

Closed-Form Solution for the Nontrivial Zeros of the Riemann Zeta Function

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(Dated: April 9, 2017)

In the year 2017 it was formally conjectured that if the Bender-Brody-Müller (BBM) Hamiltonian can be shown to be self-adjoint, then the Riemann hypothesis holds true. Herein we discuss the domain and eigenvalues of the Bender-Brody-Müller conjecture.

I. INTRODUCTION

It was recently shown in [1] that the eigenvalues of a Bender-Brody-Müller (BBM) Hamiltonian operator correspond to the nontrivial zeroes of the Riemann zeta function [2]. Although the BBM Hamiltonian is pseudo-Hermitian, it is consistent with the Berry-Keating conjecture [3, 4]. The eigenvalues of the BBM Hamiltonian are conjectured to be the imaginary parts of the nontrivial zeroes of the zeta function

$$\begin{aligned}\zeta(z) &= \sum_{k=1}^{\infty} \frac{1}{k^z} \\ &= \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{\exp(t) - 1} dt.\end{aligned}\tag{1}$$

The idea that the imaginary parts of the zeroes of Eq. (1) are given by a self-adjoint operator was conjectured by Hilbert and Pólya [5]. Formally, Hilbert and Pólya conjectured that if the eigenfunctions of a self-adjoint operator satisfy the boundary conditions $\psi_n(0) = 0 \ \forall \ n$, then the eigenvalues are the nontrivial zeroes of Eq. (1). The BBM Hamiltonian also satisfies the Berry-Keating conjecture, which states that when $\hat{x}$ and $\hat{p}$ commute, the Hamiltonian reduces to the classical $H = 2xp$.

Remark. If there are nontrivial roots of Eq. (1) for which $\Re(z) \neq 1/2$, the corresponding eigenvalues and eigenstates are degenerate [1].

II. STATEMENT OF PROBLEM

A. Bender-Brody-Müller Hamiltonian

Theorem 1. *The eigenvalues of the Hamiltonian*

$$\hat{H} = \frac{1}{1 - e^{-i\hat{p}}} (\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}})\tag{2}$$

are real, where $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$.

Corollary 1.1. [1] *Solutions to the equation $\hat{H}\psi = E\psi$ are given by the Hurwitz zeta function*

$$\begin{aligned}\psi_z(x) &= -\zeta(z, x+1) \\ &= -\sum_{n=0}^{\infty} \frac{1}{(x+1+n)^z}\end{aligned}\tag{3}$$

on the positive half line $x \in \mathbb{R}^+$ with eigenvalues $i(2z-1)$, and $z \in \mathbb{C}$, for the boundary condition $\psi_z(0) = 0$. Moreover, $\Re(z) > 1$, and $\Re(x+1) > 0$. As $-\psi_z(0)$ is the Riemann zeta function, i.e., Eq. (1), this implies that z belongs to the discrete set of zeros of the Riemann zeta function.

Proof. Let $\psi_z(x)$ be an eigenfunction of Eq. (2) with an eigenvalue $\lambda = i(2z-1)$:

$$\hat{H}\psi_z(x) = \lambda\psi_z(x).\tag{4}$$

Then we have the relation

$$\frac{1}{1 - e^{-i\hat{p}}} (\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}})\psi_z(x) = \lambda\psi_z(x).\tag{5}$$

Letting

$$\begin{aligned}\varphi_z(x) &= [1 - \exp(-\partial_x)]\psi_z(x), \\ &= \hat{\Delta}\psi_z(x),\end{aligned}\tag{6}$$

where $\hat{\Delta}\psi_z(x) = \psi_z(x) - \psi_z(x-1)$, and

$$\hat{\Delta} \equiv 1 - \exp(-\partial_x),\tag{7}$$

is a shift operator. Upon inserting Eq. (6) into Eq. (5) with $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$, we obtain

$$[-ix\partial_x - i\partial_x x]\varphi_z(x) = \lambda\varphi_z(x).\tag{8}$$

Then we have

$$\int_{\mathbb{R}^+} (x\partial_x\varphi_z(x))^*\varphi_z(x)dx + \int_{\mathbb{R}^+} (\partial_x x\varphi_z(x))^*\varphi_z(x)dx = -i\lambda^* \int_{\mathbb{R}^+} \varphi_z^*(x)\varphi_z(x)dx.\tag{9}$$

Now we integrate the first term on the LHS of Eq. (9) by parts to obtain

$$\int_{\mathbb{R}^+} x\varphi_z(x)\partial_x\varphi_z^*(x)dx = - \int_{\mathbb{R}^+} \varphi_z^*(x)\varphi_z(x)dx - \int_{\mathbb{R}^+} \varphi_z^*(x)x\frac{d}{dx}(\varphi_z(x))dx,\tag{10}$$

and the second term on the LHS of Eq. (9) by parts to obtain

$$\int_{\mathbb{R}^+} x\varphi_z(x)^*\partial_x\varphi_z(x)dx = - \int_{\mathbb{R}^+} \varphi_z(x)\varphi_z^*(x)dx - \int_{\mathbb{R}^+} \varphi_z(x)x\frac{d}{dx}(\varphi_z^*(x))dx.\tag{11}$$

Upon substituting Eqs. (10) and (11) into Eq. (9), we obtain

$$\int_{\mathbb{R}^+} \varphi_z^*(x)x\frac{d}{dx}(\varphi_z(x))dx + \int_{\mathbb{R}^+} \varphi_z(x)x\frac{d}{dx}(\varphi_z^*(x))dx = (i\lambda^* - 2)N,\tag{12}$$

where

$$N = \int_{\mathbb{R}^+} \varphi_z^*(x)\varphi_z(x)dx.\tag{13}$$

Next, we split $\varphi_z(x)$ into real and imaginary components, such that

$$\varphi = \Re(\varphi_z(x)) + i\Im(\varphi_z(x)),\tag{14}$$

and substitute Eq. (14) into Eq. (12) such that

$$\int_{\mathbb{R}^+} \Re(\varphi_z(x))x\frac{d}{dx}\Re(\varphi_z(x))dx + \int_{\mathbb{R}^+} \Im(\varphi_z(x))x\frac{d}{dx}\Im(\varphi_z(x))dx + N = \frac{i\lambda^*}{2}N.\tag{15}$$

Upon setting $\lambda = i(2z - 1)$, Eq. (15) can be written

$$\int_{\mathbb{R}^+} \Re(\varphi_z(x))x\frac{d}{dx}\Re(\varphi_z(x))dx + \int_{\mathbb{R}^+} \Im(\varphi_z(x))x\frac{d}{dx}\Im(\varphi_z(x))dx + N = \frac{1-2z}{2}N.\tag{16}$$

It can be seen that all terms on the LHS of Eq. (15) are real, thereby verifying Theorem 1.

Q.E.D.

□

Remark. If the Riemann hypothesis is correct [2], the the eigenvalues of Eq. (2) are degenerate [1].

Lemma 1.1. Under the boundary condition $\psi(0) = 0$, the n^{th} eigenstate of Eq. (2) is Eq. (3), and the nontrivial zeros of the Riemann zeta function are given by

$$z_n = \frac{1}{N} \int_{\mathbb{R}^+} \Re(\varphi_n(x))x\frac{d}{dx}\Re(\varphi_n(x))dx + \frac{1}{N} \int_{\mathbb{R}^+} \Im(\varphi_n(x))x\frac{d}{dx}\Im(\varphi_n(x))dx + \frac{3}{2}.\tag{17}$$

Proof. Given that

$$\begin{aligned}
\psi_n(x) &= \hat{\Delta}\psi_n(x) \\
&= \psi_n(x) - \psi_n(x-1) \\
&= -\sum_{n=0}^{\infty} \frac{1}{(x+1+n)^z} + \sum_{n=0}^{\infty} \frac{1}{(x+n)^z},
\end{aligned} \tag{18}$$

the second term on the RHS of Eq. (17) goes to zero, as $\Im(\varphi_n(x)) = 0$. Hence, we are left with

$$z_n = \frac{1}{N} \int_{\mathbb{R}^+} \varphi_n(x) x \frac{d}{dx} \varphi_n(x) dx + \frac{3}{2}. \tag{19}$$

Moreover, it can be seen that

$$\begin{aligned}
x \frac{d}{dx} (\varphi_n(x)) &= x \frac{d}{dx} \psi_n(x) - x \frac{d}{dx} \psi_n(x-1) \\
&= -x \frac{d}{dx} \sum_{n=0}^{\infty} \frac{1}{(x+1+n)^z} + x \frac{d}{dx} \sum_{n=0}^{\infty} \frac{1}{(x+n)^z} \\
&= xz\zeta(z+1, x+1) - xz\zeta(z+1, x).
\end{aligned} \tag{20}$$

Multiplying Eq. (20) by $\varphi_n(x)$, we obtain

$$\begin{aligned}
\varphi_n(x) xz\zeta(z+1, x+1) - \varphi_n(x) xz\zeta(z+1, x) &= \varphi_n(x) [xz\zeta(z+1, x+1) - xz\zeta(z+1, x)] \\
&= -\zeta(z, x+1) xz\zeta(z+1, x+1) \\
&\quad + \zeta(z, x+1) xz\zeta(z+1, x) \\
&\quad + \zeta(z, x) xz\zeta(z+1, x+1) \\
&\quad - \zeta(z, x) xz\zeta(z+1, x).
\end{aligned} \tag{21}$$

From the RHS of Eq. (21), it can be seen that

$$-\int_{\mathbb{R}^+} \zeta(z, x+1) xz\zeta(z+1, x+1) dx = \sum_{n=0}^{\infty} \frac{(n+x+1)^{-2z} (n+2xz+1)}{2(2z-1)} + \text{const} \tag{22}$$

$$-\int_{\mathbb{R}^+} \zeta(z, x) xz\zeta(z+1, x) dx = \sum_{n=0}^{\infty} \frac{(n+x)^{-2z} (n+2xz)}{2z(2z-1)} + \text{const} \tag{23}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^+} \zeta(z, x+1) xz\zeta(z+1, x) + \zeta(z, x) xz\zeta(z+1, x+1) dx \\
&= \sum_{n=0}^{\infty} \frac{((n+x)^{-z} (n+x+1)^{-z} ((n+x)_2 F_1(1, 1-2z, 1-z, n+x+1) - n-2xz))}{(2z-1)} + \text{const},
\end{aligned} \tag{24}$$

where the hypergeometric function is

$${}_2F_1(1, 1-2z, 1-z, n+x+1) = \sum_{n=0}^{\infty} \frac{(1)_n (1-2z)_n}{(1-z)_n} \frac{(n+x+1)^n}{n!}. \tag{25}$$

Since

$$\begin{aligned}
N &= \int_{\mathbb{R}^+} \varphi_n^*(x) \varphi_n(x) dx \\
&= \int_{\mathbb{R}^+} [\psi_n(x) - \psi_n(x-1)]^2 dx \\
&= \int_{\mathbb{R}^+} [\psi_n^2(x) - 2\psi_n(x-1)\psi_n(x) + \psi_n^2(x-1)] dx \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{R}^+} [(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z}] dx \\
&= \frac{\zeta(2z-1, x)}{(1-2z)} + \sum_{n=0}^{\infty} \frac{2(-n-x)^z (n+x)^{-z} (n+x+1)^{1-z} {}_2F_1(1-z, z, 2-z, n+x+1)}{z-1} \\
&\quad + \sum_{n=0}^{\infty} \frac{(n+x+1)^{1-2z}}{(1-2z)}, \quad \Re(z) > 1,
\end{aligned} \tag{26}$$

with the hypergeometric function

$${}_2F_1(1-z, z, 2-z, n+x+1) = \sum_{n=0}^{\infty} \frac{(1-z)_n (z)_n}{(2-z)_n} \frac{(n+x+1)^n}{n!}, \tag{27}$$

Eq. (19) can be rewritten

$$\begin{aligned}
z_n &= \left[\frac{(1-2z)}{\zeta(2z-1, x)} + \sum_{n=0}^{\infty} \frac{z-1}{2(-n-x)^z (n+x)^{-z} (n+x+1)^{1-z} {}_2F_1(1-z, z, 2-z, n+x+1)} \right. \\
&\quad + \sum_{n=0}^{\infty} \frac{(1-2z)}{(n+x+1)^{1-2z}} \left[\sum_{n=0}^{\infty} \frac{(n+x+1)^{-2z} (n+2xz+1)}{2(2z-1)} + \sum_{n=0}^{\infty} \frac{(n+x)^{-2z} (n+2xz)}{2z(2z-1)} \right. \\
&\quad \left. \left. + \sum_{n=0}^{\infty} \frac{((n+x)^{-z} (n+x+1)^{-z} ((n+x){}_2F_1(1, 1-2z, 1-z, n+x+1) - n-2xz))}{(2z-1)} \right] \right] + \frac{3}{2},
\end{aligned} \tag{28}$$

for $\Re(z) > 1$, and $\Re(x+1) > 0$. Upon imposing the boundary condition

$$\begin{aligned}
\psi_n(0) &= - \sum_{n=1}^{\infty} \frac{1}{n^z} \\
&= - \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{\exp(t)-1},
\end{aligned} \tag{29}$$

Eq. (28) are the nontrivial zeros of Eq. (1).

□

B. Convergence

For brevity, let us examine the convergence of the integral representation of the discrete nontrivial zeros of the Riemann zeta function on the positive half line $x \in \mathbb{R}^+$, $z \in \mathbb{C}$, $\Re(z) > 1$, and $\Re(x+1) > 0$. From Eq. (19), the integral representation of the discrete nontrivial zeros of the Riemann zeta function are given by

$$\begin{aligned}
z_n &= - \frac{1}{N} \int_{\mathbb{R}^+} \zeta(z, x+1) x z \zeta(z+1, x+1) dx \\
&\quad - \frac{1}{N} \int_{\mathbb{R}^+} \zeta(z, x) x z \zeta(z+1, x) dx \\
&\quad + \frac{1}{N} \int_{\mathbb{R}^+} \zeta(z, x+1) x z \zeta(z+1, x) + \zeta(z, x) x z \zeta(z+1, x+1) dx + \frac{3}{2},
\end{aligned} \tag{30}$$

where

$$N = \int_{\mathbb{R}^+} \left[(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z} \right] dx. \quad (31)$$

Lemma 1.2. *From the first term on the RHS of Eq. (30), if*

$$\int_0^t \zeta(z, x+1)xz\zeta(z+1, x+1)dx \quad (32)$$

exists for every number $t \geq 0$, then

$$\int_0^\infty \zeta(z, x+1)xz\zeta(z+1, x+1)dx = \lim_{t \rightarrow \infty} \int_0^t \zeta(z, x+1)xz\zeta(z+1, x+1)dx, \quad (33)$$

provided this limit exists as a finite number.

Proof.

$$\int_0^t \zeta(z, x+1)xz\zeta(z+1, x+1)dx = \frac{((n+t+1)^{-2z}((n+1)(\frac{t}{n+1})+1)^{2z} - n - 2tz - 1)}{(2z(2z-1))} \quad (34)$$

From L'Hospital's Rule, we have

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{((n+t+1)^{-2z}((n+1)(\frac{t}{n+1})+1)^{2z} - n - 2tz - 1)}{(2z(2z-1))} \\ &= \lim_{t \rightarrow \infty} \frac{((n+t+1)^{-2z}((n+1)(\frac{t}{n+1})+1)^{2z} - n - 2tz - 1)}{(2z(2z-1))} \cdot \frac{(n+t+1)^{-2z}}{(n+t+1)^{-2z}} \\ &= \lim_{t \rightarrow \infty} \frac{-2z(n+t+1)^{-4z-1}(n(\frac{t}{n+1})+1)^{2z} + (\frac{t}{n+1})^{2z} - n - 4tz + t - 1}{4(1-2z)z^2(n+t+1)^{-2z-1}}. \end{aligned} \quad (35)$$

Upon evaluating Eq. (35) with a series expansion at $t = \infty$, we obtain

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{(-1 - n + t + (1+n)(1 + \frac{t}{1+n})^{2z} - 4tz)}{(2(1+n+t)^{2z}z(-1+2z))} \\ &= \frac{((n+t+1)^{-2z}((n+1)(\frac{t}{n+1})+1)^{2z} - n + t(1-4z) - 1)}{(2z(2z-1))}. \end{aligned} \quad (36)$$

Hence, it can be seen that the first term on the RHS of Eq. (30) is convergent, given that the limit seen in Eq. (35) exists as a finite number as seen in Eq. (36). $\square$

Lemma 1.3. *From the second term on the RHS of Eq. (30), if*

$$\int_0^t \zeta(z, x)xz\zeta(z+1, x)dx \quad (37)$$

exists for every number $t \geq 0$, then

$$\int_0^\infty \zeta(z, x)xz\zeta(z+1, x)dx = \lim_{t \rightarrow \infty} \int_0^t \zeta(z, x)xz\zeta(z+1, x)dx, \quad (38)$$

provided this limit exists as a finite number.

Proof.

$$\int_0^t \zeta(z, x)xz\zeta(z+1, x)dx = -\frac{((n+t)^{-2z}(-n(\frac{n+t}{n})^{2z} + n + 2tz))}{(2(2z-1))} \quad (39)$$

From L'Hospital's Rule, we have

$$\begin{aligned}
& - \lim_{t \rightarrow \infty} \frac{((n+t)^{-2z}(-n(\frac{n+t}{n})^{2z} + n + 2tz))}{(2(2z-1))} \\
&= - \lim_{t \rightarrow \infty} \frac{((n+t)^{-2z}(-n(\frac{n+t}{n})^{2z} + n + 2tz))}{(2(2z-1))} \cdot \frac{(n+t)^{-2z}}{(n+t)^{-2z}} \\
&= - \lim_{t \rightarrow \infty} \frac{(n+t)^{-4z}(-n(\frac{n+t}{n})^{2z} + n + 2tz)}{2(2z-1)(n+t)^{-2z}}
\end{aligned} \tag{40}$$

Upon evaluating Eq. (40) with a series expansion at $t = \infty$, we obtain

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \frac{((n+t)^{-2z}(-n(\frac{n+t}{n})^{2z} + n + 2tz))}{((n+t)^{-2z}(-n(\frac{n+t}{n})^{2z} + n + 2tz))} \\
&= \frac{((n+t)^{-2z}(-n(\frac{n+t}{n})^{2z} + n + 2tz))}{(2(2z-1))}.
\end{aligned} \tag{41}$$

Hence, it can be seen that the second term on the RHS of Eq. (30) is convergent, given that the limit seen in Eq. (40) exists as a finite number as seen in Eq. (41). $\square$

Lemma 1.4. *From the third term on the RHS of Eq. (30), if*

$$\int_0^t \zeta(z, x+1)xz\zeta(z+1, x) + \zeta(z, x)xz\zeta(z+1, x+1)dx \tag{42}$$

exists for every number $t \geq 0$, then

$$\begin{aligned}
& \int_0^\infty \zeta(z, x+1)xz\zeta(z+1, x) + \zeta(z, x)xz\zeta(z+1, x+1)dx \\
&= \lim_{t \rightarrow \infty} \int_0^t \zeta(z, x+1)xz\zeta(z+1, x) + \zeta(z, x)xz\zeta(z+1, x+1)dx,
\end{aligned} \tag{43}$$

provided this limit exists as a finite number.

Proof. From the RHS of Eq. (24) it can be seen that

$$\begin{aligned}
& \int_0^t \zeta(z, x+1)xz\zeta(z+1, x) + \zeta(z, x)xz\zeta(z+1, x+1)dx \\
&= \frac{((n+t)^{-z}(n+t+1)^{-z}((n+t)_2F_1(1, 1-2z, 1-z, n+t+1) - n - 2tz))}{(2z-1)} \\
&- \frac{((n)^{-z}(n+1)^{-z}((n)_2F_1(1, 1-2z, 1-z, n+1) - n))}{(2z-1)}.
\end{aligned} \tag{44}$$

Since the second term on the RHS of Eq. (44) is independent of t , we are only concerned with the limit of the first term on the RHS of Eq. (44). As such, we consider the limit

$$\lim_{t \rightarrow \infty} \frac{((n+t)_2F_1(1, 1-2z, 1-z, n+t+1) - n - 2tz)}{(n+t)^z(n+t+1)^z(2z-1)}. \tag{45}$$

Here, it is useful to employ Gauss' theorem, i.e.,

$${}_2F_1(1, 1-2z, 1-z, n+t+1) = \frac{\Gamma(1-z)\Gamma(z-1)}{\Gamma(-z)\Gamma(z)} \tag{46}$$

where $\Re(z) > 1$, $n = -t$, and

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx \tag{47}$$

is the gamma function. Therefore, Eq. (45) can be written

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{((n+t) \frac{\Gamma(1-z)\Gamma(z-1)}{\Gamma(-z)\Gamma(z)} - n - 2tz)}{(n+t)^z(n+t+1)^z(2z-1)} \\ &= - \lim_{t \rightarrow \infty} \frac{(n+t)^{-z}(n+t+1)^{-z}(n+tz)}{(z-1)}. \end{aligned} \quad (48)$$

Upon evaluating Eq. (48) with a series expansion at $t = \infty$, we obtain

$$\lim_{t \rightarrow \infty} \frac{((n+t) {}_2F_1(1, 1-2z, 1-z, n+t+1) - n - 2tz)}{(n+t)^z(n+t+1)^z(2z-1)} = (n+t)^{-z}(n+t+1)^{-z} \left[-\frac{n}{(z-1)} - \frac{(tz)}{(z-1)} \right]. \quad (49)$$

Hence, it can be seen that the third term on the RHS of Eq. (30) is convergent, given that the limit seen in Eq. (45) exists as a finite number as seen in Eq. (49). $\square$

Finally, we must consider the convergence of the normalization factor N .

Lemma 1.5. *From the first three terms on the RHS of Eq. (30), if*

$$\int_0^t \left[(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z} \right] dx \quad (50)$$

exists for every number $t \geq 0$, then

$$\begin{aligned} & \int_0^\infty \left[(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z} \right] dx \\ &= \lim_{t \rightarrow \infty} \int_0^t \left[(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z} \right] dx. \end{aligned} \quad (51)$$

provided this limit exists as a finite number.

Proof.

$$\begin{aligned} & \int_0^\infty \left[(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z} \right] dx \\ &= \lim_{t \rightarrow \infty} \frac{((n+t+1)^{-2z}((n+1)(\frac{t}{n+1}) + 1)^{2z} - n - t - 1)}{(2z-1)} \\ &+ \lim_{t \rightarrow \infty} \frac{((n+t)^{-2z}(n((\frac{n+t}{n})^{2z} - 1) - t))}{(2z-1)} \\ &+ \lim_{t \rightarrow \infty} \frac{(2(-n-t)^z(n+t)^{-z}(n+t+1)^{1-z} {}_2F_1(1-z, z, 2-z, n+t+1))}{(z-1)} \\ &- \frac{(2(-n)^z(n)^{-z}(n+1)^{1-z} {}_2F_1(1-z, z, 2-z, n+1))}{(z-1)}, \end{aligned} \quad (52)$$

where the last term on the RHS of Eq. (52) omits the limit, as it is independent of t . The limits seen on the RHS of Eq. (52) can be evaluated in a similar manner to those seen in Eqs. (36), (41), and (45), respectively. $\square$

C. Domain of the Bender-Brody-Müller Hamiltonian

For the BBM Hamiltonian operator as given by Eq. (2), the Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^+, dx)$. Moreover, $\hat{p}$ and $\hat{x}$ are self-adjoint operators that act in $\mathcal{H}$. In order to study the domain of the BBM Hamiltonian operator, we first introduce an auxiliary operator $\hat{O}$, such that

$$\hat{O} = \hat{p}\hat{p} + \hat{x}\hat{x}, \quad (53)$$

where $\hat{p}\hat{p} = -\nabla^2$, and $\hat{x}\hat{x} = x^2$. The set of finite linear combinations of Hermite functions is a core of $\hat{O}$, and therefore the Schwartz space $\mathcal{S}$ is also a core of $\hat{O}$.

Lemma 1.6. [6] If φ is in $\mathcal{D}(\hat{O})$, then

$$\|\hat{p}\hat{p}\varphi\|^2 + \|\hat{x}\hat{x}\varphi\|^2 \leq \|\hat{O}\varphi\|^2 + c\|\varphi\|^2. \quad (54)$$

Proof. [6] We estimate φ for a core of $\hat{O}$ via a double commutator to make the estimate [7],

$$\begin{aligned} \hat{O}\hat{O} &= \hat{p}\hat{p}\hat{p}\hat{p} + \hat{x}\hat{x}\hat{x}\hat{x} + \hat{p}\hat{p}\hat{x}\hat{x} + \hat{x}\hat{x}\hat{p}\hat{p} \\ &= \hat{p}\hat{p}\hat{p}\hat{p} + \hat{x}\hat{x}\hat{x}\hat{x} + 2 \sum_{i=1}^n \left[\hat{x}_i \hat{p} \hat{p} \hat{x}_i + [\hat{x}_i, [\hat{x}_i, \hat{p}\hat{p}]] \right] \\ &\geq \hat{p}\hat{p}\hat{p}\hat{p} + \hat{x}\hat{x}\hat{x}\hat{x} - 2n, \end{aligned} \quad (55)$$

Therefore, in Eq. (54) $c = 2n$. $\square$

After rewriting Eq. (8) as

$$[x\partial_x + \partial_x x]\varphi = (1 - 2z)\varphi, \quad (56)$$

then $\hat{p}\hat{p} = x\partial_x$ and $f(\hat{x}) = \partial_x x$ are self-adjoint operators acting in $\mathcal{H} = L^2(\mathbb{R}^+, dx)$. Setting

$$\hat{H} = \hat{p}\hat{p} + f(\hat{x}), \quad (57)$$

defined on

$$\hat{p}\hat{p} : X \bigcap f(\hat{x}) : Y. \quad (58)$$

If $f(\hat{x})$ is local in $\mathcal{H}$, then Eq. (57) is dense and Hermitian.

Theorem 2. The BBM Hamiltonian operator in Eq. (2) is essentially self-adjoint, given that $|\nabla f(\hat{x})| \leq a|\hat{x}| + b$.

The BBM Hamiltonian operator in Eq. (2) is real-valued on the positive half line $\mathbb{R}^+$, after being reduced to Eq. (56). From $|\nabla f(\hat{x})| \leq a|\hat{x}| + b$ we have

$$\begin{aligned} |f(\hat{x})| &\leq \frac{1}{2}\hat{x}\hat{x} + b|\hat{x}| \\ &\leq c\hat{x}\hat{x} + d. \end{aligned} \quad (59)$$

Let us examine the uniqueness.

Proof. As shown in [6], if $\hat{H}$ is Hermitian, and $\hat{O}$ is a positive self-adjoint operator, then $\mathcal{C}$ is a core of $\hat{O}$ such that $\mathcal{C} \subset \mathcal{D}(\hat{H})$. As such,

$$\|(\hat{p}\hat{p} + f(\hat{x}))\varphi\|^2 \leq a\|(\hat{p}\hat{p} + \hat{x}\hat{x})\varphi\|^2 + b\|\varphi\|^2, \quad (60)$$

where $\varphi \in \mathcal{S}$. Since $(1 + \hat{x}\hat{x})\varphi \in L^2$, $f(\hat{x})\varphi \in L^2$. Therefore, $\mathcal{S} \subset \mathcal{D}(\hat{H})$. Moreover, since $f(\hat{x})^2 \leq r\hat{x}\hat{x}\hat{x}\hat{x} + s$,

$$\|f(\hat{x})\varphi\|^2 \leq r\|\hat{x}\hat{x}\varphi\|^2 + s\|\varphi\|^2. \quad (61)$$

As such, from Eq. (54), Eq. (60) is satisfied. If $\varphi \in \mathcal{S}$, then $\nabla(f(\hat{x})\varphi) \in L^2$. Since,

$$\pm i[\hat{H}, \hat{O}] \leq c\hat{O} \quad (62)$$

as quadratic forms on $\mathcal{C}$, we thus have

$$\begin{aligned} \pm i[\hat{H}, \hat{O}] &= \pm i\{[\hat{p}\hat{p}, \hat{x}\hat{x}] + [f(\hat{x}), \hat{p}\hat{p}]\} \\ &= \pm\{2(\hat{p} \cdot \hat{x} + \hat{x} \cdot \hat{p}) - (\hat{p} \cdot \nabla f(\hat{x}) + \nabla f(\hat{x}) \cdot \hat{p})\} \\ &\leq 2(\hat{p}\hat{p} + \hat{x}\hat{x}) + \hat{p}\hat{p} + (\nabla f(\hat{x}))^2 \\ &\leq 2(\hat{p}\hat{p} + \hat{x}\hat{x}) + \hat{p}\hat{p} + 2(a^2\hat{x}\hat{x} + b^2) \\ &\leq c\hat{O}, \end{aligned} \quad (63)$$

for constant c . $\square$

D. Second Quantization

We begin with the Bender-Brody-Müller (BBM) Schrödinger equation

$$-\frac{\hbar}{i} \frac{\partial}{\partial z} = \hat{\Delta}^{-1} \hat{x} \hat{p} \hat{\Delta} \psi(x, z) + \hat{\Delta} \hat{p} \hat{x} \hat{\Delta}^{-1} \psi(x, z), \quad (64)$$

where $\hat{\Delta}$ is given by Eq. (7), $\hat{x} = x$, $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, $x \in \mathbb{R}^+$, and $z \in \mathbb{C}$. Furthermore, let

$$\begin{aligned} \psi_n(x) &= -\zeta(z_n, x+1) \\ &= -\sum_{n=0}^{\infty} \frac{1}{(x+1+n)^z} \end{aligned} \quad (65)$$

be the solution of

$$\left(\hat{\Delta}^{-1} \hat{x} \hat{p} \hat{\Delta} + \hat{\Delta} \hat{p} \hat{x} \hat{\Delta}^{-1} \right) \psi_n(x) = E_n \psi_n(x), \quad (66)$$

where z_n are the nontrivial zeros of the Riemann zeta function given by Eq. (28), $\Re(z) > 1$, and $\Re(x+1) > 0$. Next, we write

$$\psi(x, z) = \sum_n b_n(z) \psi_n(x). \quad (67)$$

From Eq. (64) we find

$$\frac{d}{dz} b_n = -\frac{i}{\hbar} E_n b_n. \quad (68)$$

We now find a Hamiltonian that yields Eq. (68) as the equation of motion. Hence, we take

$$\hat{H} = \int_{\mathbb{R}^+} \psi^*(x, z) \left[\hat{\Delta}^{-1} \hat{x} \hat{p} \hat{\Delta} + \hat{\Delta} \hat{p} \hat{x} \hat{\Delta}^{-1} \right] \psi(x, z) dx \quad (69)$$

as the expectation value. Upon substituting Eq. (67) into Eq. (69) and using Eq. (66) we obtain the harmonic oscillator

$$\hat{H} = \sum_n E_n b_n^* b_n. \quad (70)$$

Taking b_n as an operator, and b_n^* as the adjoint, we obtain the usual properties:

$$\begin{aligned} [\hat{b}_n, \hat{b}_m] &= [\hat{b}_n^\dagger, \hat{b}_m^\dagger] = 0, \\ [\hat{b}_n, \hat{b}_m^\dagger] &= \delta_{nm}. \end{aligned} \quad (71)$$

From the analogous Heisenberg equations of motion,

$$\begin{aligned} -\frac{\hbar}{i} \frac{d}{dz} \hat{b}_n &= [\hat{b}_n, \hat{H}] - \\ &= \sum_m E_m \left(\hat{b}_n \hat{b}_m^\dagger \hat{b}_m - \hat{b}_m^\dagger \hat{b}_m \hat{b}_n \right) \\ &= \sum_m E_m \left(\delta_{nm} \hat{b}_m - \hat{b}_m^\dagger \hat{b}_n \hat{b}_m - \hat{b}_m^\dagger \hat{b}_m \hat{b}_n \right) \\ &= \sum_m E_m \left(\delta_{nm} \hat{b}_m + \hat{b}_m^\dagger \hat{b}_m \hat{b}_n - \hat{b}_m^\dagger \hat{b}_m \hat{b}_n \right) \\ &= E_n \hat{b}_n. \end{aligned} \quad (72)$$

The eigenvalues of $\hat{H}$ are

$$\hat{H} = \sum_n E_n N_n, \quad (73)$$

where $N_n = 0, 1, 2, 3, \dots, \infty$. Since, $E_n = i(2z_n - 1)$, we can rewrite Eq. (73) as

$$\hat{H} = i \sum_n (2z_n - 1) N_n. \quad (74)$$

However, from Eq. (72) it can be seen that

$$-\frac{\hbar}{i} \frac{d}{dz} \hat{b}_n = i(2z_n - 1) \hat{b}_n. \quad (75)$$

As such,

$$\boxed{\frac{d}{dz} \hat{b}_n = \frac{1}{\hbar} (2z_n - 1) \hat{b}_n.} \quad (76)$$

E. $\mathcal{PT}$ -symmetric Bender-Brody-Müller Hamiltonian

Theorem 3. *The eigenvalues of the Hamiltonian*

$$i\hat{H} = \frac{i}{1 - e^{-i\hat{p}}} (\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}}) \quad (77)$$

are imaginary, where $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$.

Corollary 3.1. [1] *Solutions to the equation $i\hat{H}\psi = E\psi$ are given by the Hurwitz zeta function*

$$\begin{aligned} \psi_z(x) &= -\zeta(z, x+1) \\ &= -\sum_{n=0}^{\infty} \frac{1}{(x+1+n)^z} \end{aligned} \quad (78)$$

on the positive half line $\mathbb{R}^+$ with eigenvalues $i(2z - 1)$.

Proof. Let ψ be an eigenfunction of Eq. (77) with an eigenvalue $\lambda = i(2z - 1)$:

$$i\hat{H}\psi = \lambda\psi. \quad (79)$$

Then we have the relation

$$\frac{i}{1 - e^{-i\hat{p}}} (\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}})\psi = \lambda\psi. \quad (80)$$

Letting

$$\varphi = [1 - \exp(-\partial_x)]\psi, \quad (81)$$

and inserting Eq. (81) into Eq. (80) with $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$, we obtain

$$[x\partial_x + \partial_x x]\varphi = \lambda\varphi. \quad (82)$$

Then we have

$$\int (x\partial_x \varphi)^* \varphi dx + \int (\partial_x x \varphi)^* \varphi dx = \lambda^* \int \varphi^* \varphi dx. \quad (83)$$

Now we integrate the first term on the LHS of Eq. (83) by parts to obtain

$$\int x\varphi\partial_x\varphi^* dx = -\int \varphi^*\varphi dx - \int \varphi^* x \frac{d}{dx}(\varphi) dx, \quad (84)$$

and the second term on the LHS of Eq. (83) by parts to obtain

$$\int x\varphi^*\partial_x\varphi dx = -\int \varphi\varphi^* dx - \int \varphi x \frac{d}{dx}(\varphi^*) dx. \quad (85)$$

Upon substituting Eqs. (84) and (85) into Eq. (83), we obtain

$$\int \varphi^* x \frac{d}{dx}(\varphi) dx + \int \varphi x \frac{d}{dx}(\varphi^*) dx = -(\lambda^* + 2)N, \quad (86)$$

where

$$N = \int \varphi^* \varphi dx. \quad (87)$$

Next, we split φ into real and imaginary components, such that

$$\varphi = \varphi_{\Re} + i\varphi_{\Im}, \quad (88)$$

and substitute Eq. (88) into Eq. (86) such that

$$\int \varphi_{\Re} x \frac{d}{dx} \varphi_{\Re} dx + \int \varphi_{\Im} x \frac{d}{dx} \varphi_{\Im} dx + N = -\frac{\lambda^*}{2}. \quad (89)$$

Upon setting $\lambda = i(2z - 1)$, Eq. (89) can be written

$$\int \varphi_{\Re} x \frac{d}{dx} \varphi_{\Re} dx + \int \varphi_{\Im} x \frac{d}{dx} \varphi_{\Im} dx + N = \frac{i(1 - 2z)}{2}. \quad (90)$$

It can be seen that all terms on the LHS of Eq. (89) are real, thereby verifying Theorem 3.

Q.E.D.

□

III. CONCLUSION

In this note, we have discussed the domain and eigenvalues of the BBM Hamiltonian.

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