Why Does Gravity Obey an Inverse Square Law?

This paper uncovers the reason why gravity obeys an inverse square law and not, for example, an inverse cubic law, or any other law with any other power. A relativistic approach, along with the scale law and the Plank force, are the tools I used to derive the Newton's law of universal gravitation. I also show that the approach presented here is, qualitatively, in agreement with Einstein's general relativity field equations.

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1. Introduction

When Isaac Newton, in 1666, formulated his famous law of universal gravitation, he explained how the attractive gravitational force is exerted between any two masses. Newton formulated his law as follows

$$F_G = \frac{G\, m_1 m_2}{r^2} \tag{1.1}$$

Where

F_G = Gravitational force between any two bodies of masses m_1 and m_2 (this force is also known as Newton's law of universal gravitation, force of universal gravitation, gravity, gravity force, force of gravitational attraction, force of gravity, Newtonian force of gravity, force of universal mutual gravitation, etc.)
G = Gravitational constant (also known as constant of gravitation, constant of gravity, gravitational force constant, universal constant of gravity, universal gravitational\ constant, Newtonian gravitational constant, etc.)
m_1 = mass of body or particle 1
m_2 = mass of body or particle 2
r = distance between the centers of body 1 and body 2

We can put this law into words by saying that the attractive force between two bodies is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The proportionality constant, normally denoted *G*, is used to equate the quantities of force, mass and distance used in the equation.

Furthermore, it is an experimental fact that gravity obeys an inverse square law. Now one can ask the following question: why doesn't gravity obey an inverse n-power law instead of an inverse square law?, meaning why doesn't gravity obey a law such as

$$F_G = \frac{G_n m_1 m_2}{r^n} \tag{1.2}$$

Where n can be any real number different from 2. Here I have changed the proportionality constant to G_n to produce the right units. In particular, many people have asked the question: why doesn't gravity obey an inverse cubic law?, meaning a law such as

$$F_G = \frac{G_3 m_1 m_2}{r^3} \tag{1.3}$$

In the remainder of this article I shall derive Newton's law of universal gravitation from first principles proving that the power in equation (1.2) has to be exactly 2. This is

$$n = 2 \tag{1.4}$$

The derivation, which will be carried out in the next section, will involve the scale law, the Planck force and special relativity. Appendix 1 contains the nomenclature used in this paper, Appendix 2 contains the derivation of the Planck force and Appendix 3 contains the derivation of Einsteins' formula of equivalence of mass and energy.

2. The Relativistic Derivation of The Newton's Gravity Law

I shall derive Newton's law of universal gravitation from two laws: the scale law [1] (or scale principle) and the relativistic rest energy of a body. In addition, I shall use the formula for the Planck force. The reason of using the scale law is because it gives the correct answer. So the approach followed here is "à la Feynman". Let us begin by drawing the following scale (or scaling) table

Work	Work	Energy	Energy
W_G	W	E_1	E_2

***Table 2.1**: This scale table is used to derive Newton's law of universal gravitation.*

The meaning of the quantities shown in this table is as follows:

$W_G = F_G\, r$ Unknown quantity which has units of work (The units of work and energy are the same: joules). $F_G = F_G(r)$ is the gravity force and r is the distance between the centers of the two attracting bodies or particles.

$W = F_P\, r$ Work done by the Planck force F_P (constant force) through a displacement equal to r

$E_1 = m_1\, c^2$	Relativistic rest energy of body 1
$E_2 = m_2\, c^2$	Relativistic rest energy of body 2

We assume that the speed of the bodies is much less than the speed of light and that the bodies are spherical. The bodies, of course, can be particles.

For Table 1 to work, the amounts shown must be defined as follows

$$W_G(r) = F_G\, r \tag{2.1}$$

$$W(r) = W = F_P\, r = M_P\, a_P\, r \tag{2.2}$$

$$E_1 = m_1\, c^2 \tag{2.3}$$

$$E_2 = m_2\, c^2 \tag{2.4}$$

From the table we establish the following relation

Scale law	$W_G W = S\, E_1 E_2$ or, equivalently $\frac{W_G}{E_1} = S\,\frac{E_2}{W}$	(2.5 a) (2.5 b)

Where S is the scale factor (or scaling factor). Assuming a scale factor of 1 (The reasons are explained in the conclusion section), we write the equations above as follows

Scale law *(S = 1)*	$W_G W = E_1 E_2$ or, equivalently $\frac{W_G}{E_1} = \frac{E_2}{W}$	(2.5 c) (2.5 d)

Let us review what we have done so far. The scale law, as formulated through equation (2.5 c), states that

The Scale law as a product
The product of the force of gravity, $F_G(r)$, times the distance r, times the work done by the Planck force through that distance; equals the product of the rest energy of body 1 times the rest energy of body 2. (The Planck force is a constant force).

And, equivalently, the scale law, as formulated through equation (2.5 d), states that

The Scale law as a ratio

The ratio between the force of gravity, $F_G(r)$, times the distance *r*, to the rest energy of one of the bodies; equals the ratio between the rest energy of the other body to the work done by the Planck force through the same distance.

Because gravity is a force that varies with the distance between the two bodies, it is important to observe that the product: $F_G(r)\, r$ is not the work done by gravity. However, defining this unknown amount, W_G , as $F_G r$ (whatever the physical meaning turns out to be) is correct because gravity has the property of satisfying equations (2.5 a)/(2.5 b)/(2.5c)/(2.5d) if and only if W_G is defined as: $F_G(r)\, r$ and W is defined as: $W = F_P\, r$. Now it should be clear that I have denoted this quantity W_G simply because its units are units of work (or, equivalently, energy).

Substituting variables: W_G and W by equations (2.1) and (2.2), respectively, we obtain

$$F_G\, r\, F_P\, r = E_1 E_2 \tag{2.6}$$

or

Relativistic gravity law (form 1)

$$F_G = \frac{1}{F_P} \frac{E_1 E_2}{r^2} \tag{2.7}$$

I have used three different methods to find the formula for the Planck force. One of the methods is included in Appendix 2. The other two are included in two different articles [2, 3] that I wrote before. The three methods show that the Planck force is given by

Planck force

$$F_P = \frac{c^4}{G} \tag{2.8}$$

Therefore, using this result, eq. (2.7) becomes

Relativistic gravity law (form 2)

$$F_G = \frac{G}{c^4} \frac{E_1 E_2}{r^2} \tag{2.9}$$

This is the “relativistic” form of Newton's law of gravitation. This law is relativistic because it contains the product of the rest energy of particle 1 $(E_1 = m_1\, c^2)$ times the rest energy of particle 2 $(E_2 = m_2\, c^2)$. The formula is not quantum mechanical because it does not include the Planck's constant, *h*.

Now let us replace the quantities E_1 and E_2 by the second side of equations (2.3) and (2.4), respectively. Thus, we obtain

$$F_G = \frac{G}{c^4} \frac{m_1 c^2 m_2 c^2}{r^2} \tag{2.10}$$

That, after simplification, turns out to be identical to Newton's law of gravitation

Newton's law of universal gravitation

$$F_G = \frac{G\, m_1 m_2}{r^2} \tag{2.11}$$

The form in which Newton formulated his law of universal gravitation does not allow us to appreciate the relativistic nature of this law simply because c^4 in the numerator cancels out with c^4 in the denominator as we have seen. Next, let us consider equation (2.5 d)

The work-energy ratio law of universal gravitation

$$\frac{W_G}{E_1} = \frac{E_2}{W} \tag{2.12}$$

Because equation (2.12) is another form of expressing Newton's law of gravity we need to differentiate it from Newton's classical form given by eq. (2.11). This can be achieved by using a different name for equation (2.12). *'Work'-energy ratio law of universal gravitation* (or simply *work-energy ratio law*) seems to be an appropriate name, so we shall adopt this name. This doesn't mean that we have two different laws, it simply means that we have two different ways of expressing the same physical law. However there is an important difference I should mention here. The *work-energy ratio law of universal gravitation* is more fundamental that that of Newton because it contains the work done by the Planck force: $W = F_P\, r$ through the distance *r*. But what is so special about the Planck force? As we shall see in the next section, the Planck force is a fundamental force because is part of the constant in the field equations of Einstein's general relativity.

The entire process of derivation of Newton's gravity law from relativistic principles is shown in the block diagram of Fig 2.1. It is worth noting that the blue blocks are the "fundamental laws" that allowed us to derive the Newton's gravity law.

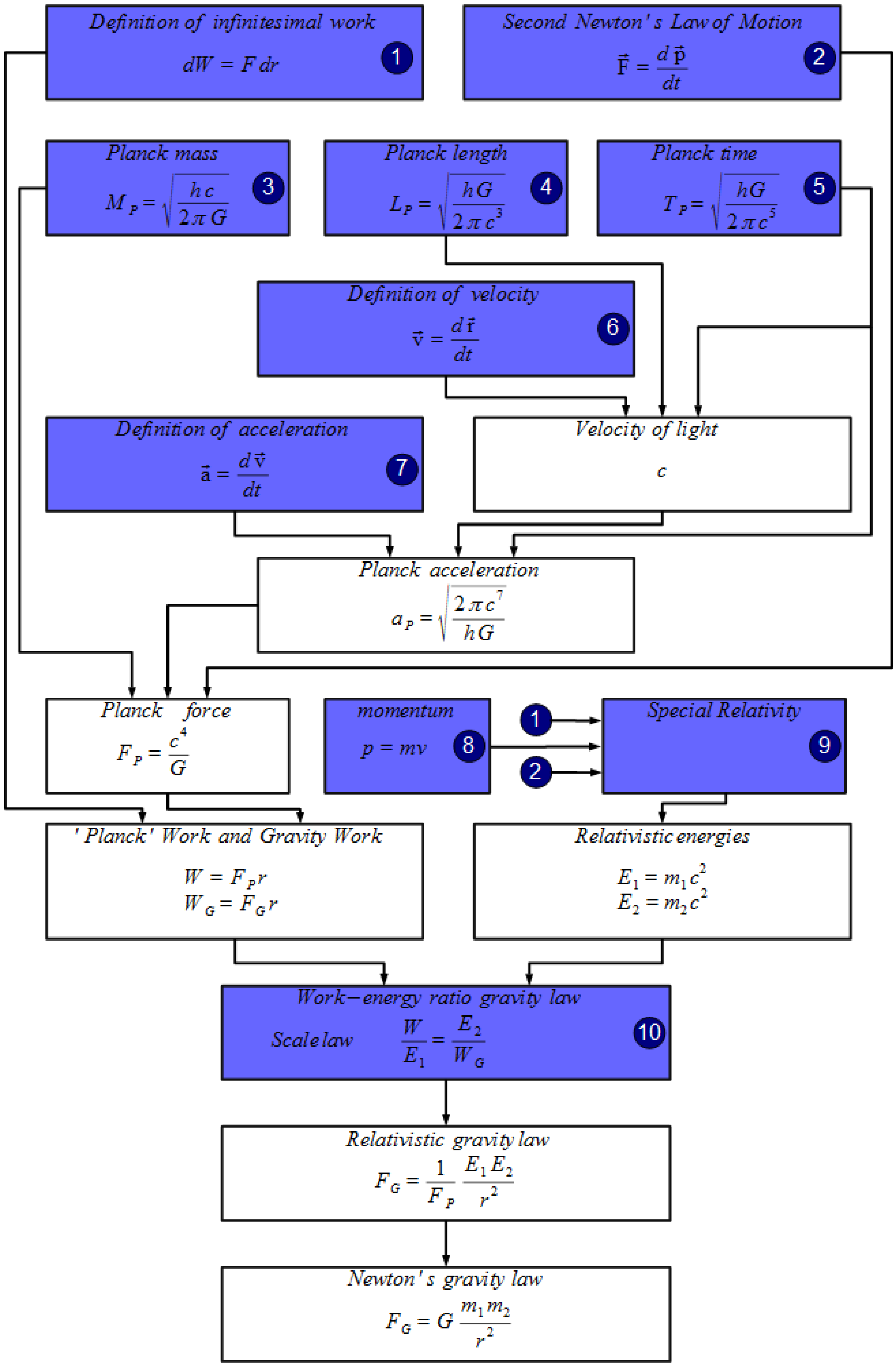

Fig 2.1: *The relativistic derivation of the Newton's gravity law. Note that the Newton's second law of motion also uses the definition of momentum (label 8). Note that, in general, the law labels in this diagram do not correspond to the law numbers in the list given in section 3.*

3. Conclusions

The derivation presented in this paper shows that the *work-energy ratio gravity law* is equivalent to *Newton's law of universal gravitation*. This fact is summarized in the following table

Form of expressing the law	Equivalent Laws	
The 'work'-energy ratio law of universal gravitation (Scale law)	$\frac{W_G}{E_1} = \frac{E_2}{W}$	(3.1)
Relativistic gravity law (form 2)	$F_G = \frac{G}{c^4} \frac{E_1 E_2}{r^2}$	(3.2)
Newton's law of universal gravitation	$F_G = \frac{G\, m_1 m_2}{r^2}$	(3.3)

Table 3.1: *Three different ways of expressing Newton's gravity law.*

One point of significance needs to be addressed here. This point is the fact that the relativistic gravity law given by equation (2.7), which I have copied below,

Relativistic gravity law (form 1)

$$F_G = \frac{1}{F_P} \frac{E_1 E_2}{r^2} \qquad (3.4) = (2.7)$$

contains: (a) the product of two energies and (b) a proportionality constant equal to the inverse of the Planck force. Similarly, the Einstein's field equations contain the stress-energy tensor and a proportionality constant equal to 8π divided by the Planck force. This can be easily seen by looking at the Einstein's field equations

Einstein's field equations

$$R_{ij} - \frac{1}{2} g_{ij} R + \Lambda g_{if} = \frac{8\pi\, G}{c^4} T_{ij} \qquad (3.5)$$

If we replace the constant G/c^4 by $1/F_P$ the field equations transform into

Einstein's field equations in terms of the Planck force

$$R_{ij} - \frac{1}{2} g_{ij} R + \Lambda g_{if} = \frac{8\pi}{F_P} T_{ij} \qquad (3.6)$$

An interesting point I would like to mention here is that Einstein never mentioned that his field equations contained the inverse of the Planck force. Maybe he thought it was too obvious and decided to remain silent about it. In summary, the similarities between equations (2.7)/(3.4) and (3.6) clearly show that the relativistic gravity law is a simplified version of the field equations of general relativity (Einstein showed that Newton's law of gravitation was a special case of general relativity).

I have shown before that because the *work-energy ratio gravity law* is equivalent to the *Newton's law of gravity*, answering the question: why does the work-energy ratio law obeys an inverse square law? is equivalent to asking: why does gravity obeys an inverse square law?. The general answer to the first question, and therefore the general answer to the second one, is that gravity obeys an inverse square law because it must satisfy the following list of laws:

(Law 1) The definition of work
(Law 2) The definition of velocity
(Law 3) The definition of acceleration
(Law 4) The definition of momentum
(Law 5) Newton's second law of motion
(Law 6) The formula for the Planck mass
(Law 7) The formula for the Planck length
(Law 8) The formula for the Planck time
(Law 9) Special Relativity
(Law 10) The scale law

Table 3.2: *List of laws gravity must satisfy.*

Although this conclusion is correct (I have omitted some laws in this list, such as the relativistic mass law, for space reasons. However, this law is included in Fig. 1 inside the block for special relativity and also in Appendix 3) it is too general and hides the specific laws that shape Newton's law of gravitation. Thus, there are two laws that are the most prominent ones: the definition of work and the scale law. This is explained at the end of this section.

One point the reader might see as an objection to this development is the fact that I have assumed that the scale law is correct without any rigorous proof. Despite this fact the reader should consider that there are a large number of examples that indicate that this law is valid. In this case, the main proof is "à la Feynman", meaning that the scale law must be true because it produces the right results as we have seen in this paper. If the scale law were true only for gravity, then I would accept that it would be a very awkward law. However, the scale law can be successfully used to derive many other physical laws as I have shown in other articles I published previously.

Another point to mention is the determination of the scale factor, *S*. In some cases the scale factor can be easily found while in some others it must be determined through a separate analysis. In this case, however, the scale factor of 1 yields the right results. But the reader may ask: is there any other reason for choosing a scale factor of 1?. Yes, there is. The Einstein's total relativistic energy equation, shown in table 3.3, and derived from the scale law, also uses a scale factor of 1 [4].

	In the form of the scale law	Special relativity
Formula for the total relativistic energy [3] (S=1)	$\frac{E+m_0c^2}{pc}=\frac{pc}{E-m_0c^2}$	$E^2=(pc)^2+(m_0c^2)^2$

Table 3.3: *Formula for Einstein's total relativistic energy.*

According to another paper I wrote [5], it seems that all equations expressed as energy ratios have scaling factors of either 1 or -1. Therefore, because gravity is an attractive force, it is natural to choose a scale factor of 1, that after all, it produces the correct result as we already know.

Although all the physical laws listed in table 2 play their role in deriving Newton's law of gravitation, I would like to point out that the definition of work and the scale law are the most "salient", "dominant" or "prominent" ones. We are aware of the specific role of these two laws by looking at eq. (2.6)

$$F_G r F_P r = E_1 E_2 \tag{3.7}$$

Which can be written as

$$F_G F_P r^2 = E_1 E_2 \tag{3.8}$$

Based on these facts, I draw the following conclusion

Conclusion

Gravity obeys an inverse square law because (a) two of the expressions used in this paper are linear in *r*:

$$W_G = F_G r$$
$$W = F_P r$$

and because (b) of the scale law. In this case the scale law dictates a multiplication of two linear expressions in *r*. As a consequence, the result is an expression that depends on *r* squared. Therefore, after some algebra, we find that gravity depends on the inverse of *r* squared.

Thus, the scale law not only provides the full derivation of Newton's law of universal gravitation but also simplifies its understanding.

It is interesting to note that if the definition of work had been different from what it really is and if all other independent physical laws had been identical to ours, then we would be living in a world governed by a different kind of gravity. But this is not all. The law of equivalence of mass and energy would have been different also. The reason for this is that, in deriving this law, we must use the definition of work. (See Appendix 3, eq: (A3.1)).

Finally, I want to point out that when we use the rest masses of the bodies in question instead of their rest energies (this is after simplification), and despite the relativistic approach used here, we still get the classical law of gravitation formulated by Newton and not a relativistic counterpart. This is so because the scale law, as it was formulated in this particular case, is an approximation to a more general law of gravitation: Einstein's General Theory of Relativity.

Appendix 1
Nomenclature

I shall use the following nomenclature for the constants and variables used in this paper

c = speed of light in vacuum
h = Planck's constant
G = Newton's gravitational constant
F_G = Gravitational force between two bodies of masses m_1 and m_2 (this force is also known as force of universal gravitation, gravity, gravity force, force of gravitational attraction, force of gravity, Newtonian force of gravity, force of universal mutual gravitation, etc.)
F = force
n = power of a hypothetical gravity law (real number)
a = acceleration
m_1 = mass of body or particle 1
m_2 = mass of body or particle 2
r = distance between the centres of body 1 and body 2. Also displacement.
dr = infinitesimal distance
L_P = Planck length
T_P = Planck time
F_P = Planck force
a_P = Planck acceleration
M_P = Planck mass
W_G = unknown quantity which has units of work
W = work done by F_P (this is a constant force)
dW = infinitesimal work
E = total relativistic energy
E_1 = rest energy of body 1
E_2 = rest energy of body 2
K = relativistic kinetic energy
S = scale factor or scaling factor
v = velocity
t = time
p = momentum

Appendix 2
Derivation of the Planck Force

There are different methods of deriving the formula for the Planck force. Three of these methods are:

(i) From Newton's second law of motion
(ii) From Coulomb's law. See reference [2]
(iii) From the Heisenberg uncertainty relations. See reference [3]

The Planck force can also be derived from Newton's gravity law, however, we have assumed that we don't know this law because we want to derive it from first principles. Consequently, we cannot use the gravity law to find the expression of the Planck force.

The method I have chosen in this paper is the first one. Thus, we start this derivation from Newton's second law of motion in its "simplified" form

Newton's second law of motion

$$F = m\,a \tag{A2.1}$$

Where we have assumed that the mass of the body or particle, m, does not vary with time. The Planck force is defined as

$$F_P \equiv M_P\, a_P \tag{A2.2}$$

Where M_P is the Planck mass and a_P is the Planck acceleration (I denote Planck acceleration a_P because I use A_P to denote Planck area). The formulas for these quantities are

Planck mass

$$M_P = \sqrt{\frac{h\,c}{2\,\pi\,G}} \tag{A2.3}$$

Planck acceleration

$$a_P = \sqrt{\frac{2\,\pi\,c^7}{h\,G}} \tag{A2.4}$$

Replacing the values of M_P and a_P in eq. (A2.2) by the second side of equations (A2.3) and (A2.4), respectively, we get

$$F_P = \sqrt{\frac{h\,c}{2\,\pi\,G}\,\frac{2\,\pi\,c^7}{h\,G}} \tag{A2.5}$$

And working algebraically we get

$$F_P = \sqrt{\frac{c^8}{G^2}} \tag{A2.6}$$

Which, finally, turns out to be

Planck force

$$F_P = \frac{c^4}{G} \tag{A2.7}$$

Appendix 3
Derivation of the Einstein's Equation of Equivalence of Mass and Energy

The starting point of this derivation is the formula for the kinetic energy of a body or particle

Kinetic energy

$$K = \int_0^r F \, dr \tag{A3.1}$$

This equation means that the work done by a net force on a given system equals the change in kinetic energy of the system.

We also need the Newton's second law of motion

Newton's second law of motion

$$F = \frac{d\,p}{dt} = \frac{d(mv)}{dt} \tag{A3.2}$$

Introducing the last side of eq. (A3.2) into (A3.1) yields

$$K = \int_0^r \frac{d(mv)}{dt} dr \tag{A3.3}$$

But the velocity of the body is

Velocity

$$v = \frac{dr}{dt} \tag{A3.4}$$

Thus, we can change the differential to *d(mv)* and the upper integration limit to *mv*. Mathematically

$$K = \int_0^{mv} v \, d(mv) \tag{A3.5}$$

Now we can introduce the relativistic mass of a body

Relativistic mass

$$m \equiv \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{A3.6}$$

to get

$$K = \int_0^v v\, d\left(\frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}} v \right) \tag{A3.7}$$

We solve the last integral by integrating by parts. The formula is

Integration by parts

$$\int x\, dy = xy - \int y\, dx \tag{A3.8}$$

where

$$x = v \tag{A3.9}$$

$$y = \frac{m_o v}{\sqrt{1-\frac{v^2}{c^2}}} \tag{A3.10}$$

Making the corresponding substitutions into eq. (A3.8) we get

$$K = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}} v^2 - \int_0^v \frac{m_o v\, dv}{\sqrt{1-\frac{v^2}{c^2}}} \tag{A3.11}$$

This integral can be solved applying the following formula

From integrals table
(we use the definite version
of this indefinite integral)

$$\int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C \tag{A3.12}$$

Then we observe that

$$\frac{d}{dv}\left(1-\frac{v^2}{c^2}\right) = -\frac{2v}{c^2} \tag{A3.13}$$

Hence

$$d\left(1-\frac{v^2}{c^2}\right) = -\frac{2v}{c^2}\, dv \tag{A3.14}$$

Because the integral (A3.11) we want to solve has the product $v\, dv$, we solve eq. (A3.14) for this product. This gives

$$v\,dv = -\frac{c^2}{2}\,d\left(1-\frac{v^2}{c^2}\right) \tag{A3.15}$$

Now, in the integral (A3.11), we replace this product with the second side of eq. (A3.15). This gives

$$K = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}\,v^2 + \frac{m_o c^2}{2}\int_0^v \frac{d\left(1-\frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} \tag{A3.16}$$

Now we make the following substitution

$$u = 1-\frac{v^2}{c^2} \tag{A3.17}$$

This gives

$$K = \frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{m_o c^2}{2}\int_0^u \frac{d\,u}{\sqrt{u}} \tag{A3.18}$$

Now we use the integral given by eq. (A3.12) to get the following expression

$$K = \frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{m_o c^2}{2}\left[2\sqrt{1-\frac{v^2}{c^2}}\right]_0^v \tag{A3.19}$$

Which is equivalent to

$$K = \frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_o c^2\left(\sqrt{1-\frac{v^2}{c^2}}-1\right) \tag{A3.20}$$

This means that

$$K = \frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_o c^2\sqrt{1-\frac{v^2}{c^2}} - m_o c^2 \tag{A3.21}$$

We take the common denominator of the two first terms on the second side of this equation to find the following equation

$$K = \frac{m_o v^2 + m_o c^2 - m_o c^2 \dfrac{v^2}{c^2}}{\sqrt{1 - \dfrac{v^2}{c^2}}} - m_0 c^2 \qquad \text{(A3.22)}$$

After simplification we get

$$K = \frac{m_o c^2}{\sqrt{1 - \dfrac{v^2}{c^2}}} - m_0 c^2 \qquad \text{(A3.23)}$$

Now we use the relativistic mass law, eq. (A3.6), to get the final result

Relativistic kinetic energy $$K = m c^2 - m_0 c^2 \qquad \text{(A3.24)}$$

The total relativistic energy is defined as

Total relativistic energy $$E \equiv m c^2 \qquad \text{(A3.25)}$$

And the rest energy is defined as

Rest energy $$E_0 \equiv m_0 c^2 \qquad \text{(A3.26)}$$

REFERENCES

[1] R. A. Frino, *The Scale Law*, viXra: 1405.0270, (2014).
[2] R. A. Frino, *Newton's Law of Universal Gravitation and the Scale Principle* viXra: 1407.0023, (2014).
[3] R. A. Frino, *Derivation of the Planck Force from the Heisenberg Uncertainty Relations,* viXra: 1705.0124, (2014-2017).
[4] R. A. Frino, *Where Do The Laws of Physics Come From?* viXra: 1407.0103, (2014).
[5] R. A. Frino, *On the Scale Factors of Energy Formulas,* viXra: 1705.0199, (2014-2017).