered.

3) One may, of course, assumed by analogy that $\Phi_{(l)}(t,\underline{x})$ at any $l\in\omega_3$ and, particularly, $\Phi_{(3)}(t,\underline{x})$, and also all its partial derivatives of any order have similar properties of temporo-spatial periodicity and hence the same properties of boundedness at $t\to\pm\infty$ as those of $\Phi_{(2)}(t,\underline{x})$ and of all

its partial derivatives. Alternatively, one can make any other assumption regarding $\Phi_{(l)}(t,\underline{x})$ at some $l \in \omega_3$. In this case, however, one should remember that if it happens that for some (strictly some or all) $l \in \omega_3$ $\phi_{(l)}(t,\underline{x})$ is unbounded as $t \to +\infty$ or $t \to -\infty$ then the asymptotic power series $\Phi_{[\infty,1]}(t;\underline{x},\varepsilon)$ of $\Phi(t,\underline{x})$, defined by (10.44), should not necessarily be divergent. In other words, there is no direct connection between the property of some or all coefficients $\phi_{(l)}(t,\underline{x})$ of the asymptotic power series $\Phi_{[\infty,1]}(t;\underline{x},\varepsilon)$ to be unbounded at $t \to +\infty$ or $t \to -\infty$ on the one hand and the property of that series either to diverge or to converge on the other hand. Here follows an example that illustrates this property.

4) It is known that for each $X \in (-\infty,+\infty)$ and each $p \in \omega_0$:

$$\frac{d^{2p}\cos X}{dX^{2p}} = (-1)^p \cos X\,,\ \frac{d^{2p+1}\cos X}{dX^{2p+1}} = (-1)^{p+1}\sin X\,, \tag{10.45}$$

$$\frac{d^{2p}\sin X}{dX^{2p}} = (-1)^p \sin X\,,\ \frac{d^{2p+1}\sin X}{dX^{2p+1}} = (-1)^{p+1}\cos X\,. \tag{10.46}$$

Therefore, given $x_0 \in (-\infty,+\infty)$, the Taylor series for $\cos(x_0 + x)$ and with respect to x, defined as

$$x \doteqdot X - x_0 \in (-\infty,+\infty), \tag{10.47}$$

about the point $x_0$ can be written as

$$\cos(x_0 + x) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{d^n \cos X}{dX^n}\right)_{X=x_0} x^n = \sum_{p=0}^{\infty}\frac{(-1)^p}{(2p)!}\left(\cos x_0 + \frac{x\sin x_0}{2p+1}\right)x^{2p}\,, \tag{10.48}$$

$$\sin(x_0 + x) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{d^n \sin X}{dX^n}\right)_{X=x_0} x^n = \sum_{p=0}^{\infty}\frac{(-1)^p}{(2p)!}\left(\sin x_0 + \frac{x\cos x_0}{2p+1}\right)x^{2p}\,, \tag{10.49}$$

where conventionally $0! \doteqdot 1$. It is known that both series (10.48) and (10.49) absolutely converge for each $x$ satisfying (10.47). If, particularly, $x_0 = 0$ so that $X = x$, then (10.48) and (10.49) turn into the known Maclaurin series for '$\cos x$' and '$\sin x$' with an infinite radius of absolute convergence, namely,

$$\cos x = \sum_{p=0}^{\infty}\frac{(-1)^p x^{2p}}{(2p)!}\,,\ \sin x = \sum_{p=0}^{\infty}\frac{(-1)^p x^{2p+1}}{(2p+1)!}\,, \text{ for each } x \in (-\infty,+\infty)\,. \tag{10.50}$$

Let $x_0 \doteqdot \omega t$ and $x \doteqdot \varepsilon t$, so that $X = (\omega + \varepsilon)t$, where '$\omega$' and '$t$' are constants, i.e. $\omega$ and $t$ are given numbers, while '$\varepsilon$' is a variable that is introduced instead of '$x$'. In this case, equations (10.48) and (10.49) become

$$\cos(\omega+\varepsilon)t = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{d^n \cos X}{dX^n}\right)_{X=\omega t} (\varepsilon t)^n = \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p)!}\left(\cos\omega t + \frac{\varepsilon t \sin\omega t}{2p+1}\right)(\varepsilon t)^{2p}, \qquad (10.51)$$

$$\sin(\omega+\varepsilon)t = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{d^n \sin X}{dX^n}\right)_{X=\omega t} (\varepsilon t)^n = \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p)!}\left(\sin\omega t + \frac{\varepsilon t \cos\omega t}{2p+1}\right)(\varepsilon t)^{2p}. \qquad (10.52)$$

Thus, in spite of the fact that both series (10.51) and (10.52) absolutely converge for each $\varepsilon t \in (-\infty,+\infty)$, each individual term of either series, except for the very first term '$\cos\omega t$' or '$\sin\omega t$' corresponding to $n = 0$, is unbounded as $t \to \pm\infty$. In this case, the domain of values both of '$\cos X$' and of '$\sin X$' is the interval $[-1,1]$, whereas the domain of values of any unbounded term of any one of the series (10.48)–(10.52) is either the entire set of real numbers, $(-\infty,+\infty)$ or one of its semi-infinite subsets $[0,+\infty)$ and $[0,-\infty)$. Therefore, it is impossible to establish from those series that the functional forms '$\cos X$' and '$\sin X$' are periodic.

5) It would be incorrect to conclude that a certain oscillatory motion of a physical system is unstable only on the base of the fact that, in a higher-order asymptotic approximation, the only coordinate, or one of the coordinates, of that motion increases indefinitely with unlimited increase of $t$ (except, perhaps, for the case when the coordinate increases exponentially with $t$). The unboundedness of the higher-order approximation may just mean that the relevant asymptotic expansion is not a suitable iterative algorithm for constructing a consistent *perturbation theory* of the phenomenon.

6) The water wave problem under discussion is a non-linearW. Consequently, the infinite asymptotic series solving the problem does not satisfy a superposition principle in the sense that the term-by-term sum of two asymptotic series corresponding to two different priming progressive, or standing waves, is not an asymptotic series generated by the sum of the two priming waves.

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