# The Homology Groups of $G_{n,m}(\mathbb{R})$

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**Abstract**

In this article, we computed the homology groups of real Grassmann manifold $G_{n,m}(\mathbb{R})$ by Witten complex.

## 1 Introduction

For a Morse function $f$ on a compact manifold $M$, and choosing a Riemannian metric $g$ on $M$. In [1], Witten introduces a chain complex and the homology groups of $M$ by studing the negative gradient vector field associated to $(f, g)$, and he claimes that these groups are the ordinary homology groups. Salamon gives a more detailed and precise interpretation to Witten's method in [2], we can also see the books [3] and [4]. Here below we give a brief description to Witten's method.

Let $M$ be a n-dimensionial compact smooth manifold and $f$ be a Morse function on $M$. We can choose a suitable Riemannian metric on $M$ such that the negative gradient vector field $-\nabla f$ to be Morse-Smale type(i.e.$f$ is a Morse-Smale function), then the connecting orbits determine the following chain complex.

First choose an orientation of the vector space $E^u(p) = T_pW^u(p)$ for every critical point $p$ of $f$ and denote by $\langle p\rangle$ the pair consisting of the critical point $p$ and the chosen orientation. For $r = 1, 2, \cdots \dim M$, denote by $C_r$ the free group $C_r = \bigoplus_p \mathbb{Z}\langle p\rangle$ where $p$ runs over all critical points of index $r$. The function $f$ being of Morse-Smale function implies that $W^u(p) \cap W^s(q)$ consists of finitely many trajectories if $\text{ind}(p) - \text{ind}(q) = 1$. In this case one can define an integer $n(p, q)$ by assigning a number $+1$ or $-1$ to every connecting orbit and taking the sum. Let $\varphi(t)$ be such a connecting orbit meaning a solution of $\dot{x} = -\nabla f$ with $\lim_{t\to-\infty} \varphi(t) = p$ and $\lim_{t\to+\infty} \varphi(t) = q$. Then $\langle p\rangle$ induces an orientation on the orthogonal complement $E^u_\varphi(p)$ of $v = \lim_{t\to-\infty} \mid \dot{\varphi}(t) \mid^{-1} \dot{\varphi}(t)$ in $E^u(p)$. Then the negative gradient flow induces an isomorphism between $E^u_\varphi(p)$ onto $E^u(p)$ and we define $n_\varphi$ to be $+1$ and $-1$ according to whether this map preserved or reverse the orientation.
Define

$$n(p, q) = \sum_\varphi n_\varphi$$

where the sum runs over all orbits of $\dot{x} = -\nabla f$ connecting $p$ and $q$. Then Witten's boundary operator $\partial : C_{r+1} \longrightarrow C_r$ of the chain complex is defined by

$$\partial\langle p\rangle = \sum_q n(p, q)\langle q\rangle$$

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where $q$ runs over all critical points of index $r$.

Witten claims that $\partial$ is a boundary operator, i.e.$\partial^2 = 0$, therefore

$$0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0$$

is a chain complex. Define for $r = 0, 1, 2, \cdots, n$,

$$H_r^W(M, \mathbb{Z}) = \frac{\ker\{\partial : C_r \longrightarrow C_{r-1}\}}{\partial(C_{r+1})}$$

1. $H_*^W(M, \mathbb{Z})$ are independent of the choice of the Riemannian metric on $M$;
2. $H_*^W(M, \mathbb{Z}) = H_*(M, \mathbb{Z})$, the usual homology groups of $M$.

To compute the homology groups of real Grassmann manifold by Witten complex comes from the work of Feng Hui-tao on $G_{5,2}(\mathbb{R})$(see [5]) for provide a nontrivial example of Witten's method. In [6], Qiao Pei-zhi gives the result on $\mathbb{R}P^n$, i.e.$G_{n+1,1}(\mathbb{R})$. In [7], Yang Ying gives the result on $G_{n,2}(\mathbb{R})$. In this article we will give the result on $G_{n,m}(\mathbb{R})$. Computation of the homology groups of $G_{n,m}(\mathbb{C})$ by Witten complex see [4]. We must point out that the comuptation of the homology groups of Grassmann manifold has been known to topologist by use Schubert calculus(see [10] and [11]).

## 2 A Morse funtion on the Grassmann manifold $G_{n,m}(\mathbb{R})$

Let the Grassmann manifold $G_{n,m}(\mathbb{R})$ is the set of all m-dimensional linear subspaces of n-dimensional real vector space $\mathbb{R}^n$. Set

$$M = \left\{ (x_{\alpha k}) = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \middle| x_{\alpha k} \in \mathbb{R}, \operatorname{rank}(x_{\alpha k}) = m \right\},$$

then the Grassmann manifold $G_{n,m}(\mathbb{R})$ can be defined by

$$G_{n,m}(\mathbb{R}) = M/GL(m, \mathbb{R}).$$

Let $\pi : M \to G_{n,m}(\mathbb{R})$ is the standard projection and define

$$U_{i_1 i_2 \cdots i_m} = \left\{ (x_{\alpha k}) \middle| x_{\alpha k} \in M, \begin{pmatrix} x_{1i_1} & x_{1i_2} & \cdots & x_{1i_m} \\ x_{2i_1} & x_{2i_2} & \cdots & x_{2i_m} \\ \cdots & \cdots & \cdots & \cdots \\ x_{mi_1} & x_{mi_2} & \cdots & x_{mi_m} \end{pmatrix} = I_m \right\}, 1 \le i_1 < i_2 < \cdots < i_m \le n.$$

where $I_m$ is the identity matrix of rank m.

$$\phi_{i_1 i_2 \cdots i_m} : U_{i_1 i_2 \cdots i_m} \to \mathbb{R}^{m(n-m)},$$

$$\phi_{i_1 i_2 \cdots i_m}(x_{\alpha k}) = (x_{1k_1}, x_{1k_2}, \cdots, x_{1k_{n-m}}, x_{2k_1}, x_{2k_2}, \cdots, x_{2k_{n-m}}, \cdots, x_{mk_1}, x_{mk_2}, \cdots, x_{mk_{n-m}}).$$

where $k_s \neq i_1, i_2, \cdots, i_m;\ s = 1, 2, \cdots, n-m;\ 1 \le k_1 < \cdots < k_{n-m} \le n$.

Set
$$\widetilde{U}_{i_1 i_2 \cdots i_m} = \pi(U_{i_1 i_2 \cdots i_m}),\ \widetilde{\phi}_{i_1 i_2 \cdots i_m} = \phi_{i_1 i_2 \cdots i_m} \circ \pi^{-1} : \widetilde{U}_{i_1 i_2 \cdots i_m} \to \mathbb{R}^{m(n-m)}.$$
Then $\{(\widetilde{U}_{i_1 i_2 \cdots i_m}, \widetilde{\phi}_{i_1 i_2 \cdots i_m}) | 1 \le i_1 < i_2 < \cdots < i_m \le n\}$ are the local coordinate covering of $G_{n,m}(\mathbb{R})$.

Here we will construct the Morse function on $G_{n,m}(\mathbb{R})$ by the same way as in [5],[6],[7].

Let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$ be fixed numbers, for any $(x_{\alpha k}) \in M$, set

$$\xi_\alpha = (x_{\alpha 1}, x_{\alpha 2}, \cdots, x_{\alpha n}),$$

$$\bar{\xi}_\alpha = (\frac{x_{\alpha 1}}{\lambda_1}, \frac{x_{\alpha 2}}{\lambda_2}, \cdots \frac{x_{\alpha n}}{\lambda_n}),\ \alpha = 1, 2, \cdots, m.$$

$$\Delta = \det \begin{pmatrix} \langle \xi_1, \xi_1 \rangle & \langle \xi_1, \xi_2 \rangle & \cdots & \langle \xi_1, \xi_m \rangle \\ \langle \xi_2, \xi_1 \rangle & \langle \xi_2, \xi_2 \rangle & \cdots & \langle \xi_2, \xi_m \rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle \xi_m, \xi_1 \rangle & \langle \xi_m, \xi_2 \rangle & \cdots & \langle \xi_m, \xi_m \rangle \end{pmatrix}$$

$$\bar{\Delta} = \det \begin{pmatrix} \langle \bar{\xi}_1, \bar{\xi}_1 \rangle & \langle \bar{\xi}_1, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_1, \bar{\xi}_m \rangle \\ \langle \bar{\xi}_2, \bar{\xi}_1 \rangle & \langle \bar{\xi}_2, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_2, \bar{\xi}_m \rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle \bar{\xi}_m, \bar{\xi}_1 \rangle & \langle \bar{\xi}_m, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_m, \bar{\xi}_m \rangle \end{pmatrix}$$

where
$$\langle \xi_i, \xi_j \rangle = x_{i1}x_{j1} + x_{i2}x_{j2} + \cdots + x_{im}x_{jm}, i, j = 1, 2, \cdots, m.$$
we define a function $f$ on $M$ by
$$f \doteq \frac{\Delta}{\bar{\Delta}}$$

**Lemma 1.** *The funtion $f$ is defined on the Grassmann manifold $G_{n,m}(\mathbb{R})$.*

*Proof.* Let $A \in GL(m, \mathbb{R})$, because

$$\begin{pmatrix} \langle \xi_1, \xi_1 \rangle & \langle \xi_1, \xi_2 \rangle & \cdots & \langle \xi_1, \xi_m \rangle \\ \langle \xi_2, \xi_1 \rangle & \langle \xi_2, \xi_2 \rangle & \cdots & \langle \xi_2, \xi_m \rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle \xi_m, \xi_1 \rangle & \langle \xi_m, \xi_2 \rangle & \cdots & \langle \xi_m, \xi_m \rangle \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}^T$$

and
$$\begin{pmatrix} \langle \bar{\xi}_1, \bar{\xi}_1 \rangle & \langle \bar{\xi}_1, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_1, \bar{\xi}_m \rangle \\ \langle \bar{\xi}_2, \bar{\xi}_1 \rangle & \langle \bar{\xi}_2, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_2, \bar{\xi}_m \rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle \bar{\xi}_m, \bar{\xi}_1 \rangle & \langle \bar{\xi}_m, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_m, \bar{\xi}_m \rangle \end{pmatrix}$$

$$= \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda_1^2} & & & & 0 \\ & \frac{1}{\lambda_2^2} & & & \\ & & \ddots & & \\ & & & \frac{1}{\lambda_{m-1}^2} & \\ 0 & & & & \frac{1}{\lambda_m^2} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}^T$$

where $\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}^T$ is the transposed matrix of $\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}$.

By computation we have

$$\begin{pmatrix} \langle A\xi_1, A\xi_1\rangle & \langle A\xi_1, A\xi_2\rangle & \cdots & \langle A\xi_1, A\xi_m\rangle \\ \langle A\xi_2, A\xi_1\rangle & \langle A\xi_2, A\xi_2\rangle & \cdots & \langle A\xi_2, A\xi_m\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle A\xi_m, A\xi_1\rangle & \langle A\xi_m, A\xi_2\rangle & \cdots & \langle A\xi_m, A\xi_m\rangle \end{pmatrix} = A \begin{pmatrix} \langle \xi_1, \xi_1\rangle & \langle \xi_1, \xi_2\rangle & \cdots & \langle \xi_1, \xi_m\rangle \\ \langle \xi_2, \xi_1\rangle & \langle \xi_2, \xi_2\rangle & \cdots & \langle \xi_2, \xi_m\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle \xi_m, \xi_1\rangle & \langle \xi_m, \xi_2\rangle & \cdots & \langle \xi_m, \xi_m\rangle \end{pmatrix} A^T$$

and

$$\begin{pmatrix} \langle A\bar\xi_1, A\bar\xi_1\rangle & \langle A\bar\xi_1, A\bar\xi_2\rangle & \cdots & \langle A\bar\xi_1, A\bar\xi_m\rangle \\ \langle A\bar\xi_2, A\bar\xi_1\rangle & \langle A\bar\xi_2, A\bar\xi_2\rangle & \cdots & \langle A\bar\xi_2, A\bar\xi_m\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle A\bar\xi_m, A\bar\xi_1\rangle & \langle A\bar\xi_m, A\bar\xi_2\rangle & \cdots & \langle A\bar\xi_m, A\bar\xi_m\rangle \end{pmatrix} = A \begin{pmatrix} \langle \bar\xi_1, \bar\xi_1\rangle & \langle \bar\xi_1, \bar\xi_2\rangle & \cdots & \langle \bar\xi_1, \bar\xi_m\rangle \\ \langle \bar\xi_2, \bar\xi_1\rangle & \langle \bar\xi_2, \bar\xi_2\rangle & \cdots & \langle \bar\xi_2, \bar\xi_m\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle \bar\xi_m, \bar\xi_1\rangle & \langle \bar\xi_m, \bar\xi_2\rangle & \cdots & \langle \bar\xi_m, \bar\xi_m\rangle \end{pmatrix} A^T$$

where $A^T$ is the transposed matrix of $A$. So we have the function $\frac{\Delta}{\bar\Delta}$ is invariant under the natural left action of the group $GL(m, \mathbb{R})$, then we get $f$ is defined on the Grassmann manifold $G_{n,m}(\mathbb{R})$.

□

**Lemma 2.** *The funtion $f$ is the Morse function on the Grassmann manifold $G_{n,m}(\mathbb{R})$, has and only has one critical point $O_{i_1 i_2 \cdots i_m} = \widetilde{\phi}^{-1}_{i_1 i_2 \cdots i_m}(\underbrace{0, 0, \cdots, 0}_{m(n-m)})$ on each $\widetilde{U}_{i_1 i_2 \cdots i_m}$, $1 \le i_1 < i_2 < \cdots < i_m \le n$.*

*Proof.* By the definition

$$f = \frac{\Delta}{\bar\Delta}$$

so on the local coordinate systems $\widetilde{U}_{i_1 i_2 \cdots i_m}$ we have

$$\frac{\partial f}{\partial x_{ij}} = \frac{\frac{\partial \Delta}{\partial x_{ij}} \cdot \bar\Delta - \Delta \cdot \frac{\partial \bar\Delta}{\partial x_{ij}}}{\bar\Delta \cdot \bar\Delta}. \tag{1}$$

where $i = 1, 2, \cdots, m$; $j = k_s \neq i_1, i_2, \cdots, i_m$. By computation we have

$$\frac{\partial \Delta}{\partial x_{ij}} = 2 \det \begin{pmatrix} \langle \xi_1, \xi_1\rangle & \langle \xi_1, \xi_2\rangle & \cdots & \langle \xi_1, \xi_{i-1}\rangle & x_{1j} & \langle \xi_1, \xi_{i+1}\rangle & \cdots & \langle \xi_1, \xi_m\rangle \\ \langle \xi_2, \xi_1\rangle & \langle \xi_2, \xi_2\rangle & \cdots & \langle \xi_2, \xi_{i-1}\rangle & x_{2j} & \langle \xi_2, \xi_{i+1}\rangle & \cdots & \langle \xi_2, \xi_m\rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle \xi_m, \xi_1\rangle & \langle \xi_m, \xi_2\rangle & \cdots & \langle \xi_m, \xi_{i-1}\rangle & x_{mj} & \langle \xi_m, \xi_{i+1}\rangle & \cdots & \langle \xi_m, \xi_m\rangle \end{pmatrix}$$

$$\frac{\partial \bar\Delta}{\partial x_{ij}} = 2 \det \begin{pmatrix} \langle \bar\xi_1, \bar\xi_1\rangle & \langle \bar\xi_1, \bar\xi_2\rangle & \cdots & \langle \bar\xi_1, \bar\xi_{i-1}\rangle & \frac{x_{1j}}{\lambda_j^2} & \langle \bar\xi_1, \bar\xi_{i+1}\rangle & \cdots & \langle \bar\xi_1, \bar\xi_m\rangle \\ \langle \bar\xi_2, \bar\xi_1\rangle & \langle \bar\xi_2, \bar\xi_2\rangle & \cdots & \langle \bar\xi_2, \bar\xi_{i-1}\rangle & \frac{x_{2j}}{\lambda_j^2} & \langle \bar\xi_2, \bar\xi_{i+1}\rangle & \cdots & \langle \bar\xi_2, \bar\xi_m\rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle \bar\xi_m, \bar\xi_1\rangle & \langle \bar\xi_m, \bar\xi_2\rangle & \cdots & \langle \bar\xi_m, \bar\xi_{i-1}\rangle & \frac{x_{mj}}{\lambda_j^2} & \langle \bar\xi_m, \bar\xi_{i+1}\rangle & \cdots & \langle \bar\xi_m, \bar\xi_m\rangle \end{pmatrix}$$

then

$$\frac{\partial \Delta}{\partial x_{ij}} \cdot \bar\Delta - \Delta \cdot \frac{\partial \bar\Delta}{\partial x_{ij}} =$$

$$2(x_{1j}A_{1i}+x_{2j}A_{2i}+\cdots+x_{mj}A_{mi})\cdot\bar{\Delta}-2(\frac{x_{1j}}{\lambda_j^2}\overline{A}_{1i}+\frac{x_{2j}}{\lambda_j^2}\overline{A}_{2i}+\cdots+\frac{x_{mj}}{\lambda_j^2}\overline{A}_{mi})\cdot\Delta \qquad (2)$$

where $A_{ij}$ is $(i,j)$ cofactor of

$$\begin{pmatrix} \langle\xi_1,\xi_1\rangle & \langle\xi_1,\xi_2\rangle & \cdots & \langle\xi_1,\xi_m\rangle \\ \langle\xi_2,\xi_1\rangle & \langle\xi_2,\xi_2\rangle & \cdots & \langle\xi_2,\xi_m\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle\xi_m,\xi_1\rangle & \langle\xi_m,\xi_2\rangle & \cdots & \langle\xi_m,\xi_m\rangle \end{pmatrix},$$

$\overline{A}_{ij}$ is $(i,j)$ cofactor of

$$\begin{pmatrix} \langle\overline{\xi}_1,\overline{\xi}_1\rangle & \langle\overline{\xi}_1,\overline{\xi}_2\rangle & \cdots & \langle\overline{\xi}_1,\overline{\xi}_m\rangle \\ \langle\overline{\xi}_2,\overline{\xi}_1\rangle & \langle\overline{\xi}_2,\overline{\xi}_2\rangle & \cdots & \langle\overline{\xi}_2,\overline{\xi}_m\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle\overline{\xi}_m,\overline{\xi}_1\rangle & \langle\overline{\xi}_m,\overline{\xi}_2\rangle & \cdots & \langle\overline{\xi}_m,\overline{\xi}_m\rangle \end{pmatrix}$$

so we have $O_{i_1i_2\cdots i_m}=\widetilde{\phi}^{-1}_{i_1i_2\cdots i_m}(\underbrace{0,0,\cdots,0}_{m(n-m)})$ is a critical point of $f$. In the following we can proof that $O_{i_1i_2\cdots i_m}$ is the only one critical point on each $\widetilde{U}_{i_1i_2\cdots i_m}$.

If there exists another critical point $p\neq O_{i_1i_2\cdots i_m}$, then for some $i,j$ we have $x_{ij}\neq 0$, and according to the definition of critical point $\frac{\partial f}{\partial x_{ij}}(p)=0$, then $\frac{\partial\Delta}{\partial x_{ij}}\cdot\bar{\Delta}-\Delta\cdot\frac{\partial\bar{\Delta}}{\partial x_{ij}}=0$, so

$$(x_{1j}A_{1i}+x_{2j}A_{2i}+\cdots+x_{mj}A_{mi})\cdot\bar{\Delta}-(\frac{x_{1j}}{\lambda_j^2}\overline{A}_{1i}+\frac{x_{2j}}{\lambda_j^2}\overline{A}_{2i}+\cdots+\frac{x_{mj}}{\lambda_j^2}\overline{A}_{mi})\cdot\Delta=0$$

If we get $i$ from 1 to $m$, by them we can get a linear system of equations, because $x_{ij}\neq 0$ so we get

$$\det\begin{pmatrix} A_{11}\cdot\bar{\Delta}-\frac{1}{\lambda_j^2}\overline{A}_{11}\cdot\Delta & A_{21}\cdot\bar{\Delta}-\frac{1}{\lambda_j^2}\overline{A}_{21}\cdot\Delta & \cdots & A_{m1}\cdot\bar{\Delta}-\frac{1}{\lambda_j^2}\overline{A}_{m1}\cdot\Delta \\ A_{12}\cdot\bar{\Delta}-\frac{1}{\lambda_j^2}\overline{A}_{12}\cdot\Delta & A_{22}\cdot\bar{\Delta}-\frac{1}{\lambda_j^2}\overline{A}_{22}\cdot\Delta & \cdots & A_{m2}\cdot\bar{\Delta}-\frac{1}{\lambda_j^2}\overline{A}_{m2}\cdot\Delta \\ \cdots & \cdots & \cdots & \cdots \\ A_{1m}\cdot\bar{\Delta}-\frac{1}{\lambda_j^2}\overline{A}_{1m}\cdot\Delta & A_{2m}\cdot\bar{\Delta}-\frac{1}{\lambda_j^2}\overline{A}_{2m}\cdot\Delta & \cdots & A_{mm}\cdot\bar{\Delta}-\frac{1}{\lambda_j^2}\overline{A}_{mm}\cdot\Delta \end{pmatrix}=0$$

we denote the above matrix by $B$ , because

$$\begin{pmatrix} \langle\xi_1,\xi_1\rangle & \langle\xi_1,\xi_2\rangle & \cdots & \langle\xi_1,\xi_m\rangle \\ \langle\xi_2,\xi_1\rangle & \langle\xi_2,\xi_2\rangle & \cdots & \langle\xi_2,\xi_m\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle\xi_m,\xi_1\rangle & \langle\xi_m,\xi_2\rangle & \cdots & \langle\xi_m,\xi_m\rangle \end{pmatrix}B$$

$$=\begin{pmatrix} \langle\overline{\xi}_1,\overline{\xi}_1\rangle-\frac{\langle\xi_1,\xi_1\rangle}{\lambda_j^2} & \langle\overline{\xi}_1,\overline{\xi}_2\rangle-\frac{\langle\xi_1,\xi_2\rangle}{\lambda_j^2} & \cdots & \langle\overline{\xi}_1,\overline{\xi}_m\rangle-\frac{\langle\xi_1,\xi_m\rangle}{\lambda_j^2} \\ \langle\overline{\xi}_2,\overline{\xi}_1\rangle-\frac{\langle\xi_2,\xi_1\rangle}{\lambda_j^2} & \langle\overline{\xi}_2,\overline{\xi}_2\rangle-\frac{\langle\xi_2,\xi_2\rangle}{\lambda_j^2} & \cdots & \langle\overline{\xi}_2,\overline{\xi}_m\rangle-\frac{\langle\xi_2,\xi_m\rangle}{\lambda_j^2} \\ \cdots & \cdots & \cdots & \cdots \\ \langle\overline{\xi}_m,\overline{\xi}_1\rangle-\frac{\langle\xi_m,\xi_1\rangle}{\lambda_j^2} & \langle\overline{\xi}_m,\overline{\xi}_2\rangle-\frac{\langle\xi_m,\xi_2\rangle}{\lambda_j^2} & \cdots & \langle\overline{\xi}_m,\overline{\xi}_m\rangle-\frac{\langle\xi_m,\xi_m\rangle}{\lambda_j^2} \end{pmatrix}C$$

where

$$C=\begin{pmatrix} \overline{A}_{11}\cdot\Delta & \overline{A}_{21}\cdot\Delta & \cdots & \overline{A}_{m1}\cdot\Delta \\ \overline{A}_{12}\cdot\Delta & \overline{A}_{22}\cdot\Delta & \cdots & \overline{A}_{m2}\cdot\Delta \\ \cdots & \cdots & \cdots & \cdots \\ \overline{A}_{1m}\cdot\Delta & \overline{A}_{2m}\cdot\Delta & \cdots & \overline{A}_{mm}\cdot\Delta \end{pmatrix}=\Delta\cdot\begin{pmatrix} \overline{A}_{11} & \overline{A}_{21} & \cdots & \overline{A}_{m1} \\ \overline{A}_{12} & \overline{A}_{22} & \cdots & \overline{A}_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ \overline{A}_{1m} & \overline{A}_{2m} & \cdots & \overline{A}_{mm} \end{pmatrix}.$$

by computation we have

$$\begin{pmatrix} \langle\bar\xi_1,\bar\xi_1\rangle-\frac{\langle\xi_1,\xi_1\rangle}{\lambda_j^2} & \langle\bar\xi_1,\bar\xi_2\rangle-\frac{\langle\xi_1,\xi_2\rangle}{\lambda_j^2} & \cdots & \langle\bar\xi_1,\bar\xi_m\rangle-\frac{\langle\xi_1,\xi_m\rangle}{\lambda_j^2} \\ \langle\bar\xi_2,\bar\xi_1\rangle-\frac{\langle\xi_2,\xi_1\rangle}{\lambda_j^2} & \langle\bar\xi_2,\bar\xi_2\rangle-\frac{\langle\xi_2,\xi_2\rangle}{\lambda_j^2} & \cdots & \langle\bar\xi_2,\bar\xi_m\rangle-\frac{\langle\xi_2,\xi_m\rangle}{\lambda_j^2} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \langle\bar\xi_m,\bar\xi_1\rangle-\frac{\langle\xi_m,\xi_1\rangle}{\lambda_j^2} & \langle\bar\xi_m,\bar\xi_2\rangle-\frac{\langle\xi_m,\xi_2\rangle}{\lambda_j^2} & \cdots & \langle\bar\xi_m,\bar\xi_m\rangle-\frac{\langle\xi_m,\xi_m\rangle}{\lambda_j^2} \end{pmatrix}$$

$$= X \begin{pmatrix} \frac{1}{\lambda_1^2}-\frac{1}{\lambda_j^2} & & & & & & & \\ & \frac{1}{\lambda_2^2}-\frac{1}{\lambda_j^2} & & & & \mathbf{0} & & \\ & & \ddots & & & & & \\ & & & \frac{1}{\lambda_{j-1}^2}-\frac{1}{\lambda_j^2} & & & & \\ & & & & \frac{1}{\lambda_{j+1}^2}-\frac{1}{\lambda_j^2} & & & \\ & & & & & \ddots & & \\ & \mathbf{0} & & & & & \frac{1}{\lambda_{n-1}^2}-\frac{1}{\lambda_j^2} & \\ & & & & & & & \frac{1}{\lambda_n^2}-\frac{1}{\lambda_j^2} \end{pmatrix} X^T$$

where

$$X=\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1(j-1)} & x_{1(j+1)} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2(j-1)} & x_{2(j+1)} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{m1} & x_{m2} & \cdots & x_{m(j-1)} & x_{m(j+1)} & \cdots & x_{mn} \end{pmatrix}$$

$X^T$ is the transposed matrix of $X$. Because $\det B = 0$, so we have

$$\det\begin{pmatrix} \langle\bar\xi_1,\bar\xi_1\rangle-\frac{\langle\xi_1,\xi_1\rangle}{\lambda_j^2} & \langle\bar\xi_1,\bar\xi_2\rangle-\frac{\langle\xi_1,\xi_2\rangle}{\lambda_j^2} & \cdots & \langle\bar\xi_1,\bar\xi_m\rangle-\frac{\langle\xi_1,\xi_m\rangle}{\lambda_j^2} \\ \langle\bar\xi_2,\bar\xi_1\rangle-\frac{\langle\xi_2,\xi_1\rangle}{\lambda_j^2} & \langle\bar\xi_2,\bar\xi_2\rangle-\frac{\langle\xi_2,\xi_2\rangle}{\lambda_j^2} & \cdots & \langle\bar\xi_2,\bar\xi_m\rangle-\frac{\langle\xi_2,\xi_m\rangle}{\lambda_j^2} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \langle\bar\xi_m,\bar\xi_1\rangle-\frac{\langle\xi_m,\xi_1\rangle}{\lambda_j^2} & \langle\bar\xi_m,\bar\xi_2\rangle-\frac{\langle\xi_m,\xi_2\rangle}{\lambda_j^2} & \cdots & \langle\bar\xi_m,\bar\xi_m\rangle-\frac{\langle\xi_m,\xi_m\rangle}{\lambda_j^2} \end{pmatrix} = 0$$

because it is independent of the value of $x_{kl}$ $(k = 1, 2, \cdots, m;\ l = 1, 2, \cdots, j-1, j+1, \cdots, n.)$, so we must have $\frac{1}{\lambda_l^2}-\frac{1}{\lambda_j^2} = 0$ for some $l$, then we get $\lambda_l = \lambda_j$, which contradicts to $\lambda_l \neq \lambda_j, (l \neq j)$. So we get $x_{ij} = 0$, there non-exists another critical point $p \neq O_{i_1 i_2 \cdots i_m}$ on $\widetilde{U}_{i_1 i_2 \cdots i_m}$. □

**Lemma 3.** *The critical point $O_{i_1 i_2 \cdots i_m}$ on each $\widetilde{U}_{i_1 i_2 \cdots i_m}$ $(1 \le i_1 < i_2 < \cdots < i_m \le n)$ is non-degenerate, and* $\operatorname{ind}(O_{i_1 i_2 \cdots i_m}) = i_1 + i_2 + \cdots + i_m - \frac{1}{2}m(m+1)$.

*Proof.* On $\widetilde{U}_{i_1 i_2 \cdots i_m}$ we have

$$\frac{\partial^2 f}{\partial x_{kl}\partial x_{ij}} = \frac{\partial}{\partial x_{kl}}\left(\frac{\frac{\partial \Delta}{\partial x_{ij}}\cdot\bar\Delta - \Delta\cdot\frac{\partial\bar\Delta}{\partial x_{ij}}}{\bar\Delta\cdot\bar\Delta}\right)$$

$$= \frac{1}{\bar\Delta^4}\left(\frac{\partial}{\partial x_{kl}}\left(\frac{\partial\Delta}{\partial x_{ij}}\cdot\bar\Delta - \Delta\cdot\frac{\partial\bar\Delta}{\partial x_{ij}}\right)\cdot\bar\Delta^2 - \left(\frac{\partial\Delta}{\partial x_{ij}}\cdot\bar\Delta - \Delta\cdot\frac{\partial\bar\Delta}{\partial x_{ij}}\right)\frac{\partial}{\partial x_{kl}}\bar\Delta^2\right)$$

$$= \frac{1}{\bar{\Delta}^4}\left(\frac{\partial^2\Delta}{\partial x_{kl}\partial x_{ij}}\cdot\bar{\Delta}^3 - \frac{\partial\Delta}{\partial x_{ij}}\frac{\partial\bar{\Delta}}{\partial x_{kl}}\cdot\bar{\Delta}^2 - \frac{\partial\Delta}{\partial x_{kl}}\frac{\partial\bar{\Delta}}{\partial x_{ij}}\cdot\bar{\Delta}^2 - \Delta\frac{\partial^2\bar{\Delta}}{\partial x_{kl}\partial x_{ij}}\bar{\Delta}^2 - 2\Delta\frac{\partial\bar{\Delta}}{\partial x_{ij}}\frac{\partial\bar{\Delta}}{\partial x_{kl}}\cdot\bar{\Delta}\right)$$

When $k < i$, $l < j$, then

$$\frac{\partial^2\Delta}{\partial x_{kl}\partial x_{ij}} =$$

$$2\det\begin{pmatrix} \langle\xi_1,\xi_1\rangle & \cdots & \langle\xi_1,\xi_k\rangle & \cdots & \langle\xi_1,\xi_{i-1}\rangle & x_{1j} & \langle\xi_1,\xi_{i+1}\rangle & \cdots & \langle\xi_1,\xi_m\rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle\xi_{k-1},\xi_1\rangle & \cdots & \langle\xi_{k-1},\xi_k\rangle & \cdots & \langle\xi_{k-1},\xi_{i-1}\rangle & x_{(k-1)j} & \langle\xi_{k-1},\xi_{i+1}\rangle & \cdots & \langle\xi_{k-1},\xi_m\rangle \\ x_{1l} & \cdots & x_{kl} & \cdots & x_{(i-1)l} & 0 & x_{(i+1)l} & \cdots & x_{ml} \\ \langle\xi_{k+1},\xi_1\rangle & \cdots & \langle\xi_{k+1},\xi_k\rangle & \cdots & \langle\xi_{k+1},\xi_{i-1}\rangle & x_{(k+1)j} & \langle\xi_{k+1},\xi_{i+1}\rangle & \cdots & \langle\xi_{k+1},\xi_m\rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle\xi_m,\xi_1\rangle & \cdots & \langle\xi_m,\xi_k\rangle & \cdots & \langle\xi_m,\xi_{i-1}\rangle & x_{mj} & \langle\xi_m,\xi_{i+1}\rangle & \cdots & \langle\xi_m,\xi_m\rangle \end{pmatrix}$$

$$+2\det\begin{pmatrix} \langle\xi_1,\xi_1\rangle & \cdots & \langle\xi_1,\xi_{k-1}\rangle & x_{1l} & \cdots & \langle\xi_1,\xi_{i-1}\rangle & x_{1j} & \cdots & \langle\xi_1,\xi_m\rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle\xi_{k-1},\xi_1\rangle & \cdots & \langle\xi_{k-1},\xi_{k-1}\rangle & x_{(k-1)l} & \cdots & \langle\xi_{k-1},\xi_{i-1}\rangle & x_{(k-1)j} & \cdots & \langle\xi_{k-1},\xi_m\rangle \\ \langle\xi_k,\xi_1\rangle & \cdots & \langle\xi_k,\xi_1\rangle & x_{kl} & \cdots & \langle\xi_k,\xi_{i-1}\rangle & x_{kj} & \cdots & \langle\xi_k,\xi_m\rangle \\ \langle\xi_{k+1},\xi_1\rangle & \cdots & \langle\xi_{k+1},\xi_{k-1}\rangle & x_{(k+1)l} & \cdots & \langle\xi_{k+1},\xi_{i-1}\rangle & x_{(k+1)j} & \cdots & \langle\xi_{k+1},\xi_m\rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle\xi_m,\xi_1\rangle & \cdots & \langle\xi_m,\xi_{k_1}\rangle & x_{ml} & \cdots & \langle\xi_m,\xi_{i-1}\rangle & x_{mj} & \cdots & \langle\xi_m,\xi_m\rangle \end{pmatrix}.$$

so we have $\frac{\partial^2\Delta}{\partial x_{kl}\partial x_{ij}}(O_{i_1i_2\cdots i_m}) = 0$. When $k > i$,$l < j$ and $k \neq i$,$l > j$ the result is the same. By the same way we can get $\frac{\partial^2\bar{\Delta}}{\partial x_{kl}\partial x_{ij}}(O_{i_1i_2\cdots i_m}) = 0$.

When $k \neq i$, $l = j$, then

$$\frac{\partial^2\Delta}{\partial x_{kj}\partial x_{ij}} =$$

$$2\det\begin{pmatrix} \langle\xi_1,\xi_1\rangle & \cdots & \langle\xi_1,\xi_k\rangle & \cdots & \langle\xi_1,\xi_{i-1}\rangle & x_{1j} & \langle\xi_1,\xi_{i+1}\rangle & \cdots & \langle\xi_1,\xi_m\rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle\xi_{k-1},\xi_1\rangle & \cdots & \langle\xi_{k-1},\xi_k\rangle & \cdots & \langle\xi_{k-1},\xi_{i-1}\rangle & x_{(k-1)j} & \langle\xi_{k-1},\xi_{i+1}\rangle & \cdots & \langle\xi_{k-1},\xi_m\rangle \\ x_{1j} & \cdots & x_{kj} & \cdots & x_{(i-1)j} & 1 & x_{(i+1)j} & \cdots & x_{mj} \\ \langle\xi_{k+1},\xi_1\rangle & \cdots & \langle\xi_{k+1},\xi_k\rangle & \cdots & \langle\xi_{k+1},\xi_{i-1}\rangle & x_{(k+1)j} & \langle\xi_{k+1},\xi_{i+1}\rangle & \cdots & \langle\xi_{k+1},\xi_m\rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle\xi_m,\xi_1\rangle & \cdots & \langle\xi_m,\xi_k\rangle & \cdots & \langle\xi_m,\xi_{i-1}\rangle & x_{mj} & \langle\xi_m,\xi_{i+1}\rangle & \cdots & \langle\xi_m,\xi_m\rangle \end{pmatrix}$$

$$= 2A_{ki}$$

because $A_{ki}(O_{i_1i_2\cdots i_m}) = 0$, so we get $\frac{\partial^2\Delta}{\partial x_{kj}\partial x_{ij}}(O_{i_1i_2\cdots i_m}) = 0$, by the same way we can get $\frac{\partial^2\bar{\Delta}}{\partial x_{kj}\partial x_{ij}}(O_{i_1i_2\cdots i_m}) = 0$.

When $k = i$, $l = j$, then

$$\frac{\partial^2\Delta}{\partial x_{ij}\partial x_{ij}} = 2\det\begin{pmatrix} \langle\xi_1,\xi_1\rangle & \cdots & \langle\xi_1,\xi_{i-1}\rangle & x_{1j} & \langle\xi_1,\xi_{i+1}\rangle & \cdots & \langle\xi_1,\xi_m\rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle\xi_{i-1},\xi_1\rangle & \cdots & \langle\xi_{i-1},\xi_{i-1}\rangle & x_{(i-1)j} & \langle\xi_{i-1},\xi_{i+1}\rangle & \cdots & \langle\xi_{i-1},\xi_m\rangle \\ x_{1j} & \cdots & x_{(i-1)j} & 1 & x_{(i+1)j} & \cdots & x_{mj} \\ \langle\xi_{i+1},\xi_1\rangle & \cdots & \langle\xi_{i+1},\xi_{i-1}\rangle & x_{(i+1)j} & \langle\xi_{i+1},\xi_{i+1}\rangle & \cdots & \langle\xi_{i+1},\xi_m\rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle\xi_m,\xi_1\rangle & \cdots & \langle\xi_m,\xi_{i-1}\rangle & x_{mj} & \langle\xi_m,\xi_{i+1}\rangle & \cdots & \langle\xi_m,\xi_m\rangle \end{pmatrix}.$$

$$\frac{\partial^2\bar{\Delta}}{\partial x_{ij}\partial x_{ij}} = 2\det\begin{pmatrix} \langle\bar{\xi}_1,\bar{\xi}_1\rangle & \cdots & \langle\bar{\xi}_1,\bar{\xi}_{i-1}\rangle & \frac{x_{1j}}{\lambda_j^2} & \langle\bar{\xi}_1,\bar{\xi}_{i+1}\rangle & \cdots & \langle\bar{\xi}_1,\bar{\xi}_m\rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle\bar{\xi}_{i-1},\bar{\xi}_1\rangle & \cdots & \langle\bar{\xi}_{i-1},\bar{\xi}_{i-1}\rangle & \frac{x_{(i-1)j}}{\lambda_j^2} & \langle\bar{\xi}_{i-1},\bar{\xi}_{i+1}\rangle & \cdots & \langle\bar{\xi}_{i-1},\bar{\xi}_m\rangle \\ \frac{x_{1j}}{\lambda_j^2} & \cdots & \frac{x_{(i-1)j}}{\lambda_j^2} & \frac{1}{\lambda_j^2} & \frac{x_{(i+1)j}}{\lambda_j^2} & \cdots & \frac{x_{mj}}{\lambda_j^2} \\ \langle\bar{\xi}_{i+1},\bar{\xi}_1\rangle & \cdots & \langle\bar{\xi}_{i+1},\bar{\xi}_{i-1}\rangle & \frac{x_{(i+1)j}}{\lambda_j^2} & \langle\bar{\xi}_{i+1},\bar{\xi}_{i+1}\rangle & \cdots & \langle\bar{\xi}_{i+1},\bar{\xi}_m\rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle\bar{\xi}_m,\bar{\xi}_1\rangle & \cdots & \langle\bar{\xi}_m,\bar{\xi}_{i-1}\rangle & \frac{x_{mj}}{\lambda_j^2} & \langle\bar{\xi}_m,\bar{\xi}_{i+1}\rangle & \cdots & \langle\bar{\xi}_m,\bar{\xi}_m\rangle \end{pmatrix}.$$

so

$$\frac{\partial^2\Delta}{\partial x_{ij}\partial x_{ij}}(O_{i_1 i_2\cdots i_m}) = 2$$

$$\frac{\partial^2\bar{\Delta}}{\partial x_{ij}\partial x_{ij}}(O_{i_1 i_2\cdots i_m}) = 2\frac{1}{\lambda_{i_1}^2\lambda_{i_2}^2\cdots\lambda_{i_{k-1}}^2\lambda_j^2\lambda_{i_{k+1}}^2\cdots\lambda_{i_m}^2}$$

then by computation we have

$$\frac{\partial^2 f}{\partial x_{ij}\partial x_{ij}}(O_{i_1 i_2\cdots i_m}) = 2\lambda_{i_1}^2\lambda_{i_2}^2\cdots\lambda_{i_m}^2\left(1-\frac{\lambda_{i_k}^2}{\lambda_j^2}\right),$$

here $j = k_s$.

So at the critical point $O_{i_1 i_2\cdots i_m}$ of $f$ on each $\widetilde{U}_{i_1 i_2\cdots i_m}$, we have the Hessian matrix $H_{O_{i_1 i_2\cdots i_m}}(f)$

$$H_{O_{i_1 i_2\cdots i_m}}(f) =$$

$$2\lambda_{i_1}^2\lambda_{i_2}^2\cdots\lambda_{i_m}^2\,\mathrm{diag}\left(1-\frac{\lambda_{i_1}^2}{\lambda_{k_1}^2},\cdots,1-\frac{\lambda_{i_1}^2}{\lambda_{k_{n-m}}^2},\cdots,1-\frac{\lambda_{i_m}^2}{\lambda_{k_1}^2},\cdots,1-\frac{\lambda_{i_m}^2}{\lambda_{k_{n-m}}^2}\right)$$

then the critical point $O_{i_1 i_2\cdots i_m}$ is non-degenerate. By the definition of Morse index of $f$, we have $\mathrm{ind}(O_{i_1 i_2\cdots i_m}) = (i_1-1)+(i_2-2)+\cdots+(i_m-m) = i_1+i_2+\cdots+i_m-\frac{1}{2}m(m+1)$. □

# 3 Riemannian metric on the Grassmann manifold $G_{n,m}(\mathbb{R})$

Lu Qi-keng introduce a Riemannian metric $g$ on the Grassmann manifold $G_{n,m}(\mathbb{R})$ in [8]. It has the following form

$$g = \mathrm{tr}[(I+ZZ^T)^{-1}dZ(I+Z^TZ)^{-1}dZ^T]$$

where $I$ is the identity matrix,

$$Z = \begin{pmatrix} x_{1k_1} & x_{1k_2} & \cdots & x_{1k_{n-m}} \\ x_{2k_1} & x_{2k_2} & \cdots & x_{2k_{n-m}} \\ \cdots & \cdots & \cdots & \cdots \\ x_{mk_1} & x_{mk_2} & \cdots & x_{mk_{n-m}} \end{pmatrix},$$

$Z^T$ is the transposed matrix of $Z$.

we have

$$I+ZZ^T=\begin{pmatrix}\langle\xi_1,\xi_1\rangle & \langle\xi_1,\xi_2\rangle & \cdots & \langle\xi_1,\xi_m\rangle\\ \langle\xi_2,\xi_1\rangle & \langle\xi_2,\xi_2\rangle & \cdots & \langle\xi_2,\xi_m\rangle\\ \cdots & \cdots & \cdots & \cdots\\ \langle\xi_m,\xi_1\rangle & \langle\xi_m,\xi_2\rangle & \cdots & \langle\xi_m,\xi_m\rangle\end{pmatrix}$$

$$(I+ZZ^T)^{-1}=\frac{1}{\Delta}\begin{pmatrix}A_{11} & A_{21} & \cdots & A_{m1}\\ A_{12} & A_{22} & \cdots & A_{m2}\\ \cdots & \cdots & \cdots & \cdots\\ A_{1m} & A_{2m} & \cdots & A_{mm}\end{pmatrix}$$

Set

$$\begin{cases}\eta_{k_1}=(x_{1k_1},x_{2k_1},\cdots,x_{mk_1},1,0,\cdots,0)\\ \eta_{k_2}=(x_{1k_2},x_{2k_2},\cdots,x_{mk_2},0,1,\cdots,0)\\ \cdots\\ \eta_{k_{n-m}}=(x_{1k_{n-m}},x_{2k_{n-m}},\cdots,x_{mk_{n-m}},0,0,\cdots,1)\end{cases}$$

so

$$I+Z^TZ=\begin{pmatrix}\langle\eta_{k_1},\eta_{k_1}\rangle & \langle\eta_{k_1},\eta_{k_2}\rangle & \cdots & \langle\eta_{k_1},\eta_{k_{n-m}}\rangle\\ \langle\eta_{k_2},\eta_{k_1}\rangle & \langle\eta_{k_2},\eta_{k_2}\rangle & \cdots & \langle\eta_{k_2},\eta_{k_{n-m}}\rangle\\ \cdots & \cdots & \cdots & \cdots\\ \langle\eta_{k_{n-m}},\eta_{k_1}\rangle & \langle\eta_{k_{n-m}},\eta_{k_2}\rangle & \cdots & \langle\eta_{k_{n-m}},\eta_{k_{n-m}}\rangle\end{pmatrix}$$

$$R\doteq(I+Z^TZ)^{-1}=\frac{1}{\det(I+Z^TZ)}\begin{pmatrix}D_{11} & D_{21} & \cdots & D_{(n-m)1}\\ D_{12} & D_{22} & \cdots & D_{(n-m)2}\\ \cdots & \cdots & \cdots & \cdots\\ D_{1(n-m)} & D_{2(n-m)} & \cdots & D_{(n-m)(n-m)}\end{pmatrix}$$

where $D_{ij}$ is $(i,j)$ cofactor of $I+Z^TZ$.

**Lemma 4.** *On $\{(\widetilde{U}_{i_1i_2\cdots i_m},\widetilde{\phi}_{i_1i_2\cdots i_m})|1\le i_1<i_2<\cdots<i_m\le n\}$, the local matrix expression of the Riemannian metric $g$ of $G_{n,m}(\mathbb{R})$ is given by*

$$G=\frac{1}{\Delta}\begin{pmatrix}A_{11}(I+Z^TZ)^{-1} & A_{21}(I+Z^TZ)^{-1} & \cdots & A_{m1}(I+Z^TZ)^{-1}\\ A_{12}(I+Z^TZ)^{-1} & A_{22}(I+Z^TZ)^{-1} & \cdots & A_{m2}(I+Z^TZ)^{-1}\\ \cdots & \cdots & \cdots & \cdots\\ A_{1m}(I+Z^TZ)^{-1} & A_{2m}(I+Z^TZ)^{-1} & \cdots & A_{mm}(I+Z^TZ)^{-1}\end{pmatrix}$$

*Proof.* By definition

$$g=\operatorname{tr}[(I+ZZ^T)^{-1}dZ(I+Z^TZ)^{-1}dZ^T]$$

$$=\operatorname{tr}\left[\frac{1}{\Delta}\begin{pmatrix}A_{11} & A_{21} & \cdots & A_{m1}\\ A_{12} & A_{22} & \cdots & A_{m2}\\ \cdots & \cdots & \cdots & \cdots\\ A_{1m} & A_{2m} & \cdots & A_{mm}\end{pmatrix}\begin{pmatrix}dX_1\\ dX_2\\ \vdots\\ dX_m\end{pmatrix}(I+Z^TZ)^{-1}\begin{pmatrix}dX_1^T & dX_2^T & \cdots & dX_m^T\end{pmatrix}\right]$$

$$=\frac{1}{\Delta}\begin{pmatrix}dX_1 & dX_2 & \cdots & dX_m\end{pmatrix}\begin{pmatrix}A_{11}R & A_{21}R & \cdots & A_{m1}R\\ A_{12}R & A_{22}R & \cdots & A_{m2}R\\ \cdots & \cdots & \cdots & \cdots\\ A_{1m}R & A_{2m}R & \cdots & A_{mm}R\end{pmatrix}\begin{pmatrix}dX_1^T\\ dX_2^T\\ \vdots\\ dX_m^T\end{pmatrix}$$

where $X_\alpha = row_\alpha Z,\ \alpha = 1, 2, \cdots, m$. So we get the result

$$G = \frac{1}{\Delta}\begin{pmatrix} A_{11}(I+Z^TZ)^{-1} & A_{21}(I+Z^TZ)^{-1} & \cdots & A_{m1}(I+Z^TZ)^{-1} \\ A_{12}(I+Z^TZ)^{-1} & A_{22}(I+Z^TZ)^{-1} & \cdots & A_{m2}(I+Z^TZ)^{-1} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1m}(I+Z^TZ)^{-1} & A_{2m}(I+Z^TZ)^{-1} & \cdots & A_{mm}(I+Z^TZ)^{-1} \end{pmatrix}$$

we can see that the matrix $G$ is rank of $m(n-m)$. □

By computation we have

$$G^{-1} = \begin{pmatrix} \langle\xi_1,\xi_1\rangle(I+Z^TZ) & \langle\xi_1,\xi_2\rangle(I+Z^TZ) & \cdots & \langle\xi_1,\xi_m\rangle(I+Z^TZ) \\ \langle\xi_2,\xi_1\rangle(I+Z^TZ) & \langle\xi_2,\xi_2\rangle(I+Z^TZ) & \cdots & \langle\xi_2,\xi_m\rangle(I+Z^TZ) \\ \cdots & \cdots & \cdots & \cdots \\ \langle\xi_m,\xi_1\rangle(I+Z^TZ) & \langle\xi_m,\xi_2\rangle(I+Z^TZ) & \cdots & \langle\xi_m,\xi_m\rangle(I+Z^TZ) \end{pmatrix} \tag{3}$$

By simple computation, it is obviously that $G$ is the unit matrix of $m(n-m) \times m(n-m)$ at critical point $O_{i_1 i_2 \cdots i_m}$.

# 4 The study of the dynamical system $\dot{x} = -\nabla f$

We can use the above-mentioned Riemannian metric to define the negative gradient vector field $-\nabla f$ on $G_{n,m}(\mathbb{R})$ for the Morse function $f \doteq \frac{\Delta}{\bar{\Delta}} : G_{n,m}(\mathbb{R}) \to \mathbb{R}$, which has the following local expression on $\{(\widetilde{U}_{i_1 i_2 \cdots i_m}, \widetilde{\phi}_{i_1 i_2 \cdots i_m}) | 1 \le i_1 < i_2 < \cdots < i_m \le n\}$

$$-\nabla f = -((\nabla f)_{1k_1}, (\nabla f)_{1k_2}, \cdots, (\nabla f)_{1k_{n-m}}, \cdots, (\nabla f)_{mk_1}, (\nabla f)_{mk_2}, \cdots, (\nabla f)_{mk_{n-m}}).$$

we can get the computation formula of $-\nabla f$ by $g(\nabla f, V) = Vf$,

$$-\nabla f = -g^{ij}\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i}$$

where $g^{ij}$ is the inverse of the matrix expression of the Riemannian metric $g$.

By computation we have

$$\begin{aligned} (\nabla f)_{ik_s} &= \langle\xi_i,\xi_1\rangle\langle\eta_{k_s},\eta_{k_1}\rangle\frac{\partial f}{\partial x_{1k_1}} + \cdots + \langle\xi_i,\xi_1\rangle\langle\eta_{k_s},\eta_{k_{n-m}}\rangle\frac{\partial f}{\partial x_{1k_{n-m}}} \\ &+ \langle\xi_i,\xi_2\rangle\langle\eta_{k_s},\eta_{k_1}\rangle\frac{\partial f}{\partial x_{2k_1}} + \cdots + \langle\xi_i,\xi_2\rangle\langle\eta_{k_s},\eta_{k_{n-m}}\rangle\frac{\partial f}{\partial x_{2k_{n-m}}} \\ &+ \cdots \\ &+ \langle\xi_i,\xi_m\rangle\langle\eta_{k_s},\eta_{k_1}\rangle\frac{\partial f}{\partial x_{mk_1}} + \cdots + \langle\xi_i,\xi_m\rangle\langle\eta_{k_s},\eta_{k_{n-m}}\rangle\frac{\partial f}{\partial x_{mk_{n-m}}}. \end{aligned}$$

Because (1) and (2) we have

$$\frac{\partial f}{\partial x_{ij}} = \frac{1}{\bar{\Delta}^2}\left(2(x_{1j}A_{1i} + x_{2j}A_{2i} + \cdots + x_{mj}A_{mi}) \cdot \bar{\Delta} - 2(\frac{x_{1j}}{\lambda_j^2}\overline{A}_{1i} + \frac{x_{2j}}{\lambda_j^2}\overline{A}_{2i} + \cdots + \frac{x_{mj}}{\lambda_j^2}\overline{A}_{mi}) \cdot \Delta\right)$$

so we get

$$
\begin{aligned}
(\nabla f)_{ik_s} &= \frac{2}{\bar{\Delta}^2}\sum_{t=1}^{n-m}[\langle\xi_i,\xi_1\rangle\langle\eta_{k_s},\eta_{k_t}\rangle(x_{1k_t}A_{11}+x_{2k_t}A_{21}+\cdots+x_{mk_t}A_{m1})\cdot\bar{\Delta}\\
&+\langle\xi_i,\xi_2\rangle\langle\eta_{k_s},\eta_{k_t}\rangle(x_{1k_t}A_{12}+x_{2k_t}A_{22}+\cdots+x_{mk_t}A_{m2})\cdot\bar{\Delta}+\cdots\\
&+\langle\xi_i,\xi_m\rangle\langle\eta_{k_s},\eta_{k_t}\rangle(x_{1k_t}A_{1m}+x_{2k_t}A_{2m}+\cdots+x_{mk_t}A_{mm})\cdot\bar{\Delta}]\\
&-\frac{2}{\bar{\Delta}^2}\sum_{t=1}^{n-m}[\langle\xi_i,\xi_1\rangle\langle\eta_{k_s},\eta_{k_t}\rangle(\frac{x_{1k_t}}{\lambda_{k_t}^2}\overline{A}_{11}+\frac{x_{2k_t}}{\lambda_{k_t}^2}\overline{A}_{21}+\cdots+\frac{x_{mk_t}}{\lambda_{k_t}^2}\overline{A}_{m1})\cdot\Delta\\
&+\langle\xi_i,\xi_2\rangle\langle\eta_{k_s},\eta_{k_t}\rangle(\frac{x_{1k_t}}{\lambda_{k_t}^2}\overline{A}_{12}+\frac{x_{2k_t}}{\lambda_{k_t}^2}\overline{A}_{22}+\cdots+\frac{x_{mk_t}}{\lambda_{k_t}^2}\overline{A}_{m2})\cdot\Delta+\cdots\\
&+\langle\xi_i,\xi_m\rangle\langle\eta_{k_s},\eta_{k_t}\rangle(\frac{x_{1k_t}}{\lambda_{k_t}^2}\overline{A}_{1m}+\frac{x_{2k_t}}{\lambda_{k_t}^2}\overline{A}_{2m}+\cdots+\frac{x_{mk_t}}{\lambda_{k_t}^2}\overline{A}_{mm})\cdot\Delta]\\
&=\frac{2}{\bar{\Delta}^2}[\langle\xi_i,\xi_1\rangle\cdot\Delta\cdot\bar{\Delta}\cdot x_{1k_s}+\langle\xi_i,\xi_2\rangle\cdot\Delta\cdot\bar{\Delta}\cdot x_{2k_s}+\cdots+\langle\xi_i,\xi_m\rangle\cdot\Delta\cdot\bar{\Delta}\cdot x_{mk_s}]\\
&-\frac{2}{\bar{\Delta}^2}[\langle\xi_i,\xi_1\rangle\cdot\Delta\cdot\bar{\Delta}\cdot x_{1k_s}+\langle\xi_i,\xi_1\rangle\cdot\Delta\left(\overline{A}_{11}(\frac{1}{\lambda_{k_s}^2}-\frac{1}{\lambda_{i_1}^2})x_{1k_s}+\cdots+\overline{A}_{m1}(\frac{1}{\lambda_{k_s}^2}-\frac{1}{\lambda_{i_m}^2})x_{mk_s}\right)\\
&+\langle\xi_i,\xi_2\rangle\cdot\Delta\cdot\bar{\Delta}\cdot x_{2k_s}+\langle\xi_i,\xi_2\rangle\cdot\Delta\left(\overline{A}_{12}(\frac{1}{\lambda_{k_s}^2}-\frac{1}{\lambda_{i_1}^2})x_{1k_s}+\cdots+\overline{A}_{m2}(\frac{1}{\lambda_{k_s}^2}-\frac{1}{\lambda_{i_m}^2})x_{mk_s}\right)+\cdots\\
&+\langle\xi_i,\xi_m\rangle\cdot\Delta\cdot\bar{\Delta}\cdot x_{mk_s}+\langle\xi_i,\xi_m\rangle\cdot\Delta\left(\overline{A}_{1m}(\frac{1}{\lambda_{k_s}^2}-\frac{1}{\lambda_{i_1}^2})x_{1k_s}+\cdots+\overline{A}_{mm}(\frac{1}{\lambda_{k_s}^2}-\frac{1}{\lambda_{i_m}^2})x_{mk_s}\right)]\\
&=-\frac{2\Delta}{\bar{\Delta}^2}[(\frac{1}{\lambda_{k_s}^2}-\frac{1}{\lambda_{i_1}^2})\cdot(\langle\xi_i,\xi_1\rangle\overline{A}_{11}+\langle\xi_i,\xi_2\rangle\overline{A}_{12}+\cdots+\langle\xi_i,\xi_m\rangle\overline{A}_{1m})\cdot x_{1k_s}\\
&+(\frac{1}{\lambda_{k_s}^2}-\frac{1}{\lambda_{i_2}^2})\cdot(\langle\xi_i,\xi_1\rangle\overline{A}_{21}+\langle\xi_i,\xi_2\rangle\overline{A}_{22}+\cdots+\langle\xi_i,\xi_m\rangle\overline{A}_{2m})\cdot x_{2k_s}+\cdots\\
&+(\frac{1}{\lambda_{k_s}^2}-\frac{1}{\lambda_{i_m}^2})\cdot(\langle\xi_i,\xi_1\rangle\overline{A}_{m1}+\langle\xi_i,\xi_2\rangle\overline{A}_{m2}+\cdots+\langle\xi_i,\xi_m\rangle\overline{A}_{mm})\cdot x_{mk_s}]
\end{aligned}
$$

by the computation we have

$$
\begin{pmatrix}(\nabla f)_{1k_1}\\(\nabla f)_{1k_2}\\\vdots\\(\nabla f)_{1k_{n-m}}\\\vdots\\(\nabla f)_{mk_1}\\(\nabla f)_{mk_2}\\\vdots\\(\nabla f)_{mk_{n-m}}\end{pmatrix}=-\frac{2\Delta}{\bar{\Delta}^2}\begin{pmatrix}\langle\xi_1,\xi_1\rangle I_{n-m}&\cdots&\langle\xi_1,\xi_m\rangle I_{n-m}\\\langle\xi_2,\xi_1\rangle I_{n-m}&\cdots&\langle\xi_2,\xi_m\rangle I_{n-m}\\\cdots&\cdots&\cdots\\\langle\xi_m,\xi_1\rangle I_{n-m}&\cdots&\langle\xi_m,\xi_m\rangle I_{n-m}\end{pmatrix}\cdot F\cdot H\cdot X \qquad (4)
$$

where

$$
X=(x_{1k_1},x_{1k_2},\cdots,x_{1k_{n-m}},x_{2k_1},x_{2k_2},\cdots,x_{2k_{n-m}},\cdots,x_{mk_1},x_{mk_2},\cdots,x_{mk_{n-m}})^T.
$$

$$
F=\begin{pmatrix}\overline{A}_{11}I_{n-m}&\overline{A}_{21}I_{n-m}&\cdots&\overline{A}_{m1}I_{n-m}\\\overline{A}_{12}I_{n-m}&\overline{A}_{22}I_{n-m}&\cdots&\overline{A}_{m1}I_{n-m}\\\cdots&\cdots&\cdots&\cdots\\\overline{A}_{1m}I_{n-m}&\overline{A}_{2m}I_{n-m}&\cdots&\overline{A}_{mm}I_{n-m}\end{pmatrix}
$$

$$H=\begin{pmatrix} H_{11} & & & 0 \\ & H_{22} & & \\ & & \ddots & \\ 0 & & & H_{(n-m)(n-m)} \end{pmatrix}$$

where

$$H_{ss}=\begin{pmatrix} (\frac{1}{\lambda_{k_s}^2}-\frac{1}{\lambda_{i_1}^2}) & & & 0 \\ & (\frac{1}{\lambda_{k_s}^2}-\frac{1}{\lambda_{i_2}^2}) & & \\ & & \ddots & \\ 0 & & & (\frac{1}{\lambda_{k_s}^2}-\frac{1}{\lambda_{i_m}^2}) \end{pmatrix}$$

$s=1,2,\cdots,n-m$.

**Lemma 5.** *$O_{i_1i_2\cdots i_m}$ is a hyperbolic singular point of $\dot{x}=-\nabla f$, and the linear part of $\dot{x}=-\nabla f$ at $O_{i_1i_2\cdots i_m}$ has the expression $AX$ on $(\widetilde{U}_{i_1i_2\cdots i_m},\widetilde{\phi}_{i_1i_2\cdots i_m})$ where*

$$A=-2\lambda_{i_1}^2\lambda_{i_2}^2\cdots\lambda_{i_m}^2\mathrm{diag}\left(1-\frac{\lambda_{i_1}^2}{\lambda_{k_1}^2},\cdots,1-\frac{\lambda_{i_1}^2}{\lambda_{k_{n-m}}^2},\cdots,1-\frac{\lambda_{i_m}^2}{\lambda_{k_1}^2},\cdots,1-\frac{\lambda_{i_m}^2}{\lambda_{k_{n-m}}^2}\right)$$

*and $X=(x_{1k_1},x_{1k_2},\cdots,x_{1k_{n-m}},x_{2k_1},x_{2k_2},\cdots,x_{2k_{n-m}},\cdots,x_{mk_1},x_{mk_2},\cdots,x_{mk_{n-m}})^T$.*

*Proof.* Because at $O_{i_1i_2\cdots i_m}$ by comptutation $\frac{2\Delta}{\Delta^2}(O_{i_1i_2\cdots i_m})=2\lambda_{i_1}^4\lambda_{i_2}^4\cdots\lambda_{i_m}^4$,

$$F(O_{i_1i_2\cdots i_m})=\begin{pmatrix} (\frac{1}{\lambda_{i_2}^2}\frac{1}{\lambda_{i_3}^2}\cdots\frac{1}{\lambda_{i_m}^2})I_{n-m} & & & 0 \\ & (\frac{1}{\lambda_{i_1}^2}\frac{1}{\lambda_{i_3}^2}\cdots\frac{1}{\lambda_{i_m}^2})I_{n-m} & & \\ & & \ddots & \\ 0 & & & (\frac{1}{\lambda_{i_1}^2}\frac{1}{\lambda_{i_2}^2}\cdots\frac{1}{\lambda_{i_{m-1}}^2})I_{n-m} \end{pmatrix}$$

then by (4) we can get

$$A=-2\lambda_{i_1}^2\lambda_{i_2}^2\cdots\lambda_{i_m}^2\mathrm{diag}\left(1-\frac{\lambda_{i_1}^2}{\lambda_{k_1}^2},\cdots,1-\frac{\lambda_{i_1}^2}{\lambda_{k_{n-m}}^2},\cdots,1-\frac{\lambda_{i_m}^2}{\lambda_{k_1}^2},\cdots,1-\frac{\lambda_{i_m}^2}{\lambda_{k_{n-m}}^2}\right)$$

□

Let $M$ be a compact smooth Riemannian manifold with metirc $g$, and $f$ be a Morse function on $M$, then the negative gradient vector field $-\nabla f$ determines a smooth flow $\varphi:\mathbb{R}\times M\to M$, and $\varphi_t$ is a diffeomorphism of $M$ for all $t\in\mathbb{R}$(see [4]).

**Definition 1** (see [4] )**.** *Let $p\in M$ be a non-degenerate critical point of Morse function f, the stable manifold of p is defined to be*

$$W^s(p)=\{x\in M\mid \lim_{t\to+\infty}\varphi_t(x)=p\}$$

*the unstable manifold of p is defined to be*

$$W^u(p)=\{x\in M\mid \lim_{t\to-\infty}\varphi_t(x)=p\}$$

Obviously

$$\partial_{1k_1}, \partial_{1k_2}, \cdots, \partial_{1k_{n-m}}, \partial_{2k_1}, \cdots, \partial_{2k_{n-m}}, \partial_{mk_1}, \cdots, \partial_{mk_{n-m}}$$

is the orthonormal basis of the tangent space on $\widetilde{U}_{i_1 i_2 \cdots i_m}$, where $\partial_{ij} = \frac{\partial}{\partial x_{ij}}$.

By the computation in lemma 3., we have linear part $A = -H_{O_{i_1 i_2 \cdots i_m}}(f)$, so use the stabel and unstable manifold theorem for a Morse function(see [4]), we identified the positive definite eigenvalue subspace

$$E^s(O_{i_1 i_2 \cdots i_m}) =$$

$$\mathrm{span}_{\mathbb{R}}\{\partial_{1(\mathrm{i}_1+1)}, \partial_{1(\mathrm{i}_1+2)}, \cdots, \widehat{\partial_{1\mathrm{i}_2}}, \cdots, \widehat{\partial_{1\mathrm{i}_3}}, \cdots, \widehat{\partial_{1\mathrm{i}_\mathrm{m}}}, \cdots, \partial_{1\mathrm{n}},$$

$$\partial_{2(i_2+1)}, \partial_{2(i_2+2)}, \cdots, \widehat{\partial_{2i_3}}, \cdots, \widehat{\partial_{2i_m}}, \cdots, \partial_{2n},$$

$$\partial_{3(i_3+1)}, \partial_{3(i_3+1)}, \cdots, \widehat{\partial_{3i_4}}, \cdots, \widehat{\partial_{3i_m}}, \cdots, \partial_{3n},$$

$$\cdots,$$

$$\partial_{m(i_m+1)}, \partial_{m(i_m+2)}, \cdots, \partial_{mn}\}$$

and negative definite eigenvalue subspace

$$E^u(O_{i_1 i_2 \cdots i_m}) =$$

$$\mathrm{span}_{\mathbb{R}}\{\partial_{11}, \partial_{12}, \cdots, \partial_{1(\mathrm{i}_1-1)},$$

$$\partial_{21}, \partial_{22}, \cdots, \widehat{\partial_{2i_1}}, \cdots, \partial_{2(i_2-1)},$$

$$\partial_{31}, \partial_{32}, \cdots, \widehat{\partial_{3i_1}}, \cdots, \widehat{\partial_{3i_2}}, \cdots, \partial_{3(i_3-1)},$$

$$\cdots,$$

$$\partial_{m1}, \partial_{m2}, \cdots, \widehat{\partial_{mi_1}}, \cdots, \widehat{\partial_{mi_2}}, \cdots, \widehat{\partial_{mi_3}}, \cdots, \widehat{\partial_{mi_{m-1}}}, \cdots, \partial_{m(i_m-1)}\}$$

and

$$E^s(O_{i_1 i_2 \cdots i_m}) = T_{O_{i_1 i_2 \cdots i_m}} W^s(O_{i_1 i_2 \cdots i_m}),\ E^u(O_{i_1 i_2 \cdots i_m}) = T_{O_{i_1 i_2 \cdots i_m}} W^u(O_{i_1 i_2 \cdots i_m}).$$

where $\widehat{\partial_{ij}}$ means this one is empty.

**Definition 2** (see [6])**.** *An invariant manifold $N$ of a vector field $V$ on a manifold $M$ and of the corresponding differential equation $\dot{x} = V(x)$ is defined to be a submanifold of $M$ which is tangent to the vector field $V$ at each of its points.*

*An invariant manifold $N$ is global if the initial value problem*

$$\dot{x} = V(x),\ x(0) = p$$

*has a global solution $x = x(t)$, $(-\infty < t < +\infty)$ for any $p \in N$.*

**Lemma 6.** *The following sets as global invariant manifold of $\dot{x} = -\nabla f$ on $G_{n,m}(\mathbb{R})$,*

$$\widetilde{U}_{i_1 i_2 \cdots i_m}(x_{\beta k_{s_1}}, x_{\beta k_{s_2}}, \cdots, x_{\beta k_{s_t}})$$

$$= \pi(\{(x_{\alpha k}) \in U_{i_1 i_2 \cdots i_m} \mid x_{\alpha k} = 0, (\alpha, k) \neq (\beta, k_{s_p}), p = 1, \cdots, t\}),$$

$$\widetilde{U}_{i_1 i_2 \cdots i_m}(x_{\beta_1 k_{s_1^1}}, x_{\beta_1 k_{s_2^1}}, \cdots, x_{\beta_1 k_{s_{t_1}^1}}, x_{\beta_2 k_{s_1^2}}, x_{\beta_2 k_{s_2^2}}, \cdots, x_{\beta_2 k_{s_{t_2}^2}})$$

$$= \pi(\{(x_{\alpha k}) \in U_{i_1 i_2 \cdots i_m} \mid x_{\alpha k} = 0, (\alpha, k) \neq (\beta_j, k_{s_p^j}), j = 1, 2; p = 1, \cdots, t_j\}),$$

$$\widetilde{U}_{i_1i_2\cdots i_m}(x_{\beta_1 k_{s_1^1}}, x_{\beta_1 k_{s_2^1}}, \cdots, x_{\beta_1 k_{s_{t_1}^1}}, x_{\beta_2 k_{s_1^2}}, x_{\beta_2 k_{s_2^2}}, \cdots, x_{\beta_2 k_{s_{t_2}^2}}, x_{\beta_3 k_{s_1^3}}, x_{\beta_3 k_{s_2^3}}, \cdots, x_{\beta_3 k_{s_{t_3}^3}})$$
$$= \pi(\{(x_{\alpha k}) \in U_{i_1i_2\cdots i_m} \mid x_{\alpha k} = 0, (\alpha, k) \neq (\beta_j, k_{s_p^j}), j = 1, 2, 3; p = 1, \cdots, t_j\}),$$

$$\cdots\cdots\cdots$$

$$\widetilde{U}_{i_1i_2\cdots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \cdots, x_{\beta_j k_{s_{t_j}^j}})$$
$$= \pi(\{(x_{\alpha k}) \in U_{i_1i_2\cdots i_m} \mid x_{\alpha k} = 0, (\alpha, k) \neq (\beta_j, k_{s_p^j}), j = 1, 2, 3, \cdots, m; p = 1, \cdots, t_j\}).$$

*Proof.* For any $\widetilde{U}_{i_1i_2\cdots i_m}(\cdots)$, when $x_{\beta k} \notin (\cdots)$, then $x_{\beta k} = 0$, so $\dot{x}_{\beta k}(t) = 0$ on $\widetilde{U}_{i_1i_2\cdots i_m}(\cdots)$ and $-(\nabla f)_{\beta k} \mid_{\widetilde{U}_{i_1i_2\cdots i_m}(\cdots)} = \dot{x}_{\beta k}(t) = 0$, then $-(\nabla f) \mid_{\widetilde{U}_{i_1i_2\cdots i_m}(\cdots)}$ is tangent vector field on $\widetilde{U}_{i_1i_2\cdots i_m}(\cdots)$. So $\widetilde{U}_{i_1i_2\cdots i_m}(\cdots)$ is invariant manifold of $\dot{x} = -\nabla f$ on $G_{n,m}(\mathbb{R})$.

Because $G_{n,m}(\mathbb{R})$ is compact smooth manifold, so the initial value problem

$$\dot{x} = -\nabla f, \ x(0) = p$$

on $G_{n,m}(\mathbb{R})$ has global solution.

$$\widetilde{U}_{i_1i_2\cdots i_m}(\cdots) \in G_{n,m}(\mathbb{R})$$

the initial value problem

$$\dot{x} = -(\nabla f) \mid_{\widetilde{U}_{i_1i_2\cdots i_m}(\cdots)}, \ x(0) = p, \ p \in \widetilde{U}_{i_1i_2\cdots i_m}(\cdots)$$

on $\widetilde{U}_{i_1i_2\cdots i_m}(\cdots)$ also has global solution. Then $\widetilde{U}_{i_1i_2\cdots i_m}(\cdots)$ is the global invariant manifold. □

**Lemma 7** (see [6])**.** *Let $V$ be a smooth vector field on a manifold $M$ and $p \in M$ be a hyperbolic singular point of $V$. Let $N$ be a global invariant manifold of $V$ in $M$ and $p \in N$. Then we have the following sets equalities*

$$W_N^s(p) = W^s(p) \cap N, \ W_N^u(p) = W^u(p) \cap N,$$

*where $W_N^s(p)$,$W_N^u(p)$ are the stable and unstable manifold of $V \mid_N$, the restriction of $V$ on $N$ at $p$, particularly we have*

$$W_N^s(p) = W^s(p) \quad if \ \dim W_N^s(p) = \dim W^s(p)$$

$$W_N^u(p) = W^u(p) \quad if \ \dim W_N^u(p) = \dim W^s(p).$$

**Lemma 8.** *The stable and unstable manifold of $O_{i_1i_2\cdots i_m}$, have the following results*

**a)**

$$W^s(O_{i_1i_2\cdots i_m}) \subset \widetilde{U}_{i_1i_2\cdots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \cdots, x_{\beta_j k_{s_{t_j}^j}})$$

*where $j = 1, 2, \cdots, m$;*
*when $j = 1$, let $\beta_1 = 1$, $k_{s_p^1} = i_1 + 1, i_1 + 2, \cdots, \widehat{i_2}, \cdots, \widehat{i_3}, \cdots, \widehat{i_m}, \cdots, n$;*
*when $j = 2$, let $\beta_2 = 2$, $k_{s_p^2} = i_2 + 1, i_2 + 2, \cdots, \widehat{i_3}, \cdots, \widehat{i_4}, \cdots, \widehat{i_m}, \cdots, n$;*

$$\cdots\cdots\cdots$$

*when $j = m$, let $\beta_m = m$, $k_{s_p^m} = i_m + 1, i_m + 2, \cdots, n$.*

**b)**

$$W^u(O_{i_1i_2\cdots i_m}) \subset \widetilde{U}_{i_1i_2\cdots i_m}(x_{\beta_jk_{s_1^j}}, x_{\beta_jk_{s_2^j}}, \cdots, x_{\beta_jk_{s_{t_j}^j}})$$

*where* $j = 1, 2, \cdots, m$;
*when* $j = 1$, *let* $\beta_1 = 1$, $k_{s_p^1} = 1, 2, \cdots, i_1 - 1$;
*when* $j = 2$, *let* $\beta_2 = 2$, $k_{s_p^2} = 1, 2, \cdots, \widehat{i_1}, \cdots, i_2 - 1$;

$$\cdots\cdots\cdots$$

*when* $j = m$, *let* $\beta_m = m$, $k_{s_p^m} = 1, 2, \cdots, \widehat{i_1}, \cdots, \widehat{i_2}, \cdots, \widehat{i_{m-1}}, \cdots, i_m - 1$.

*Proof.* Because $O_{i_1i_2\cdots i_m} \in \widetilde{U}_{i_1i_2\cdots i_m}(x_{\beta_jk_{s_1^j}}, x_{\beta_jk_{s_2^j}}, \cdots, x_{\beta_jk_{s_{t_j}^j}})$, where $j = 1, 2, \cdots, m$; so $O_{i_1i_2\cdots i_m}$ is a singular point of $-\nabla f\ |_{\widetilde{U}_{i_1i_2\cdots i_m}(x_{\beta_jk_{s_1^j}}, x_{\beta_jk_{s_2^j}}, \cdots, x_{\beta_jk_{s_{t_j}^j}})}$.

By lemma 6., $\widetilde{U}_{i_1i_2\cdots i_m}(x_{\beta_jk_{s_1^j}}, x_{\beta_jk_{s_2^j}}, \cdots, x_{\beta_jk_{s_{t_j}^j}})$ is a global invariant manifold, so $O_{i_1i_2\cdots i_m}$ is the hyperbolic singular point of $-\nabla f\ |_{\widetilde{U}_{i_1i_2\cdots i_m}(x_{\beta_jk_{s_1^j}}, x_{\beta_jk_{s_2^j}}, \cdots, x_{\beta_jk_{s_{t_j}^j}})}$ and the linear part at $O_{i_1i_2\cdots i_m}$ is

$$A = -2\lambda_{i_1}^2\lambda_{i_2}^2\cdots\lambda_{i_m}^2 \mathrm{diag}\left(1 - \frac{\lambda_{i_1}^2}{\lambda_{k_1^1}^2}, \cdots, 1 - \frac{\lambda_{i_1}^2}{\lambda_{k_{t_1}^1}^2}, \cdots, 1 - \frac{\lambda_{i_m}^2}{\lambda_{k_1^m}^2}, \cdots, 1 - \frac{\lambda_{i_m}^2}{\lambda_{k_{t_m}^m}^2}\right)$$

where $k_{t_p}^1 = i_1 + 1, i_1 + 2, \cdots, \widehat{i_2}, \cdots, \widehat{i_3}, \cdots, \widehat{i_m}, \cdots, n$;
$k_{t_p}^2 = i_2 + 1, i_2 + 2, \cdots, \widehat{i_3}, \cdots, \widehat{i_4}, \cdots, \widehat{i_m}, \cdots, n$; $\cdots\cdots$; $k_{t_p}^m = i_m + 1, i_m + 2, \cdots, n$.
Then the positive definite eigenvalue subspace is

$$E^s_{\widetilde{U}_{i_1i_2\cdots i_m}(x_{\beta_jk_{s_1^j}}, x_{\beta_jk_{s_2^j}}, \cdots, x_{\beta_jk_{s_{t_j}^j}})}(O_{i_1i_2\cdots i_m}) =$$

$$\begin{gathered}\mathrm{span}_{\mathbb{R}}\{\partial_{1(i_1+1)}, \partial_{1(i_1+2)}, \cdots, \widehat{\partial_{1i_2}}, \cdots, \widehat{\partial_{1i_3}}, \cdots, \widehat{\partial_{1i_m}}, \cdots, \partial_{1n},\\ \partial_{2(i_2+1)}, \partial_{2(i_2+2)}, \cdots, \widehat{\partial_{2i_3}}, \cdots, \widehat{\partial_{2i_m}}, \cdots, \partial_{2n},\\ \partial_{3(i_3+1)}, \partial_{3(i_3+1)}, \cdots, \widehat{\partial_{3i_4}}, \cdots, \widehat{\partial_{3i_m}}, \cdots, \partial_{3n},\\ \cdots,\\ \partial_{m(i_m+1)}, \partial_{m(i_m+2)}, \cdots, \partial_{mn}\}.\end{gathered}$$

So we have $E^s_{\widetilde{U}_{i_1i_2\cdots i_m}(x_{\beta_jk_{s_1^j}}, x_{\beta_jk_{s_2^j}}, \cdots, x_{\beta_jk_{s_{t_j}^j}})}(O_{i_1i_2\cdots i_m}) = E^s(O_{i_1i_2\cdots i_m})$, and we get

$$\dim W^s_{\widetilde{U}_{i_1i_2\cdots i_m}(x_{\beta_jk_{s_1^j}}, x_{\beta_jk_{s_2^j}}, \cdots, x_{\beta_jk_{s_{t_j}^j}})}(O_{i_1i_2\cdots i_m}) = \dim E^s_{\widetilde{U}_{i_1i_2\cdots i_m}(x_{\beta_jk_{s_1^j}}, x_{\beta_jk_{s_2^j}}, \cdots, x_{\beta_jk_{s_{t_j}^j}})}(O_{i_1i_2\cdots i_m})$$

$$= \dim E^s(O_{i_1i_2\cdots i_m}) = \dim W^s(O_{i_1i_2\cdots i_m})$$

by Lemma 7., we have

$$W^s(O_{i_1i_2\cdots i_m}) \subset \widetilde{U}_{i_1i_2\cdots i_m}(x_{\beta_jk_{s_1^j}}, x_{\beta_jk_{s_2^j}}, \cdots, x_{\beta_jk_{s_{t_j}^j}});$$

where $j=1,2,\cdots,m$;
when $j=1$, let $\beta_1=1$, $k_{s_p^1}=i_1+1,i_1+2,\cdots,\widehat{i_2},\cdots,\widehat{i_3},\cdots,\widehat{i_m},\cdots,n$;
when $j=2$, let $\beta_2=2$, $k_{s_p^2}=i_2+1,i_2+2,\cdots,\widehat{i_3},\cdots,\widehat{i_4},\cdots,\widehat{i_m},\cdots,n$;

$$\cdots\cdots\cdots$$

when $j=m$, let $\beta_m=m$, $k_{s_p^m}=i_m+1,i_m+2,\cdots,n$.

By the same way, we can get

$$W^u(O_{i_1i_2\cdots i_m})\subset\widetilde{U}_{i_1i_2\cdots i_m}(x_{\beta_jk_{s_1^j}},x_{\beta_jk_{s_2^j}},\cdots,x_{\beta_jk_{s_{t_j}^j}});$$

where $j=1,2,\cdots,m$;
when $j=1$, let $\beta_1=1$, $k_{s_p^1}=1,2,\cdots,i_1-1$;
when $j=2$, let $\beta_2=2$, $k_{s_p^2}=1,2,\cdots,\widehat{i_1},\cdots,i_2-1$;

$$\cdots\cdots\cdots$$

when $j=m$, let $\beta_m=m$, $k_{s_p^m}=1,2,\cdots,\widehat{i_1},\cdots,\widehat{i_2},\cdots,\widehat{i_{m-1}},\cdots,i_m-1$. □

## 5 The Morse-Smale transversality condition

In this section we will to proof $f=\frac{\Delta}{\Delta}$ is a Morse-Smale funtion. For all critical points $O_{i_1i_2\cdots i_m}$, $O_{l_1l_2\cdots l_m}$ of $f=\frac{\Delta}{\Delta}$ we need to proof the stable and unstable manifolds of $f$ intersect transversally(see [4] and [9]).

**Lemma 9.** *Let $O_{i_1i_2\cdots i_m}$, $O_{l_1l_2\cdots l_m}$ be the different critical points of $f$, and let $l_k\geq i_k$ for some $k\in 1,2,\cdots,m$, then have the following result*

$$W^u(O_{i_1i_2\cdots i_m})\cap W^s(O_{l_1l_2\cdots l_m})=\emptyset$$

*Proof.* If $W^u(O_{i_1i_2\cdots i_m})\cap W^s(O_{l_1l_2\cdots l_m})\neq\emptyset$, there is $p\in W^u(O_{i_1i_2\cdots i_m})\cap W^s(O_{l_1l_2\cdots l_m})$ and $\exists\varphi(t)(-\infty<t<+\infty)$ is the solution of $\dot{x}=-\nabla f$, with $\varphi(0)=p$ and

$$\lim_{t\to+\infty}\varphi(t)=O_{l_1l_2\cdots l_m},\ \lim_{t\to-\infty}\varphi(t)=O_{i_1i_2\cdots i_m}$$

so $\varphi(t)\in W^u(O_{i_1i_2\cdots i_m})\cap W^s(O_{l_1l_2\cdots l_m})$.

Because $O_{l_1l_2\cdots l_m}\in\widetilde{U}_{l_1l_2\cdots l_m}$, so $\exists t_0>0$ with $t>t_0$,$\varphi(t)\in\widetilde{U}_{l_1l_2\cdots l_m}$. By Lemma 8., we have

$$\varphi(t)\in\widetilde{U}_{l_1l_2\cdots l_m}\cap\widetilde{U}_{i_1i_2\cdots i_m}(x_{\beta_jk_{s_1^j}},x_{\beta_jk_{s_2^j}},\cdots,x_{\beta_jk_{s_{t_j}^j}}),\ t>t_0,$$

where $j=1,2,\cdots,m$;
when $j=1$, let $\beta_1=1$, $k_{s_p^1}=1,2,\cdots,i_1-1$;
when $j=2$, let $\beta_2=2$, $k_{s_p^2}=1,2,\cdots,\widehat{i_1},\cdots,i_2-1$;

$$\cdots\cdots\cdots$$

when $j=m$, let $\beta_m=m$, $k_{s_p^m}=1,2,\cdots,\widehat{i_1},\cdots,\widehat{i_2},\cdots,\widehat{i_{m-1}},\cdots,i_m-1$.

But because $l_k\geq i_k$ for some $k\in 1,2,\cdots,m$, so we have

$$\widetilde{U}_{l_1l_2\cdots l_m}\cap\widetilde{U}_{i_1i_2\cdots i_m}(x_{\beta_jk_{s_1^j}},x_{\beta_jk_{s_2^j}},\cdots,x_{\beta_jk_{s_{t_j}^j}})=\emptyset,$$

where $j = 1, 2, \cdots, m$;
when $j = 1$, let $\beta_1 = 1$, $k_{s_p^1} = 1, 2, \cdots, i_1 - 1$;
when $j = 2$, let $\beta_2 = 2$, $k_{s_p^2} = 1, 2, \cdots, \widehat{i_1}, \cdots, i_2 - 1$;

$$\cdots\cdots\cdots$$

when $j = m$, let $\beta_m = m$, $k_{s_p^m} = 1, 2, \cdots, \widehat{i_1}, \cdots, \widehat{i_2}, \cdots, \widehat{i_{m-1}}, \cdots, i_m - 1$.

Then we get

$$W^u(O_{i_1 i_2 \cdots i_m}) \cap W^s(O_{l_1 l_2 \cdots l_m}) = \emptyset$$

. □

**Lemma 10.** *The function $f = \frac{\Delta}{\Delta}$ is Morse-Smale funtion.*

*Proof.* Let $O_{i_1 i_2 \cdots i_m}$, $O_{l_1 l_2 \cdots l_m}$ be any two different critical points of $f$, and with

$$W^u(O_{i_1 i_2 \cdots i_m}) \cap W^s(O_{l_1 l_2 \cdots l_m}) \neq \emptyset.$$

By the lemma 9., we know that $l_1 < i_1, l_2 < i_2, \cdots, l_m < i_m$. Let $p \in W^u(O_{i_1 i_2 \cdots i_m}) \cap W^s(O_{l_1 l_2 \cdots l_m})$, the tangent space of $W^u(O_{i_1 i_2 \cdots i_m})$ at $p$ is

$$T_p W^u(O_{i_1 i_2 \cdots i_m}) = \mathrm{span}_{\mathbb{R}}\{\partial_{11}, \partial_{12}, \cdots, \partial_{1(i_1-1)},$$

$$\partial_{21}, \partial_{22}, \cdots, \widehat{\partial_{2i_1}}, \cdots, \partial_{2(i_2-1)},$$

$$\partial_{31}, \partial_{32}, \cdots, \widehat{\partial_{3i_1}}, \cdots, \widehat{\partial_{3i_2}}, \cdots, \partial_{3(i_3-1)},$$

$$\cdots,$$

$$\partial_{m1}, \partial_{m2}, \cdots, \widehat{\partial_{mi_1}}, \cdots, \widehat{\partial_{mi_2}}, \cdots, \widehat{\partial_{mi_3}}, \cdots, \widehat{\partial_{mi_{m-1}}}, \cdots, \partial_{m(i_m-1)}\};$$

The tangent space of $W^s(O_{l_1 l_2 \cdots l_m})$ at $p$ is

$$T_p W^s(O_{l_1 l_2 \cdots l_m}) = \mathrm{span}_{\mathbb{R}}\{\partial_{1(l_1+1)}, \partial_{1(l_1+2)}, \cdots, \widehat{\partial_{1l_2}}, \cdots, \widehat{\partial_{1l_3}}, \cdots, \widehat{\partial_{1l_m}}, \cdots, \partial_{1n},$$

$$\partial_{2(l_2+1)}, \partial_{2(l_2+2)}, \cdots, \widehat{\partial_{2l_3}}, \cdots, \widehat{\partial_{2l_m}}, \cdots, \partial_{2n},$$

$$\partial_{3(l_3+1)}, \partial_{3(l_3+1)}, \cdots, \widehat{\partial_{3l_4}}, \cdots, \widehat{\partial_{3l_m}}, \cdots, \partial_{3n},$$

$$\cdots,$$

$$\partial_{m(l_m+1)}, \partial_{m(l_m+2)}, \cdots, \partial_{mn}\}.$$

So we have

$$T_p G_{n,m}(\mathbb{R}) \subset T_p W^u(O_{i_1 i_2 \cdots i_m}) + T_p W^s(O_{l_1 l_2 \cdots l_m})$$

and because

$$W^u(O_{i_1 i_2 \cdots i_m}) \subset G_{n,m}(\mathbb{R}),\ W^s(O_{l_1 l_2 \cdots l_m}) \subset G_{n,m}(\mathbb{R})$$

so

$$T_p G_{n,m}(\mathbb{R}) \supset T_p W^u(O_{i_1 i_2 \cdots i_m}) + T_p W^s(O_{l_1 l_2 \cdots l_m})$$

Then we get

$$T_p G_{n,m}(\mathbb{R}) = T_p W^u(O_{i_1 i_2 \cdots i_m}) + T_p W^s(O_{l_1 l_2 \cdots l_m}).$$

It means that the stable and unstable manifolds of $f$ intersect transversally, $f = \frac{\Delta}{\Delta}$ is a Morse-Smale funtion.

□

## 6 The trajectories connect the critical points

**Theorem 1.** *Let $O_{i_1 i_2 \cdots i_m}$, $O_{l_1 l_2 \cdots l_m}$ be any two different critical points of $f$, with*

$$\mathrm{ind}(O_{i_1 i_2 \cdots i_m}) - \mathrm{ind}(O_{l_1 l_2 \cdots l_m}) = 1,$$

*and $i_k - l_k > 1$ for some $k \in 1, 2, \cdots, m$; then*

$$W^u(O_{i_1 i_2 \cdots i_m}) \cap W^s(O_{l_1 l_2 \cdots l_m}) = \emptyset$$

*Proof.* If $W^u(O_{i_1 i_2 \cdots i_m}) \cap W^s(O_{l_1 l_2 \cdots l_m}) \neq \emptyset$, there is $p \in W^u(O_{i_1 i_2 \cdots i_m}) \cap W^s(O_{l_1 l_2 \cdots l_m})$ and $\exists \varphi(t)(-\infty < t < +\infty)$ is the solution of $\dot{x} = -\nabla f$, with $\varphi(0) = p$ and

$$\lim_{t \to +\infty} \varphi(t) = O_{l_1 l_2 \cdots l_m}, \ \lim_{t \to -\infty} \varphi(t) = O_{i_1 i_2 \cdots i_m}$$

so $\varphi(t) \in W^u(O_{i_1 i_2 \cdots i_m}) \cap W^s(O_{l_1 l_2 \cdots l_m})$.

Because $O_{l_1 l_2 \cdots l_m} \in \widetilde{U}_{l_1 l_2 \cdots l_m}$, so $\exists t_0 > 0$ with $t > t_0$,$\varphi(t) \in \widetilde{U}_{l_1 l_2 \cdots l_m}$. By Lemma 8., we have

$$\varphi(t) \in \widetilde{U}_{l_1 l_2 \cdots l_m} \cap \widetilde{U}_{i_1 i_2 \cdots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \cdots, x_{\beta_j k_{s_{t_j}^j}}), \ t > t_0,$$

where $j = 1, 2, \cdots, m$;
when $j = 1$, let $\beta_1 = 1$, $k_{s_p^1} = 1, 2, \cdots, i_1 - 1$;
when $j = 2$, let $\beta_2 = 2$, $k_{s_p^2} = 1, 2, \cdots, \widehat{i_1}, \cdots, i_2 - 1$;

. . . . . . . . .

when $j = m$, let $\beta_m = m$, $k_{s_p^m} = 1, 2, \cdots, \widehat{i_1}, \cdots, \widehat{i_2}, \cdots, \widehat{i_{m-1}}, \cdots, i_m - 1$.
Because $\mathrm{ind}(O_{i_1 i_2 \cdots i_m}) - \mathrm{ind}(O_{l_1 l_2 \cdots l_m}) = 1$, so we have $i_1 + i_2 + \cdots + i_m - (l_1 + l_2 + \cdots + l_m) = 1$, and by $i_k - l_k > 1$ for some $k \in 1, 2, \cdots, m$, we get there is a $j \neq k$, $j \in 1, 2, \cdots, m$ with $l_j > i_j$. By it, we have

$$\widetilde{U}_{l_1 l_2 \cdots l_m} \cap \widetilde{U}_{i_1 i_2 \cdots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \cdots, x_{\beta_j k_{s_{t_j}^j}}) = \emptyset,$$

where $j = 1, 2, \cdots, m$;
when $j = 1$, let $\beta_1 = 1$, $k_{s_p^1} = 1, 2, \cdots, i_1 - 1$;
when $j = 2$, let $\beta_2 = 2$, $k_{s_p^2} = 1, 2, \cdots, \widehat{i_1}, \cdots, i_2 - 1$;

. . . . . . . . .

when $j = m$, let $\beta_m = m$, $k_{s_p^m} = 1, 2, \cdots, \widehat{i_1}, \cdots, \widehat{i_2}, \cdots, \widehat{i_{m-1}}, \cdots, i_m - 1$. which contradicts to

$$\varphi(t) \in \widetilde{U}_{l_1 l_2 \cdots l_m} \cap \widetilde{U}_{i_1 i_2 \cdots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \cdots, x_{\beta_j k_{s_{t_j}^j}}), \ t > t_0.$$

So we get

$$W^u(O_{i_1 i_2 \cdots i_m}) \cap W^s(O_{l_1 l_2 \cdots l_m}) = \emptyset.$$

□

By Theorem 1., we know that if $\mathrm{ind}(O_{i_1 i_2 \cdots i_m}) - \mathrm{ind}(O_{l_1 l_2 \cdots l_m}) = 1$ and $W^u(O_{i_1 i_2 \cdots i_m}) \cap W^s(O_{l_1 l_2 \cdots l_m}) \neq \emptyset$, so $\exists k$ with $i_k - l_k = 1$, where $k \in 1, 2, \cdots, m$.

**Theorem 2.** *Let $O_{i_1i_2\cdots i_m}$, $O_{l_1l_2\cdots l_m}$ be any two different critical points of $f$, with*

$$\mathrm{ind}(O_{i_1i_2\cdots i_m}) - \mathrm{ind}(O_{l_1l_2\cdots l_m}) = 1,$$

*Then*

$$W^u(O_{i_1i_2\cdots i_m}) \cap W^s(O_{i_1i_2\cdots(i_k-1)\cdots i_m}) = \widetilde{U}_{i_1i_2\cdots i_m}(x_{k(i_k-1)}; x_{k(i_k-1)} \neq 0)$$

*Proof.* By Lemma 8., $\forall p \in W^u(O_{i_1i_2\cdots i_m}) \cap W^s(O_{i_1i_2\cdots(i_k-1)\cdots i_m})$, the coordinate in $\widetilde{U}_{i_1i_2\cdots i_m}$ is

$$(x_{11}(p), x_{12}(p), \cdots, x_{1(i_1-1)}(p), 1,$$
$$x_{21}(p), x_{22}(p), \cdots, \widehat{x_{2i_1}(p)}, \cdots, x_{2(i_2-1)}(p), 1,$$
$$x_{31}(p), x_{32}(p), \cdots, \widehat{x_{3i_1}(p)}, \cdots, \widehat{x_{3i_2}(p)}, \cdots, x_{3(i_3-1)}(p), 1,$$
$$\cdots,$$
$$x_{k1}(p), x_{k2}(p), \cdots, \widehat{x_{ki_1}(p)}, \cdots, \widehat{x_{ki_2}(p)}, \cdots, \widehat{x_{ki_3}(p)}, \cdots, \widehat{x_{ki_{k-1}}(p)}, \cdots, x_{k(i_k-1)}(p), 1)$$
$$\cdots,$$
$$x_{m1}(p), x_{m2}(p), \cdots, \widehat{x_{mi_1}(p)}, \cdots, \widehat{x_{mi_2}(p)}, \cdots, \widehat{x_{mi_3}(p)}, \cdots, \widehat{x_{mi_{m-1}}(p)}, \cdots, x_{m(i_m-1)}(p), 1);$$

the coordinate in $\widetilde{U}_{i_1i_2\cdots(i_k-1)\cdots i_m}$ is

$$(1, y_{1(i_1+1)}(p), y_{1(i_1+2)}(p), \cdots, \widehat{y_{1i_2}(p)}, \cdots, \widehat{y_{1i_3}(p)}, \cdots, \widehat{y_{1i_m}(p)}, \cdots, y_{1n}(p),$$
$$1, y_{2(i_2+1)}(p), y_{2(i_2+2)}(p), \cdots, \widehat{y_{2i_3}(p)}, \cdots, \widehat{y_{2i_m}(p)}, \cdots, y_{2n}(p),$$
$$1, y_{3(i_3+1)}(p), y_{3(i_3+1)}(p), \cdots, \widehat{y_{3i_4}(p)}, \cdots, \widehat{y_{3i_m}(p)}, \cdots, y_{3n}(p),$$
$$\cdots,$$
$$1, y_{ki_k}(p), y_{k(i_k+1)}(p), \cdots, \widehat{y_{ki_{k+1}}(p)}, \cdots, \widehat{y_{ki_m}(p)}, \cdots, y_{kn}(p),$$
$$\cdots,$$
$$1, y_{m(i_m+1)}(p), y_{m(i_m+2)}(p), \cdots, y_{mn}(p)).$$

So the change of the coordinate is

$$x_{k(i_k-1)}(p) = \frac{1}{y_{ki_k}(p)} \neq 0,\ 1 = \frac{y_{ki_k}(p)}{y_{ki_k}(p)};$$

Then we get

$$W^u(O_{i_1i_2\cdots i_m}) \cap W^s(O_{i_1i_2\cdots(i_k-1)\cdots i_m}) = \widetilde{U}_{i_1i_2\cdots i_m}(x_{k(i_k-1)}; x_{k(i_k-1)} \neq 0)$$

□

By the theorem 2., we can get the following corollery

**Corollery 1.** *Let $O_{i_1i_2\cdots i_m}$, $O_{l_1l_2\cdots l_m}$ be any two different critical points of $f$, with*

$$\mathrm{ind}(O_{i_1i_2\cdots i_m}) - \mathrm{ind}(O_{l_1l_2\cdots l_m}) = 1.$$

*Let $\Gamma(O_{i_1i_2\cdots i_m}, O_{l_1l_2\cdots l_m})$ be the numbers of trajectories connectting $O_{i_1i_2\cdots i_m}$ to $O_{l_1l_2\cdots l_m}$, then*

$$\Gamma(O_{i_1i_2\cdots i_m}, O_{l_1l_2\cdots l_m}) = \begin{cases} 2,\ i_1 = l_1, i_2 = l_2, \cdots, i_{k-1} = l_{k-1}, i_k = l_k + 1, i_{k+1} = l_{k+1}, \cdots, i_m = l_m; \\ 0,\ \textit{otherwise}. \end{cases}$$

*where $k \in 1, 2, \cdots, m$.*

# 7 The homology groups of $G_{n,m}(\mathbb{R})$

Now we choose an orientation of the vector space $E^u(O_{i_1i_2\cdots i_m}) = T_{O_{i_1i_2\cdots i_m}}W^u(O_{i_1i_2\cdots i_m})$ for every critical point $O_{i_1i_2\cdots i_m}$ and denote by $\langle O_{i_1i_2\cdots i_m}\rangle$ the pair consisting of the critical point $O_{i_1i_2\cdots i_m}$ and the orientation.

$$\begin{gathered}\langle O_{i_1i_2\cdots i_m}\rangle = \{O_{i_1i_2\cdots i_m}; (\partial_{11}, \partial_{12}, \cdots, \partial_{1(i_1-1)},\\ \partial_{21}, \partial_{22}, \cdots, \widehat{\partial_{2i_1}}, \cdots, \partial_{2(i_2-1)},\\ \partial_{31}, \partial_{32}, \cdots, \widehat{\partial_{3i_1}}, \cdots, \widehat{\partial_{3i_2}}, \cdots, \partial_{3(i_3-1)},\\ \cdots,\\ \partial_{m1}, \partial_{m2}, \cdots, \widehat{\partial_{mi_1}}, \cdots, \widehat{\partial_{mi_2}}, \cdots, \widehat{\partial_{mi_3}}, \cdots, \widehat{\partial_{mi_{m-1}}}, \cdots, \partial_{m(i_m-1)})\}\end{gathered}$$

By theorem 2., between the critical points $O_{i_1i_2\cdots i_m}$ and $O_{i_1i_2\cdots(i_k-1)\cdots i_m}$, we have two orbits $\varphi^+(t)$ and $\varphi^-(t)$ as following in the coordinate system $\widetilde{U}_{i_1i_2\cdots i_m}$

$$\varphi^+(t) = (0,0,0,\cdots,0,x_{k(i_k-1)}(t),0,\cdots,0,0,0),\ x_{k(i_k-1)}(t) > 0$$

where $\lim_{t\to-\infty} x_{k(i_k-1)}(t) = 0$, $\lim_{t\to-\infty}\varphi^+(t) = O_{i_1i_2\cdots i_m}$;

$$\varphi^-(t) = (0,0,0,\cdots,0,x_{k(i_k-1)}(t),0,\cdots,0,0,0),\ x_{k(i_k-1)}(t) < 0$$

where $\lim_{t\to-\infty} x_{k(i_k-1)}(t) = 0$, $\lim_{t\to-\infty}\varphi^-(t) = O_{i_1i_2\cdots i_m}$; and in the coordinate system $\widetilde{U}_{i_1i_2\cdots(i_k-1)\cdots i_m}$

$$\varphi^+(t) = (0,0,0,\cdots,0,\frac{1}{x_{k(i_k-1)}(t)},0,\cdots,0,0,0),\ x_{k(i_k-1)}(t) > 0$$

where $\lim_{t\to+\infty} x_{k(i_k-1)}(t) = +\infty$, $\lim_{t\to+\infty}\varphi^+(t) = O_{i_1i_2\cdots(i_k-1)\cdots i_m}$;

$$\varphi^-(t) = (0,0,0,\cdots,0,\frac{1}{x_{k(i_k-1)}(t)},0,\cdots,0,0,0),\ x_{k(i_k-1)}(t) < 0$$

where $\lim_{t\to+\infty} x_{k(i_k-1)}(t) = -\infty$, $\lim_{t\to+\infty}\varphi^+(t) = O_{i_1i_2\cdots(i_k-1)\cdots i_m}$.

**Theorem 3.**

$$\begin{gathered}n_{\varphi^+(O_{i_1i_2\cdots i_m},O_{i_1i_2\cdots(i_k-1)\cdots i_m})} = 1,\\ n_{\varphi^-(O_{i_1i_2\cdots i_m},O_{i_1i_2\cdots(i_k-1)\cdots i_m})} = (-1)^{i_k-k}.\ (1\le k\le m)\end{gathered}$$

*Proof.* Because the tangent vector of $\varphi^{\pm}(O_{i_1i_2\cdots i_m}, O_{i_1i_2\cdots(i_k-1)\cdots i_m})$ at $O_{i_1i_2\cdots i_m}$ is $\pm\partial_{k(i_k-1)}$, then $\langle O_{i_1i_2\cdots i_m}\rangle$ induces an orientation on the orthogonal complement $E^u_{\varphi^\pm}(O_{i_1i_2\cdots i_m})$ of $\pm\partial_{k(i_k-1)}$ in $E^u(O_{i_1i_2\cdots i_m})$ as follow

$$\begin{gathered}(\pm\partial_{11}, \partial_{12}, \cdots, \partial_{1(i_1-1)},\\ \partial_{21}, \partial_{22}, \cdots, \widehat{\partial_{2i_1}}, \cdots, \partial_{2(i_2-1)},\\ \partial_{31}, \partial_{32}, \cdots, \widehat{\partial_{3i_1}}, \cdots, \widehat{\partial_{3i_2}}, \cdots, \partial_{3(i_3-1)},\\ \cdots,\\ \partial_{k1}, \partial_{k2}, \cdots, \widehat{\partial_{ki_1}}, \cdots, \widehat{\partial_{ki_2}}, \cdots, \partial_{k(i_k-2)},\end{gathered}$$

$$\cdots,$$

$$\partial_{m1},\partial_{m2},\cdots,\widehat{\partial_{mi_1}},\cdots,\widehat{\partial_{mi_2}},\cdots,\widehat{\partial_{mi_3}},\cdots,\widehat{\partial_{mi_{m-1}}},\cdots,\partial_{m(i_m-1)}).$$

Because

$$\langle O_{i_1i_2\cdots(i_k-1)\cdots i_m}\rangle=\{O_{i_1i_2\cdots(i_k-1)\cdots i_m};(\partial_{11},\partial_{12},\cdots,\partial_{1(i_1-1)},$$

$$\partial_{21},\partial_{22},\cdots,\widehat{\partial_{2i_1}},\cdots,\partial_{2(i_2-1)},$$

$$\partial_{31},\partial_{32},\cdots,\widehat{\partial_{3i_1}},\cdots,\widehat{\partial_{3i_2}},\cdots,\partial_{3(i_3-1)},$$

$$\cdots,$$

$$\partial_{k1},\partial_{k2},\cdots,\widehat{\partial_{ki_1}},\cdots,\widehat{\partial_{ki_2}},\cdots,\partial_{k(i_k-2)},$$

$$\cdots,$$

$$\partial_{m1},\partial_{m2},\cdots,\widehat{\partial_{mi_1}},\cdots,\widehat{\partial_{mi_2}},\cdots,\widehat{\partial_{mi_3}},\cdots,\widehat{\partial_{mi_{m-1}}},\cdots,\partial_{m(i_m-1)})\}$$

to compute $n_{\varphi^\pm(O_{i_1i_2\cdots i_m},O_{i_1i_2\cdots(i_k-1)\cdots i_m})}$, we compute the Jacobian of the coordinate transformation from $\widetilde{U}_{i_1i_2\cdots i_m}$ to $\widetilde{U}_{i_1i_2\cdots(i_k-1)\cdots i_m}$, we denote it by $J_{n_{\varphi^\pm(O_{i_1i_2\cdots i_m},O_{i_1i_2\cdots(i_k-1)\cdots i_m})}}$,

$$J_{n_{\varphi^\pm(O_{i_1i_2\cdots i_m},O_{i_1i_2\cdots(i_k-1)\cdots i_m})}}=$$

$$\mathrm{diag}(1,\cdots,1,\frac{1}{x_{k(i_k-1)}},\cdots,\frac{1}{x_{k(i_k-1)}},-\frac{1}{x^2_{k(i_k-1)}},\frac{1}{x_{k(i_k-1)}},\cdots,\frac{1}{x_{k(i_k-1)}},1,\cdots,1)$$

$$J_{n_{\varphi^\pm(O_{i_1i_2\cdots i_m},O_{i_1i_2\cdots(i_k-1)\cdots i_m})}}(\pm\partial_{11},\partial_{12},\cdots,\partial_{1(i_1-1)},$$

$$\partial_{21},\partial_{22},\cdots,\widehat{\partial_{2i_1}},\cdots,\partial_{2(i_2-1)},$$

$$\partial_{31},\partial_{32},\cdots,\widehat{\partial_{3i_1}},\cdots,\widehat{\partial_{3i_2}},\cdots,\partial_{3(i_3-1)},$$

$$\cdots,$$

$$\partial_{k1},\partial_{k2},\cdots,\widehat{\partial_{ki_1}},\cdots,\widehat{\partial_{ki_2}},\cdots,\partial_{k(i_k-2)},$$

$$\cdots,$$

$$\partial_{m1},\partial_{m2},\cdots,\widehat{\partial_{mi_1}},\cdots,\widehat{\partial_{mi_2}},\cdots,\widehat{\partial_{mi_3}},\cdots,\widehat{\partial_{mi_{m-1}}},\cdots,\partial_{m(i_m-1)})$$

$$=(\pm\partial_{11},\partial_{12},\cdots,\partial_{1(i_1-1)},$$

$$\partial_{21},\partial_{22},\cdots,\widehat{\partial_{2i_1}},\cdots,\partial_{2(i_2-1)},$$

$$\partial_{31},\partial_{32},\cdots,\widehat{\partial_{3i_1}},\cdots,\widehat{\partial_{3i_2}},\cdots,\partial_{3(i_3-1)},$$

$$\cdots,$$

$$\frac{\partial_{k1}}{x_{k(i_k-1)}},\frac{\partial_{k2}}{x_{k(i_k-1)}},\cdots,\widehat{\partial_{ki_1}},\cdots,\widehat{\partial_{ki_2}},\cdots,\frac{\partial_{k(i_k-2)}}{x_{k(i_k-1)}},$$

$$\cdots,$$

$$\partial_{m1},\partial_{m2},\cdots,\widehat{\partial_{mi_1}},\cdots,\widehat{\partial_{mi_2}},\cdots,\widehat{\partial_{mi_3}},\cdots,\widehat{\partial_{mi_{m-1}}},\cdots,\partial_{m(i_m-1)})$$

So we get, if $x_{k(i_k-1)}>0$ then $n_{\varphi^+(O_{i_1i_2\cdots i_m},O_{i_1i_2\cdots(i_k-1)\cdots i_m})}=1$; if $x_{k(i_k-1)}<0$ then $n_{\varphi^-(O_{i_1i_2\cdots i_m},O_{i_1i_2\cdots(i_k-1)\cdots i_m})}=(-1)^{i_k-k}$. □

By Theorem 3., we have

$$n(O_{i_1i_2\cdots i_m}, O_{i_1i_2\cdots(i_k-1)\cdots i_m}) = n_{\varphi^+(O_{i_1i_2\cdots i_m},O_{i_1i_2\cdots(i_k-1)\cdots i_m})} + n_{\varphi^-(O_{i_1i_2\cdots i_m},O_{i_1i_2\cdots(i_k-1)\cdots i_m})}$$

Then we can get the result about Witten's boundary operator

**Lemma 11.**

$$\partial\langle O_{i_1i_2\cdots i_m}\rangle = \sum_k (1+(-1)^{i_k-k})\langle O_{i_1i_2\cdots(i_k-1)\cdots i_m}\rangle,\ (1\le k\le m)$$

*and if $i_1-1, i_2-2, \cdots, i_m-m$ all is odd, then*

$$\partial\langle O_{i_1i_2\cdots i_m}\rangle = 0$$

*Proof.* Because $n_{\varphi^+(O_{i_1i_2\cdots i_m},O_{i_1i_2\cdots(i_k-1)\cdots i_m})} + n_{\varphi^-(O_{i_1i_2\cdots i_m},O_{i_1i_2\cdots(i_k-1)\cdots i_m})} = 1+(-1)^{i_k-k}$ so $n(O_{i_1i_2\cdots i_m}, O_{i_1i_2\cdots(i_k-1)\cdots i_m}) = 1+(-1)^{i_k-k}$, and by the definition of Witten's boundary operator, we get

$$\partial\langle O_{i_1i_2\cdots i_m}\rangle = \sum_k (1+(-1)^{i_k-k})\langle O_{i_1i_2\cdots(i_k-1)\cdots i_m}\rangle.$$

When $i_1-1, i_2-2, \cdots, i_m-m$ all is odd, then $1+(-1)^{i_k-k}=0$ so get the result. □

Now we can give the most important result in this article,

**Theorem 4.** *The homology groups of real Grassmann manifold $G_{n,m}(\mathbb{R})$ with integral coefficients by Witten complex is*

$$H_r(G_{n,m}(\mathbb{R}),\mathbb{Z}) = \bigoplus_k \Big(\bigoplus_{\substack{j_1+j_2+\cdots+j_m=r\\ j_k+1-k=odd}} \mathbb{Z}_2 \oplus \bigoplus_{\substack{j_1+j_2+\cdots+j_m=r\\ j_k+1-k=even}} \mathbb{Z}\Big)$$

*where $k=1,2,\cdots,m$.*

*Proof.* Since Witten chain complex

$$0\to C_{m(n-m)}\to C_{m(n-m)-1}\to\cdots\to C_1\to C_0\to 0$$

where

$$C_r = \bigoplus_{i_1+i_2+\cdots+i_m=r} \mathbb{Z}\langle O_{i_1i_2\cdots i_m}\rangle.$$

By lemma 11., we have

$$\begin{aligned}\partial(C_{r+1}) &= \bigoplus_{i_1+i_2+\cdots+i_m=r+1} \mathbb{Z}\partial\langle O_{i_1i_2\cdots i_m}\rangle\\ &= \bigoplus_{i_1+i_2+\cdots+i_m=r+1} \mathbb{Z}\Big(\sum_k (1+(-1)^{i_k-k})\langle O_{i_1i_2\cdots(i_k-1)\cdots i_m}\rangle\Big);\end{aligned}$$

$\ker\{\partial: C_r \longrightarrow C_{r-1}\} = \bigoplus_{j_1+j_2+\cdots+j_m=r}\mathbb{Z}\langle O_{j_1j_2\cdots j_m}\rangle$, where $j_1-1, j_2-2, \cdots, j_m-m$ all is odd. By the definition of homology groups

$$H_r^W(M,\mathbb{Z}) = \frac{\ker\{\partial: C_r\longrightarrow C_{r-1}\}}{\partial(C_{r+1})},$$

then

$$H_r^W(G_{n,m}(\mathbb{R}),\mathbb{Z}) = \bigoplus_k \Big(\bigoplus_{\substack{j_1+j_2+\cdots+j_m=r\\ j_k+1-k=odd}} \mathbb{Z}_2 \oplus \bigoplus_{\substack{j_1+j_2+\cdots+j_m=r\\ j_k+1-k=even}} \mathbb{Z}\Big)$$

where $k=1,2,\cdots,m$. And because $H_r^W(G_{n,m}(\mathbb{R}),\mathbb{Z}) = H_r(G_{n,m}(\mathbb{R}),\mathbb{Z})$, so we get the result. □

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