Why Does Gravity Obey an Inverse Square Law?

This paper uncovers the reason why gravity obeys an inverse square law and not, for example, an inverse cubic law, or any other law with any other power. A relativistic approach is sufficient, if not necessary, to understand the answer to this question as the answer is not obvious.

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1. Introduction

When Isaac Newton, in 1666, formulated his famous law of universal gravitation, he explained how the attractive gravitational force is exerted between any two masses. Newton formulated his law as follows

$$F_G = \frac{G\, m_1 m_2}{r^2} \tag{1.1}$$

Where

F_G = Gravitational force between two any bodies of masses m_1 and m_2 (this force is also known as force of universal gravitation, gravity, gravity force, force of gravitational attraction, force of gravity, Newtonian force of gravity, force of universal mutual gravitation, etc.)
G = Gravitational constant (also known as constant of gravitation, constant of gravity, gravitational force constant, universal constant of gravity, universal gravitational\ constant, Newtonian gravitational constant, etc.)
m_1 = mass of body or particle 1
m_2 = mass of body or particle 2
r = distance between the centers of body 1 and body 2

We can put this law into words by saying that the attractive force between two bodies is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The proportionality constant, normally denoted by G, is used to be able to equate the quantities of force, mass and distance used in the equation. It is an experimental fact that gravity obeys an inverse square law. Now one can ask the following question: why doesn't gravity obey an inverse n-power law instead of an inverse square law?, meaning why doesn't gravity obeys a law such as

$$F_G = \frac{G_n m_1 m_2}{r^n} \tag{1.2}$$

Where n can be any real number different from 2. Where I have changed the proportionality constant to G_n to yield the right units. In particular, many people have asked the question: why doesn't gravity obey an inverse cubic law?, meaning a law such as

$$F_G = \frac{G_3 m_1 m_2}{r^3} \tag{1.3}$$

In the remainder of this article I shall prove that the power in equation (1.2) has to be 2, and not any other real number. This is

$$n = 2 \tag{1.4}$$

The answer I have found is not intuitive and so, in order to explain it, I shall use a relativistic approach. This is carried out in the next section. Appendix 1 contains the nomenclature. Appendix 2 contains the derivation of the Planck force. Appendix 3 contains the derivation of Einsteins' formula of equivalence of mass and energy

2. The Relativistic Derivation of The Newton's Gravity Law

I shall derive Newton's law of universal gravitation from the scale law [1]. The reason of using the scale law is because it yields the correct answer. So the approach followed here is "a la Feynman". In order to do that I shall draw the following scale table

Work	Work	Energy	Energy
W_G	W	E_1	E_2

***Table 1**: This scale table (or scaling table) is used to derive Newton's law of universal gravitation.*

$W_G = F_G\, r$	Unknown quantity which has units of work
$W = F_P\, r$	Work done by F_P
$E_1 = m_1\, c^2$	Relativistic energy of body 1
$E_2 = m_2\, c^2$	Relativistic energy of body 2

We assume that the speed of the body is much less than the velocity of light
For table 1 to work, the quantities shown on this table must be defined as follows

$$W_G(r) = F_G r \tag{2.1}$$

$$W(r) = W = F_P r = M_P a_P r \tag{2.2}$$

$$E_1 = m_1 c^2 \tag{2.3}$$

$$E_2 = m_2 c^2 \tag{2.4}$$

From the table we establish the following relationship

Scale law

$$W_G W = E_1 E_2 \tag{2.5}$$

Where we have assumed a scale factor equal to 1 (The reasons are explained in the conclusion section). Because gravity is a force that varies with the distance, *r*, between the two bodies, it is important to observe that $F_G(r)r$ is not the work done by gravity. However, defining this unknown quantity, W_G, as $F_G r$ (whatever the physical meaning turns out to be) is correct because gravity has the property of satisfying eq. (2.5) if and only if W_G is defined this way. Now it should be clear that I have denoted this quantity with W_G simply because its units are units of work. In other words, gravity is a force that satisfies eq. (2.5) when W_G is defined as $W_G(r) \equiv F_G r$ and when W is defined as $W \equiv F_P r$.

Replacing the variables: W_G, W by equations (2.1) and (2.2), respectively, we get

$$F_G r F_P r = E_1 E_2 \tag{2.6}$$

or

$$F_G = \frac{1}{F_P}\frac{E_1 E_2}{r^2} \tag{2.7}$$

In Appendix 2 and in another article I wrote [2] I have shown that the Planck force is given by

Planck force

$$F_P = \frac{c^4}{G} \tag{2.8}$$

Therefore, using this result we may write

Relativistic gravity law

$$F_G = \frac{G}{c^4}\frac{E_1 E_2}{r^2} \tag{2.9}$$

This is the relativistic form of Newton's law of gravity. Thus, this law is relativistic because it contains the product of the rest energy of particle 1 $\left(E_1 = m_1 c^2\right)$ times the

rest energy of particle 2 $\left(E_2 = m_2 c^2\right)$. The formula is not quantum mechanical because the formula for the Planck force does not contain the Planck's constant.

Now we replace the quantities E_1 and E_2 by the second side of equations (2.3) and (2.4) respectively and we get

$$F_G = \frac{G}{c^4}\frac{m_1 c^2 m_2 c^2}{r^2} \tag{2.10}$$

Which, after simplification, turns out to be identical to the Newton's gravity law

Newton's law of universal gravitation

$$F_G = \frac{G m_1 m_2}{r^2} \tag{2.11}$$

The way Newton formulated his law of gravitation does not allow us to appreciate the relativistic nature of this law simply because c^4 in the denominator cancels out with c^4 in the numerator as we have seen. Equation (2.5) can be written as

The work-energy ratio law of universal gravitation

$$\frac{W_G}{E_1} = \frac{E_2}{W} \tag{2.12}$$

Because equation (2.12) is another form of expressing the Newton's law of gravitation we need to differentiate it from the classical Newton's form given by eq. (2.11). This can be achieved by calling equation (2.12): the *'work'-energy ratio law of universal gravitation* (or simply *work-energy ratio law*). This doesn't mean that we have two different laws, it simply means that we have two different forms of expressing the same law. However there is a difference. The *work-energy ratio law of universal gravitation* is more fundamental that the Newton's gravity law because it contains the work done by the Planck force: $W = F_P r$.

The entire process of derivation of Newton's gravity law from relativistic principles is shown in the block diagram of Fig 1. It is worthwhile to observe that the blue blocks are the fundamental laws that allowed us to derive the Newton's gravity law.

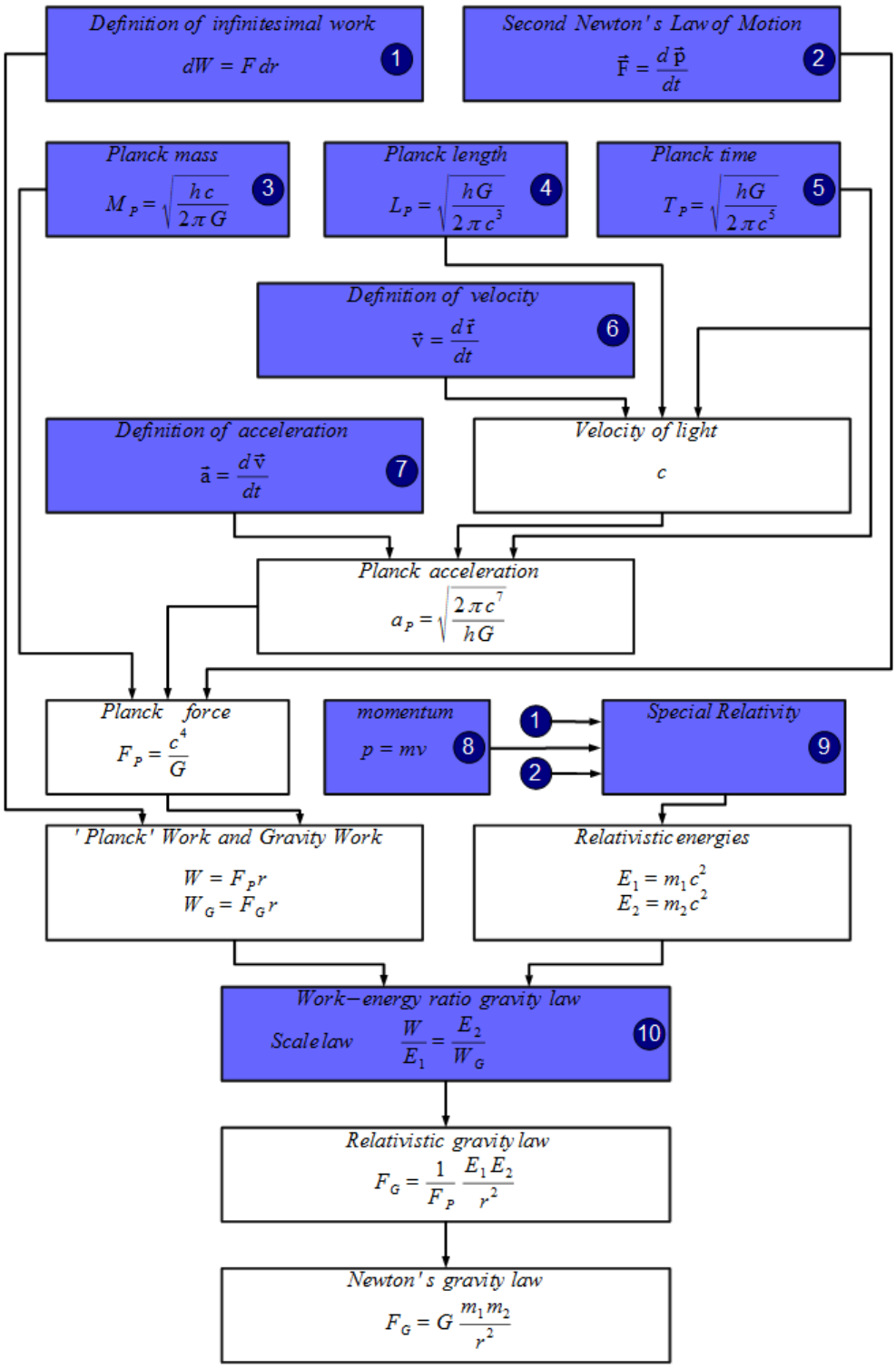

Fig 1: *The relativistic derivation of the Newton's gravity law. Note that the Newton's second law of motion also uses the definition of momentum (label 8). Note that, in general, the law labels in this diagram do not correspond to the law numbers in the list given in section 3.*

3. Conclusions

The derivation presented here showed that the *work-energy ratio gravity law* is equivalent to *Newton's law of universal gravitation*. This fact is summarized in the following table

Form of expressing the law	Equivalent Laws	
The work-energy ratio law of universal gravitation (Scale law)	$\frac{W_G}{E_1}=\frac{E_2}{W}$	(3.1)
Relativistic gravity law	$F_G=\frac{G}{c^4}\frac{E_1E_2}{r^2}$	(3.2)
Newton's law of universal gravitation	$F_G=\frac{G\,m_1m_2}{r^2}$	(3.3)

I have shown before that because the *work-energy ratio gravity law* is equivalent to the *Newton's law of gravity*, answering the question: why does the work-energy ratio law obeys an inverse square law? is equivalent to asking: why does gravity obeys an inverse square law?. The general answer to the first question, and therefore the general answer to the second one, is that gravity must satisfy the following list of laws:

(Law 1) The definition of work
(Law 2) The definition of velocity
(Law 3) The definition of acceleration
(Law 4) The definition of momentum
(Law 5) Newton's second law of motion
(Law 6) The formula for the Planck mass
(Law 7) The formula for the Planck length
(Law 8) The formula for the Planck time
(Law 9) Special Relativity
(Law 10) The scale law

Although this conclusion is correct (I have omitted some laws in this list, such as the relativistic mass law, for space reasons. However, this law is included in Fig. 1 inside the block for Special relativity and also in Appendix 3) it is too general and hides the specific laws that shape Newton's gravity law. Thus, there are two laws that are the most salient ones: the definition of work and the scale law which are shown in blue. This is explained at the end of this section.

One point the reader might see as an objection to this development is the fact that I have assumed that the scale law is correct without any rigorous proof. Despite this fact we should consider that there are a large number of examples that indicate that this law is valid. In this case, the main proof is “a la Feynman”, meaning that the scale law must be true because it produces the right results as we have seen in this paper. If the scale law were true only for gravity, then I would accept that it would be a very awkward law. However, the scale law can be successfully used to derive many other physical laws (if not all) as I have shown in other articles I published previously.

Another point to mention is the determination of the scale factor. In some cases the scale factor can be easily found while in some others it must be determine through a separate analysis. In this case, however, the scale factor of 1 yields the right results. There is still another question to consider: are there any other reasons to choose a scale factor of 1?. The answer is yes. The derivation of Einstein's formula of the total relativistic energy

	In the form of the scale law	Special relativity
Formula of the total relativistic energy	$\frac{E+m_0c^2}{pc}=\frac{pc}{E-m_0c^2}$	$E^2=(pc)^2+(m_0c^2)^2$

based on the scale law also uses a scale factor of 1 [3]. So it seems that all equations dealing with energy have scale factors of 1. Thus it seems natural to choose that scale factor here, that after all, yields the right result.

Although all the laws listed in the table play their roll in the derivation of Newton's gravity law, I want to point out that the definition of work and the scale law are the most "salient" or "dominant" ones. We realize the specific roll of these two laws by looking at eq. (2.6) (which was renumbered as 3.9 in this section)

$$F_G r F_P r = E_1 E_2 \tag{3.9}$$

Which can be written as

$$F_G F_P r^2 = E_1 E_2 \tag{3.10}$$

Based on these facts, I draw the following conclusion

Conclusion

Gravity obeys an inverse square law because (a) two of the expressions used in this article are linear in *r*:

$$W_G = F_G r$$
$$W = F_P r$$

and because of (b) the scale law. The scale law dictates, in this case, a multiplication of two expressions linear in *r*. As a consequence, the result is an expression that depends on *r* square.

Thus, the scale law simplifies the understanding of Newton's gravity law.

Finally, it is worthwhile to observe that if the definition of work would have been different to what it actually is, then we would be living in a world governed by a different kind of gravity law. But, of course to get the exact form of this hypothetical law, we should consider all the relating physical laws that could also be different to ours.

Appendix 1
Nomenclature

I shall use the following nomenclature for the constants and variables used in this paper

c = speed of light in vacuum
h = Planck's constant
G = Newton's gravitational constant
F_G = Gravitational force between two bodies of masses m_1 and m_2 (this force is also known as force of universal gravitation, gravity, gravity force, force of gravitational attraction, force of gravity, Newtonian force of gravity, force of universal mutual gravitation, etc.)
F = force
n = power of a hypothetical gravity law
a = acceleration
m_1 = mass of body or particle 1
m_2 = mass of body or particle 2
r = distance between the centers of body 1 and body 2. Also displacement.
dr = infinitesimal distance
L_P = Planck length
T_P = Planck time
F_P = Planck force
a_P = Planck acceleration
M_P = Planck mass
W_G = unknown quantity which has units of work
W = work done by F_P (this is a constant force)
dW = infinitesimal work
E = total relativistic energy
E_1 = relativistic energy of body 1 (or rest energy if the body is at rest)
E_2 = relativistic energy of body 2 (or rest energy if the body is at rest)
K = relativistic kinetic energy
S = scale factor or scaling factor
v = velocity
t = time
p = momentum
$\circ$ = dot product used in the definition of work

Appendix 2
Derivation of the Planck Force

We start this derivation from Newton's second law of motion in its simplified form

$$F = m\,a \tag{A2.1}$$

Where we have assumed that the mass of the body or particle, *m*, does not vary with time. The Planck force is defined as

$$F_P \equiv M_P\,a_P \tag{A2.2}$$

Where a_P is the Planck acceleration. Therefore equation (2.6) may be rewritten as

$$F_G\,r\,M_P\,a_P\,r = S\,m_1 c^2\,m_2 c^2 \tag{A2.3}$$

Hence, solving for the Planck mass

$$M_P = S\frac{c^4}{F_G\,a_P}\frac{m_1 m_2}{r^2} \tag{A2.4}$$

In virtue of Newton's gravity law and making the scaling factor, *S,* equal to 1, we get

$$M_P = \frac{c^4}{G\,a_P} \tag{A2.5}$$

Therefore the ratio c^4/G must be a force, more precisely it must be the Planck force (Bear in mind that there are other ways of getting this result)

$$F_P = \frac{c^4}{G} \tag{A2.6}$$

Thus, we have derived the formula for the Planck force without using the formula for the Planck mass. Because the Planck mass is the Planck force divided by the Planck acceleration, all we need to do now is to calculate the Planck acceleration. To calculate the Planck acceleration (which is constant) we use the concept of acceleration which is defined as the change in the speed of a given body divided by the time taken to complete the change ($a = dv/dt$ for the expert). Then we define the Planck acceleration as

$$a_P = \frac{c-0}{T_P} = \frac{c}{T_P} \tag{A2.7}$$

Thus, the Planck acceleration is the acceleration of a body or particle when its velocity changes from zero (e.g. from rest) to the speed of light (the fastest speed according to Einstein) in a time interval equal to the Planck time (the shortest time interval). Thus, the Planck acceleration, according to the Einsteinian philosophy, is the fastest possible

acceleration in the universe. Because, according to Einstein, massive bodies cannot travel at the speed of light or faster than this speed, we draw the conclusion that the Planck acceleration cannot be achieved by any massive body or particle. However, it seems there are particles that could achieve this acceleration when they come into existence. These particles are photons (or gravitons if they really exist). If photons are really massless particles, then a photon can achieve the Planck acceleration when is generated (when "is born") provided that it is created in a time equal to the Planck time. But, how do we get the formula for the Planck acceleration? To get the formula for the Planck acceleration we use equation (A2.7) where we replace the Planck time by the corresponding formula. This yields

$$a_P = \frac{c}{\sqrt{\frac{hG}{2\pi c^5}}} \tag{A2.8}$$

Finally, we get the equation for the Planck acceleration

$$a_P = \sqrt{\frac{2\pi c^7}{hG}} \tag{A2.9}$$

Because we have derived the expressions for both the Planck force and the Planck acceleration independently from the Planck mass (without using the formula for the Planck mass), we might define the Planck mass from equation (A2.2) by solving it for the Planck mass. This gives

$$M_P \equiv \frac{F_P}{a_P} \tag{A2.10}$$

Appendix 3
Derivation of the Einstein's Equation of Equivalence of Mass and Energy

The starting point of this derivation is the formula for the kinetic energy of a body or particle

Kinetic energy

$$K = \int_0^r F\,dr \tag{A3.1}$$

We also need the Newton's second law of motion

Newton's second law of motion

$$F = \frac{d\,p}{dt} = \frac{d\,(mv)}{dt} \tag{A3.2}$$

Introducing the last side of eq. (A3.2) into (A3.1) yields

$$K = \int_0^r \frac{d(mv)}{dt}\, dr \tag{A3.3}$$

But the velocity of the body is

Velocity

$$v = \frac{dr}{dt} \tag{A3.4}$$

Thus, we can change the differential to *d(mv)* and the upper integration limit to *mv*. Mathematically

$$K = \int_0^{mv} v\, d(mv) \tag{A3.5}$$

Now we can introduce the relativistic mass of a body

Relativistic mass

$$m \equiv \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{A3.6}$$

to get

$$K = \int_0^v v\, d\left(\frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} v \right) \tag{A3.7}$$

We solve the last integral by integrating by parts. The formula is

Integration by parts

$$\int x\, dy = xy - \int y\, dx \tag{A3.8}$$

where

$$x = v \tag{A3.9}$$

$$y = \frac{m_o v}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{A3.10}$$

Making the corresponding substitutions into eq. (A3.8) we get

$$K = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}} v^2 - \int_0^v \frac{m_o v\, dv}{\sqrt{1-\frac{v^2}{c^2}}} \tag{A3.11}$$

This integral can be solved applying the following formula

From the integrals table (we use the definite version of this indefinite integral)

$$\int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C \tag{A3.12}$$

Then we observe that

$$\frac{d}{dv}\left(1-\frac{v^2}{c^2}\right) = -\frac{2v}{c^2} \tag{A3.13}$$

Hence

$$d\left(1-\frac{v^2}{c^2}\right) = -\frac{2v}{c^2} dv \tag{A3.14}$$

Because the integral (A3.11) we want to solve has the product $v\,dv$, we solve eq. (A3.14) for this product. This gives

$$v\,dv = -\frac{c^2}{2}\, d\left(1-\frac{v^2}{c^2}\right) \tag{A3.15}$$

Now, in the integral (A3.11), we replace this product with the second side of eq. (A3.15). This gives

$$K = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}} v^2 + \frac{m_o c^2}{2} \int_0^v \frac{d\left(1-\frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} \tag{A3.16}$$

Now we make the following substitution

$$u = 1-\frac{v^2}{c^2} \tag{A3.17}$$

This gives

$$K = \frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{m_o c^2}{2}\int_0^v \frac{d\,u}{\sqrt{u}} \tag{A3.18}$$

Now we use the integral given by eq. (A3.12) to get the following expression

$$K = \frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{m_o c^2}{2}\left[2\sqrt{1-\frac{v^2}{c^2}}\right]_0^v \tag{A3.19}$$

Which is equivalent to

$$K = \frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_o c^2\left(\sqrt{1-\frac{v^2}{c^2}}-1\right) \tag{A3.20}$$

This means that

$$K = \frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_o c^2\sqrt{1-\frac{v^2}{c^2}} - m_o c^2 \tag{A3.21}$$

We take the common denominator of the two first terms on the second side of this equation to find the following equation

$$K = \frac{m_o v^2 + m_o c^2 - m_o c^2\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} - m_0 c^2 \tag{A3.22}$$

After simplification we get

$$K = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0 c^2 \tag{A3.23}$$

Now we use the relativistic mass law, eq. (A3.6), to get the final result

Relativistic kinetic energy $$K = m c^2 - m_0 c^2 \tag{A3.24}$$

The total relativistic energy is defined as

Total relativistic energy $$E \equiv m c^2 \tag{A3.25}$$

And the rest energy is defined as

Rest energy $$E_0 \equiv m_0 c^2 \tag{A3.26}$$

REFERENCES

[1] R. A. Frino, *The Scale Law*, viXra: 1405.0270, (2014).
[2] R. A. Frino, *Newton's Law of Universal Gravitation and the Scale Principle* viXra: 1407.0023, (2014).
[3] R. A. Frino, *Where Do The Laws of Physics Come From?* viXra: 1407.0103, (2014).