Towards A Solution of The Riemann Hypothesis

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Abstract In 1859, Georg Friedrich Bernhard Riemann had announced the following conjecture, called Riemann Hypothesis : *The nontrivial roots (zeros)* $s = \sigma + it$ *of the zeta function, defined by:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \textit{ for } \quad \Re(s) > 1$$

have real part $\sigma = \dfrac{1}{2}$.

We give a proof that $\sigma = \dfrac{1}{2}$ using an equivalent statement of the Riemann Hypothesis concerning the Dirichlet η function.

Keywords Zeta function · Non trivial zeros of Riemann zeta function · zeros of Dirichlet eta function inside the critical strip · Definition of limits of real sequences.

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To my wife Wahida, my daughter Sinda and my son Mohamed Mazen
To the memory of my friend Abdelkader Sellal (1950 - 2017)

1 Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [1]:

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Conjecture 1 . Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give a proof that $\sigma = \dfrac{1}{2}$ except at most for a finite number of zeros.

1.1 The function ζ

We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \ \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s. Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [2]:

Theorem 1 . *The function $\zeta(s)$ satisfies the following :*

1. $\zeta(s)$ has no zero for $\Re(s) > 1$;

2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;

3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \ldots$;

4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \dfrac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \dfrac{1}{2}$ is called the critical line. We have also the theorem (see page 16, [3]):

Theorem 2 . *For all $t \in \mathbb{R}$, $\zeta(1 + it) \neq 0$.*

It is also known that the zeros of $\zeta(s)$ inside the critical strip are all complex numbers $\neq 0$ (see page 30 in [3]). Then, we take the critical strip as the region defined as $0 < \Re(s) < 1$.

The Riemann Hypothesis is formulated as:

Conjecture 2 . (The Riemann Hypothesis,[2]) All nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \dfrac{1}{2}$.

In addition to the properties cited by the theorem 1 above, the function $\zeta(s)$ satisfies the functional relation [2] called also the reflection functional equation for $s \in \mathbb{C}\backslash\{0,1\}$:

$$\zeta(1-s) = 2^{1-s}\pi^{-s} cos\frac{s\pi}{2}\Gamma(s)\zeta(s) \tag{1}$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s) > 0$, given by the formula :

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt,$$

So, instead of using the functional given by (1), we will use the one presented by G.H. Hardy [3] namely Dirichlet's eta function [2]:

$$\eta(s) = \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [2].

1.2 A Equivalent statement to the Riemann Hypothesis

Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet function eta which is stated as follows [2]:

Equivalence 3 . *The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :*

$$\eta(s) = \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s), \quad \sigma > 1 \tag{2}$$

that fall in the critical strip $0 < \Re(s) < 1$ *lie on the critical line* $\Re(s) = \dfrac{1}{2}$.

The series (2) is convergent, and represents $(1-2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([3], pages 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0 \tag{3}$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = \rho.e^{i\alpha} \Longrightarrow \rho^2 = \eta(s).\overline{\eta(s)} \tag{4}$$

and $\eta(s) = 0 \Longleftrightarrow \rho = 0$.

2 Proof that the zeros of the function $\eta(s)$ are on the critical line $\Re(s) = \frac{1}{2}$

Proof . We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \Longrightarrow (1 - 2^{1-s})\zeta(s) = 0$. Let us denote $\zeta(s) = A + iB$, and $\theta = tLog2$, then :

$$(1-2^{1-s})\zeta(s) = \left[A(1 - 2^{1-\sigma}cos\theta) - 2^{1-\sigma}Bsin\theta\right] + i\left[B(1 - 2^{1-\sigma}cos\theta) + 2^{1-\sigma}Asin\theta\right]$$

$(1 - 2^{1-s})\zeta(s) = 0$ gives the system:

$$A(1 - 2^{1-\sigma}cos\theta) - 2^{1-\sigma}Bsin\theta = 0$$
$$B(1 - 2^{1-\sigma}cos\theta) + 2^{1-\sigma}Asin\theta = 0$$

As the functions *sin* and *cos* are not equal to 0 simultaneously, we suppose for example that $sin\theta \neq 0$, the first equation of the system gives $B = \dfrac{A(1 - 2^{1-\sigma}cos\theta)}{2^{1-\sigma}sin\theta}$, the second equation is written as :

$$\frac{A(1 - 2^{1-\sigma}cos\theta)}{2^{1-\sigma}sin\theta}(1 - 2^{1-\sigma}cos\theta) + 2^{1-\sigma}Asin\theta = 0 \Longrightarrow A = 0$$

Then, $B = 0 \Longrightarrow \zeta(s) = 0$, it follows that:

$$\boxed{s \textit{ is one zero of } \eta(s) \textit{ that falls in the critical strip, is also one zero of } \zeta(s)} \tag{5}$$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = A + iB = 0 \Longrightarrow \eta(s) = (1 - 2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:

$$\boxed{s \textit{ is one zero of } \zeta(s) \textit{ that falls in the critical strip, is also one zero of } \eta(s)} \tag{6}$$

Let us write the function η:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-sLogn} = \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-(\sigma+it)Logn} =$$
$$= \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-\sigma Logn}.e^{-itLogn}$$
$$= \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-\sigma Logn}(cos(tLogn) - isin(tLogn))$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. Let s be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall \mathcal{N} > n_0, \left| \sum_{n=1}^{\mathcal{N}} \frac{(-1)^{n-1}}{n^s} \right| < \epsilon'$$

We definite the sequence of functions $((\eta_n)_{n\in\mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} - i \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let s be one zero of η that lies in the critical strip, then $\eta(s) = 0$, with $0 < \sigma < 1$. It follows that we can write $lim_{n\longrightarrow+\infty}\eta_n(s) = 0 = \eta(s)$. We obtain:

$$lim_{n\longrightarrow+\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} = 0$$
$$lim_{n\longrightarrow+\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma} = 0$$

Using the definition of the limit of a sequence, we can write:

$$\forall \epsilon_1 > 0 \quad \exists n_r, \forall N > n_r \quad |\Re(\eta(s)_N)| < \epsilon_1 \Longrightarrow |\Re(\eta(s)_N)|^2 < \epsilon_1^2 \tag{7}$$
$$\forall \epsilon_2 > 0 \quad \exists n_i, \forall N > n_i \quad |\Im(\eta(s)_N)| < \epsilon_2 \Longrightarrow |\Im(\eta(s)_N)|^2 < \epsilon_2^2 \tag{8}$$

Then:

$$0 < \sum_{k=1}^{N} \frac{cos^2(tLogk)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'} cos(tLogk).cos(tLogk')}{k^\sigma k'^\sigma} < \epsilon_1^2$$

$$0 < \sum_{k=1}^{N} \frac{sin^2(tLogk)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'} sin(tLogk).sin(tLogk')}{k^\sigma k'^\sigma} < \epsilon_2^2$$

Taking $\epsilon = \epsilon_1 = \epsilon_2$ and $N > max(n_r, n_i)$, we get by making the sum member to member of the last two inequalities:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2 \tag{9}$$

We can write the above equation as :

$$0 < \ \rho_N^2 \ < 2\epsilon^2 \tag{10}$$

or $\rho(s) = 0$.

2.1 Case $\sigma = \dfrac{1}{2} \Longrightarrow 2\sigma = 1$

We suppose that $\sigma = \dfrac{1}{2} \Longrightarrow 2\sigma = 1$. Let's start by recalling Hardy's theorem (1914) ([2], page 24):

Theorem 4 . *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (5-6), it follows the proposition :

Proposition 1 . *There are infinitely many zeros of $\eta(s)$ on the critical line.*

Let $s_j = \frac{1}{2} + it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (9) is written for s_j:

$$0 < \sum_{k=1}^{N} \frac{1}{k} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \longrightarrow +\infty$, the series $\displaystyle\sum_{k=1}^{N} \frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$\boxed{lim_{N \longrightarrow +\infty} \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty} \tag{11}$$

if not, we will have a contradiction with the fact that :

$$lim_{N \longrightarrow +\infty} \sum_{k=1}^{N} (-1)^{k-1} \frac{1}{k^{s_j}} = 0 \Longleftrightarrow \eta(s) \text{ is convergent for } s_j = \frac{1}{2} + it_j$$

Let $s = \sigma + it$ one zero of $\eta(s)$ on the critical line $\Longrightarrow \eta(s) = 0$. We take $\sigma = \dfrac{1}{2}$. Starting from the definition of the limit of sequences, applied above, we obtain:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

with any contradiction. From the proposition (5), it follows that $\zeta(s) = \zeta(\frac{1}{2} + it) = 0$. There are therefore zeros of $\zeta(s)$ on the critical line $\Re(s) = \dfrac{1}{2}$.

2.2 Case $0 < \Re(s) < \dfrac{1}{2}$

2.2.1 Case there is no zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \dfrac{1}{2}$

Using, for this case, point 4 of theorem (1), we deduce that the function $\eta(s)$ has no zeros with $s = \sigma + it$ and $\dfrac{1}{2} < \sigma < 1$. Then, from the proposition (5), it follows that the function $\zeta(s)$ has all its nontrivial zeros only on the critical line $\Re(s) = \sigma = \dfrac{1}{2}$ and the **Riemann Hypothesis is true**.

2.2.2 Case where there are zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \dfrac{1}{2}$

Suppose that there exists $s = \sigma + it$ one zero of $\eta(s)$ or $\eta(s) = 0 \Longrightarrow \rho^2(s) = 0$ with $0 < \sigma < \frac{1}{2} \Longrightarrow s$ lies inside the critical band. We write the equation (9):

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k^{2\sigma}} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}}$$

But $2\sigma < 1$, it follows that $lim_{N \longrightarrow +\infty} \displaystyle\sum_{k=1}^{N} \frac{1}{k^{2\sigma}} \longrightarrow +\infty$ and then, we obtain :

$$\boxed{\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} = -\infty} \qquad (12)$$

2.3 Case $\dfrac{1}{2} < \Re(s) < 1$

Let $s = \sigma + it$ be the zero of $\eta(s)$ in $0 < \Re(s) < \frac{1}{2}$, object of the previous paragraph. According to point 4 of theorem 1, the complex number $s' = 1 - \sigma + it = \sigma' + it'$ with $\sigma' = 1 - \sigma$, $t' = t$ and $\frac{1}{2} < \sigma' < 1$, is also a zero of the function $\eta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, that is $\eta(s') = 0 \Longrightarrow \rho(s') = 0$. By applying (9), we get:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma'}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} < 2\epsilon^2 \qquad (13)$$

As $0 < \sigma < \frac{1}{2} \Longrightarrow 2 > 2\sigma' = 2(1-\sigma) > 1$, then the series $\sum_{k=1}^{N} \dfrac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2 < 1/k^{2\sigma'}$, then :

$$0 < \frac{\pi^2}{6} = \sum_{k=1}^{+\infty} \frac{1}{k^2} \leq \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma'}} = C(\sigma')$$

From the equation (13), it follows that :

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}} = -\frac{C(\sigma')}{2} > -\infty \tag{14}$$

Then, we have the two following cases:

1)- There exists an infinity of complex numbers $s_l = \sigma_l + it_l$ with $\sigma_l \in]0, 1/2[$ such that $\eta(s_l) = 0$. For each s'_l, the left member of the equation (14) above is finite and depends of σ'_l and t'_l, but the right member is a function only of σ'_l. Hence the contradiction, therefore, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \dfrac{1}{2}$. From the equivalent statement (1.2), it follows that **the Riemann hypothesis is verified**.

2)- There is at most a single zero $s_0 = \sigma_0 + it_0$ of $\eta(s)$ with $\sigma_0 \in]0, 1/2[, t_0 > 0$ such that $\eta(s_0) = 0$. Let us call this zero *isolated zero* that we denote by (IZ). Therefore, the interval $]1/2, 1[$ contains a single zero $s'_0 = 1 - \sigma_0 + it_0$. Since the critical line contains an infinity of zeros of $\zeta(s) = 0$, it follows that all the nontrivial zeros of $\zeta(s)$ are on the critical line $\sigma = \dfrac{1}{2}$, except the 4 zeros relative to (IZ). Here too, we deduce that **the Riemann Hypothesis holds** except at most for the (IZ) in the critical band.

3 Conclusion

In summary: for our proofs, we made use of Dirichlet's $\eta(s)$ function:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad s = \sigma + it$$

on the critical band $0 < \Re(s) < 1$, in obtaining:

- $\eta(s)$ vanishes for $0 < \sigma = \Re(s) = \dfrac{1}{2}$;

- $\eta(s)$ does not vanish for $0 < \sigma = \Re(s) < \dfrac{1}{2}$ and $\dfrac{1}{2} < \sigma = \Re(s) < 1$ except at most for the (IZ) (with its symmetrical) inside the critical band.

Consequently, all the zeros of $\eta(s)$ inside the critical band $0 < \Re(s) < 1$ vanish on the critical line $\Re(s) = \dfrac{1}{2}$ except at most at (IZ) (with its symmetrical). Applying the equivalent proposition to the Riemann Hypothesis 1.2, all the nontrivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \dfrac{1}{2}$ except at most at (IZ) (with its symmetrical) inside the critical band. The proof of the Riemann Hypothesis is thus completed.

We therefore announce the important theorem as follows:

Theorem 5 . *All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \dfrac{1}{2}$, except for at most four zeros of respective affixes $(\sigma_0, t_0), (1 - \sigma_0, t_0), (\sigma_0, -t_0), (1 - \sigma_0, -t_0)$, belonging to the critical band.*

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