Draft on a Problem in Euler and Navier-Stokes Equations

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Abstract – A brief draft respect to a problem found in the equations of Euler and Navier-Stokes, whose adequate treatment solves a centennial problem about the solution of these equations and a most correct modeling of fluid movement.

Keywords – Euler equations, Navier-Stokes equations, Eulerian description, Lagrangian description.

§ 1

Motived by my work on Lagrangian and Eulerian descriptions in Euler[1] and Navier-Stokes[2] equations, where I used for velocity's components the relation

(1) $$\begin{cases} \frac{\partial u_i}{\partial x_j} = 0, \ i \neq j, \\ \partial x_i = u_i \partial t \end{cases}$$

because the construction of the non-linear terms $u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z}$ in these equations was based on the 2nd law of Newton, $F = ma$, making (with mass density ρ equal to 1)

(2) $$a = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt},$$

with

(3) $$\begin{cases} \frac{dx}{dt} = u_1 \\ \frac{dy}{dt} = u_2 \\ \frac{dz}{dt} = u_3 \end{cases}$$

I now realize that it is possible, or better said, it is necessary for a more appropriate modeling of fluids in motion, the simultaneous use of both velocities, in the Lagrangian and Eulerian descriptions, in the same equation (Euler equations or Navier-Stokes equations), what we will see in § 3.

The equations (3), writing synthetically as $\frac{dx_i}{dt} = u_i$, with $x_1 \equiv x,\ x_2 \equiv y$, $x_3 \equiv z$, show us that the velocity's component u_i is dependent only of coordinate x_i, regardless of the values of others $x_j,\ j \neq i$, justifying the use of (1).

Following this idea, the original system

$$
(4)\quad \begin{cases} \dfrac{\partial p}{\partial x} + \dfrac{\partial u_1}{\partial t} + u_1\dfrac{\partial u_1}{\partial x} + u_2\dfrac{\partial u_1}{\partial y} + u_3\dfrac{\partial u_1}{\partial z} = \nu\nabla^2 u_1 + \frac{1}{3}\nu\nabla_1(\nabla\cdot u) + f_1 \\ \dfrac{\partial p}{\partial y} + \dfrac{\partial u_2}{\partial t} + u_1\dfrac{\partial u_2}{\partial x} + u_2\dfrac{\partial u_2}{\partial y} + u_3\dfrac{\partial u_2}{\partial z} = \nu\nabla^2 u_2 + \frac{1}{3}\nu\nabla_2(\nabla\cdot u) + f_2 \\ \dfrac{\partial p}{\partial z} + \dfrac{\partial u_3}{\partial t} + u_1\dfrac{\partial u_3}{\partial x} + u_2\dfrac{\partial u_3}{\partial y} + u_3\dfrac{\partial u_3}{\partial z} = \nu\nabla^2 u_3 + \frac{1}{3}\nu\nabla_3(\nabla\cdot u) + f_3 \end{cases}
$$

can be transformed in

$$
(5)\quad \begin{cases} \dfrac{1}{u_1}\dfrac{\partial p}{\partial t} + \dfrac{Du_1}{Dt} = \nu(\nabla^2 u_1)|_t + \frac{1}{3}\nu(\nabla_1(\nabla\cdot u))|_t + f_1|_t \\ \dfrac{1}{u_2}\dfrac{\partial p}{\partial t} + \dfrac{Du_2}{Dt} = \nu(\nabla^2 u_2)|_t + \frac{1}{3}\nu(\nabla_2(\nabla\cdot u))|_t + f_2|_t \\ \dfrac{1}{u_3}\dfrac{\partial p}{\partial t} + \dfrac{Du_3}{Dt} = \nu(\nabla^2 u_3)|_t + \frac{1}{3}\nu(\nabla_3(\nabla\cdot u))|_t + f_3|_t \end{cases}
$$

thus (4) and (5) are equivalent systems, according validity of (2) and (3), since that the partial derivatives of the pressure and velocities were correctly transformed to the variable time, using $\partial x = u_1\partial t$, $\partial y = u_2\partial t$, $\partial z = u_3\partial t$. The nabla and Laplacian operators are considered calculated in Lagrangian formulation, i.e., in the variable time. Likewise for the calculation of $\frac{Du}{Dt}$, following (2), and external force f, using $x = x(t), y = y(t), z = z(t)$. The integration of the system (5) shows that anyone of its equations can be used for solve it, and the results must be equals each other. Then this is a condition to the occurrence of solutions.

We use the following transformations (omitting the use of $|_t$, the calculation at time t of the position (x, y, z) of the moving particle):

$$
(6.1)\quad \frac{\partial u_i}{\partial x_j} = \begin{cases} \dfrac{\partial u_i/\partial t}{\partial x_i/\partial t} = \dfrac{1}{u_i}\dfrac{\partial u_i}{\partial t}, \ i = j \\ 0, \ i \neq j \end{cases}
$$

$$
(6.2)\quad \nabla\cdot u = \sum_{j=1}^{3}\frac{\partial u_j}{\partial x_j} = \sum_{j=1}^{3}\frac{1}{u_j}\frac{\partial u_j}{\partial t}
$$

$$
(6.3)\quad \nabla_i(\nabla\cdot u) = \frac{\partial}{\partial x_i}\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right) = \frac{\partial}{\partial x_i}\frac{\partial u_i}{\partial x_i} = \frac{\partial/\partial t}{\partial x_i/\partial t}\frac{1}{u_i}\frac{\partial u_i}{\partial t} = \frac{1}{u_i^2}\left[-\frac{1}{u_i}\left(\frac{\partial u_i}{\partial t}\right)^2 + \frac{\partial^2 u_i}{\partial t^2}\right]
$$

and

$$
(7.1)\quad \frac{\partial^2 u_i}{\partial x_j^2} = \begin{cases} \dfrac{1}{u_i^2}\left[-\dfrac{1}{u_i}\left(\dfrac{\partial u_i}{\partial t}\right)^2 + \dfrac{\partial^2 u_i}{\partial t^2}\right], \ i = j \\ 0, \ i \neq j \end{cases}
$$

$$(7.2) \qquad \nabla^2 u_i = \frac{\partial^2 u_i}{\partial x_i^2} = \frac{1}{u_i^2}\left[-\frac{1}{u_i}\left(\frac{\partial u_i}{\partial t}\right)^2 + \frac{\partial^2 u_i}{\partial t^2}\right]$$

and thus the system (5) can be integrated, finding the pressure p.

§ 2

Without passing through the Lagrangian formulation, given a velocity $u(x,y,z,t)$ at least two times differentiable with respect to spatial coordinates and one respect to time and an integrable external force $f(x,y,z,t)$, perhaps the better expression for the solution of the equation (4) is

$$(8) \qquad p(x,y,z,t) = \int_L S \cdot dl + \theta(t) = \sum_{i=1}^{3} \int_{P_i^0}^{P_i} S_i dx_i + \theta(t),$$

$$S = (S_1,\ S_2,\ S_3),$$

$$S_i = -\left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}\right) + \nu(\nabla^2 u_i) + \frac{1}{3}\nu\big(\nabla_i(\nabla \cdot u)\big) + f_i,$$

supposing possible the integrations and that the vector $S = -\left[\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right] + \nu\nabla^2 u + \frac{1}{3}\nu\nabla(\nabla \cdot u) + f$ is a gradient function, where it is necessary that

$$(9) \qquad \frac{\partial S_i}{\partial x_j} = \frac{\partial S_j}{\partial x_i}.$$

This is the development of the solution of (4) for the specific path L going parallely (or perpendicularly) to axes X, Y and Z from $(x_1^0, x_2^0, x_3^0) \equiv (x_0, y_0, z_0)$ to $(x_1, x_2, x_3) \equiv (x, y, z)$, since that the solution (8) is valid for any piecewise smooth path L. We can choose $P_1^0 = (x_0, y_0, z_0)$, $P_2^0 = (x, y_0, z_0)$, $P_3^0 = (x, y, z_0)$ for the origin points and $P_1 = (x, y_0, z_0)$, $P_2 = (x, y, z_0)$, $P_3 = (x, y, z)$ for the destination points. $\theta(t)$ is a generic time function, physically and mathematically reasonable, for example with $\theta(0) = 0$ or adjustable for some given condition. Again we have seen that the system of Navier-Stokes equations has no unique solution, only given initial conditions, supposing that there is some solution. We can choose different velocities that have the same initial velocity and also result, in general, in different pressures.

The remark given for the system (5), when used in (4), leads us to the following conclusion: the integration of the system (4), confronting with (5), shows that anyone of its equations can be used for solve it, and the results must be equals each other. Then again this is a condition to the occurrence of solutions, which shows to us the possibility of existence of "breakdown" solutions, as defined in [3].

By other side, using the first condition (1), $\frac{\partial u_i}{\partial x_j} = 0$ if $i \neq j$, due to Lagrangian formulation, where $u_i = \frac{dx_i}{dt}$, the original system (4) is simplified as

$$(10)\qquad \begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} = \frac{4}{3}\nu \frac{\partial^2 u_1}{\partial x^2} + f_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial y} = \frac{4}{3}\nu \frac{\partial^2 u_2}{\partial y^2} + f_2 \\ \frac{\partial p}{\partial z} + \frac{\partial u_3}{\partial t} + u_3 \frac{\partial u_3}{\partial z} = \frac{4}{3}\nu \frac{\partial^2 u_3}{\partial z^2} + f_3 \end{cases}$$

where u_i is a function only of the respective x_i and t, but not x_j if $j \neq i$. When is required the incompressibility condition, $\nabla \cdot u = \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right) = 0$, then the constant $\frac{4}{3}$ in (10) should be replaced by 1.

If the external force has potential, $f = \nabla V$, then the system (10) has solution

$$(11)\qquad \begin{aligned} p &= \sum_{i=1}^{3} \int_{P_i^0}^{P_i} \left[-\left(\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x_i}\right) + \frac{4}{3}\nu \frac{\partial^2 u_i}{\partial x_i^2} + f_i \right] dx_i + \theta(t) \\ &= V + \sum_{i=1}^{3} \int_{x_i^0}^{x_i} \left[-\left(\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x_i}\right) + \frac{4}{3}\nu \frac{\partial^2 u_i}{\partial x_i^2} \right] dx_i + \theta(t), \end{aligned}$$

$V = \int_L f \cdot dl$, which although similar to (8) has the solubility guaranteed by the special functional dependence of the components of the vector u, i.e., $u_i = u_i(x_i, t)$, with $\frac{\partial u_i}{\partial x_j} = 0$ if $i \neq j$, supposing u, its derivatives and f integrable vectors. In this case the vector S described in (8) is always a gradient function, i.e., the relation (9) is satisfied. Note that if f is not an irrotational or gradient vector, i.e., if it does not have a potential, then the system (10), with $u_i = u_i(x_i, t)$, it has no solution, the case of "breakdown" solution in [3].

When the incompressibility condition is imposed ($\nabla \cdot u = 0$) we have, using (1), a small variety of possible solutions for velocity, of the form

$$(12)\qquad u_i(x_i, t) = \alpha_i(t) x_i + \beta_i(t),$$

$\alpha_i, \beta_i \in C^\infty([0, \infty[)$. In this case is valid $\nabla^2 u = 0$, i.e., the system of equations has a solution for velocity independent of viscosity coefficient, equal to Euler equations, and except when $u = 0$ (for some or all $t \geq 0$) we have always $\int_{\mathbb{R}^3} |u|^2\, dxdydz \to \infty$, the occurrence of unbounded or unlimited energy, what is not difficult to see.

§ 3

The system (3), for the sake of mathematical rigor, needs to be replaced by

$$(13) \qquad \begin{cases} \frac{dx}{dt} = u_1(t) \\ \frac{dy}{dt} = u_2(t) \\ \frac{dz}{dt} = u_3(t) \end{cases}$$

emphasizing that the velocity components that appear as the time derivative of the coordinate (x, y, z) are legitimate functions of time, i.e., can be considered as representative of the Lagrangian description, $u_i(t)$, unlike the derivatives from u_i in $\frac{\partial u_i}{\partial t}$, $\frac{\partial u_i}{\partial x_j}$, $\nabla \cdot u$ and $\nabla^2 u_i$, that are in the Eulerian description, function of (x, y, z, t).

Representing the Eulerian velocity and respective components with the letter E indicated as upper index, and the corresponding Lagrangian components with the letter L, the system (4) is rewritten as

$$(14) \quad \begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1^E}{\partial t} + u_1^L \frac{\partial u_1^E}{\partial x} + u_2^L \frac{\partial u_1^E}{\partial y} + u_3^L \frac{\partial u_1^E}{\partial z} = \nu \nabla^2 u_1^E + \frac{1}{3} \nu \nabla_1 (\nabla \cdot u^E) + f_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2^E}{\partial t} + u_1^L \frac{\partial u_2^E}{\partial x} + u_2^L \frac{\partial u_2^E}{\partial y} + u_3^L \frac{\partial u_2^E}{\partial z} = \nu \nabla^2 u_2^E + \frac{1}{3} \nu \nabla_2 (\nabla \cdot u^E) + f_2 \\ \frac{\partial p}{\partial z} + \frac{\partial u_3^E}{\partial t} + u_1^L \frac{\partial u_3^E}{\partial x} + u_2^L \frac{\partial u_3^E}{\partial y} + u_3^L \frac{\partial u_3^E}{\partial z} = \nu \nabla^2 u_3^E + \frac{1}{3} \nu \nabla_3 (\nabla \cdot u^E) + f_3 \end{cases}$$

being the pressure p and external force f implicitly defined in the Eulerian description. A more concise notation for (14) is simply, for $i = 1, 2, 3$,

$$(15) \quad \frac{\partial p}{\partial x_i} + \frac{\partial u_i}{\partial t} + \alpha_1 \frac{\partial u_i}{\partial x} + \alpha_2 \frac{\partial u_i}{\partial y} + \alpha_3 \frac{\partial u_i}{\partial z} = \nu \nabla^2 u_i + \frac{1}{3} \nu \nabla_i (\nabla \cdot u) + f_i,$$

where p, f_i, u and u_i are in Eulerian description and $\alpha_i = \alpha_i(t)$ in Lagrangian description, i.e., $\alpha_i = \frac{dx_i}{dt}$, with the radius vector $r = (x_1, x_2, x_3) = (x, y, z)$ function of time and indicating a motion of a specific particle of fluid.

The equations (14) and (15) shows us that the nonlinear terms disappear, facilitating the obtaining of its solutions, transforming when $\nabla \cdot u = 0$ into a linear and second-order partial differential equation of the parabolic type, already well-studied[4]. If $\nu = 0$ (Euler equations) we have equations of first order, obviously, which is also widely studied[5]. We realize that for each possible value of α_i it is possible to obtain different values of u_i, i.e., there is not an one-one correspondence between α_i and u_i, thus it is convenient choose more easy time functions for the $\alpha_i(t)$, provided that compatible with the physical problem to be studied.

References

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