The formulation and interpretation of the Lorentz transformation

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Abstract. The Lorentz transformation of the special theory of relativity (STR) describes the time dilation between two reference frames moving relative to each other at a constant speed. The present work focuses on the fact that the time observed on the 'other' system depends on the location of the clocks used for time comparisons. First, the paper presents a unified framework for the various 'observational principles', *i.e.* the specification of which clocks to apply; throughout restricting to consider a single space coordinate. Next we suggest that the time can be defined by a two-dimensional variables, involving both 'clock time' and position. This two-dimensional (time, space)-vector intends to alleviate the inherent paradox of time dilation. One suggestion is to formulate this vector as a complex number. Here both the absolute value, as well as the real and imaginary part have simple interpretations. The real part is denoted 'positional time' and is actually invariant under the Lorentz transformation.

Key words: Lorentz transformation; time dilation; symmetry; observational principle; positional time.

1. Introduction

The Lorentz transformation provides the mathematical description of space-time for two reference frames moving relative to each other at constant speed; *i.e.* the situation described in the special theory of relativity (STR). The present work strives to explore the Lorentz transformation, and the interpretation of time dilation. The first part of the paper (Ch. 2-4) presents an abridged and modified version of [1]; discussing essentially well-known results. This includes some critical remarks to the current narrative on time dilation.

The basis for the discussions is the standard theoretical experiment, with two co-ordinate systems (reference frames), K and K' moving relative to each other at speed, v. We investigate the relation between space and time parameters, (x, t) on system K and the corresponding parameters (x', t') on system K', (thus, restricting to have just one space coordinate). The relation is provided by the Lorentz transformation.

A vast amount of literature exists on this topic. As a background we consider a small sample, authored by experienced scientists in the field: books by Bridgman,[2], Giulini, [3] and Mermin, [4]; further, some web pages; Andrew Hamilton, [5] and Pössel, ('Einstein Online'), [6]. These references mainly address non-experts. But it is of interest to see how the main ideas of the STR are presented.

Definition of *simultaneity* becomes crucial when clocks are moving relative to each other. However, we will in this paper restrict to consider events which occur at the same location *and* time. We will assume that each reference frame has a set of calibrated clocks, located at virtually any position. So in principle we can at any position compare the clocks of the two reference frames. Thus, any convention to define simultaneity across reference frames by use of light rays is not considered in the present paper.

The question of *symmetry* is interesting. The STR essentially describes a symmetric situation for the two systems/observers moving relative to each other. And for instance the reference [5] specifies an experiment of complete symmetry, referring to two spaceships moving relative to each other. Other references, however, are not found that explicit. Some will for instance include examples, like the 'travelling twin', *e.g.* [4] - which clearly involves asymmetry - as an example of a 'true time dilation' occurring even under the periods of constant speed.

Actually, I find the sources somewhat ambiguous regarding the very interpretation of *time dilation*. In what sense – and under which precise conditions – is time dilation to be considered a true physical phenomenon? So, how should we interpret the common statement: 'Moving clock goes slower'? Many authors apply the expression 'as seen' by the observer on the other reference system, perhaps indicating that it is an apparent effect, not a physical reality, without elaborating on the interpretation of 'as seen'. However, others stress that 'everything goes slower' on the 'moving system', not only the clocks; truly stating that the time dilation represents a physical reality also under the conditions of STR, (no gravitation *etc*.). On the other side Giulini [3] in Section 3.3 of his book states: 'Moving clocks slow down' is 'potentially misleading and should not be taken too literally'. However, the expression 'not be taken too literally' is not very precise.

Further, I miss a more thorough discussion of the multitude of (time) solutions offered by the Lorentz transformation. As pointed out *e.g.* by Pössel [6] the phenomenon of time dilation stems from the fact that clocks of the two systems have to be compared at least twice, so it cannot be the same two clocks being compared. Mermin [4] states that what 'moves' is decided by which clocks are chosen to be synchronized. This seems to be is in line with the views of the present work: the procedure of clock synchronization and clock comparison decides which reference system has the time moving faster/slower.

In the present paper we specify the various expressions for the time dilation following from the Lorentz transformation. In doing so, we introduce the concept of observational principle; that is, the specification of which clocks to apply for the required time comparisons. A unified framework for these observational principles is given, stressing that time measured for 'the other system' is given by the location where the time reading/comparison is carried out.

So rather than specifying *one* single time dilation formula – which is typically based on a somewhat arbitrary definition of simultaneity – we will in the present work look at the total picture of *all* expressions for time dilation.

This multitude of expressions/solutions represents a kind of paradox: There is a clock synchronization both locally on the reference frames and between reference frames; but nevertheless clock comparisons show different values. So in the second part of this work we investigate alternative formulations of the (time, space) variable. One formulation applies complex numbers, and both the absolute value and the real and imaginary parts have simple interpretations. The real part depends on both the clock reading of the reference frame and the location, and we refer to this as positional time. Further, this real part is invariant under the Lorentz transformation; thus, assigning the same value to the time *at this position* for all reference frame of constant relative speed.

The present work represents the views of a non-physicist. The considerations are essentially of mathematical nature, exploring the model suggested by the Lorentz transformation. However, it is hoped that the results can be of interest also when the physical interpretation of time dilation is discussed.

2. The Lorentz transformation

2.1 Basic assumptions

The main focus of this paper is the Lorentz transformation, describing two reference frames, K and K' passing each other at a relative speed, v. We restrict to consider just one space co-ordinate, (x-axis), with speed of light (c) observed to be constant for all reference frames. We apply 'Newtonian/classical' arguments for events relative to a specific coordinate system, and the model represents a completely idealized situation. Further, the discussions are based on the following specifications:

- There is a complete *symmetry* between the two co-ordinate systems, K and K'; the systems being identical in all respects.
- On both reference frames there is an arbitrary number of identical, synchronized clocks, located at various positions where it is required to measure time. Thus, simultaneity of events occurring on one specific reference frame is decided by comparing clock readings of the relevant clocks.

- When we consider two different reference system, simultaneity of events will mean that they occur at the same time *and* at the same location; *i.e.* clocks at different systems are only compared when they are at the same location.
- At time $t = t' = 0$, clocks at the location $x = 0$ on K and location $x' = 0'$ on K' are synchronized. This represents the definite starting point, from which all events are measured; it is the 'point of initiation'.
- We may choose the *perspective* of one of the systems, say K, and refer to this as the *primary* system. This could in general refer to the system where the (main) experimental equipment is located. In the present context it will simply mean that the time on this system, for any position, x, is given as a constant, $t(x) \equiv t$. Consequently, the time t' observed on the 'other' system at a specific instant will depend on the location, x.

2.2 Formulation of the Lorentz transformation

The Lorentz transformation represents the fundament for our discussion of time dilation. The so-called length contraction along the x-axis equals,

$$k_x = \sqrt{1 - (\frac{v}{c})^2} \tag{1}$$

The Lorentz transformation for time and one space co-ordinate is given by

$$x' = \frac{x - vt}{\sqrt{1 - (\frac{v}{c})^2}} \tag{2}$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - (\frac{v}{c})^2}} \tag{3}$$

This transformation relates simultaneous time readings, t and t' performed at identical locations x on K and x' on K'. We note that an observer at K and an observer at K', who at an instant in time are at the same location - actually passing each other at the moment in question - will agree both on the time t at K and on the time t' at K'; these observed values being specified by the above Lorentz transformation. They will, however, (usually) observe $t \neq t'$.

3 Observational principles and time dilation

3.1 Two main observational principle

We now point to some direct consequences of the Lorentz transformation. A first time comparison is carried out at $t = t' = 0$, for clocks located at $x' = 0$, $x = 0$, respectively. Now we present two different approaches for the next time comparison on the two systems.

- *Principle A*. Assume a set of clocks and observational equipment are located along the x-axis of K, allowing observers to follow and compare time readings with a fixed clock located on K'. At the moment when the clock at position, $x' \equiv 0'$ on K' passes a position, x (on K), we observe both the time t' on the clock on K' and the time t on the clock K. Now at time t on K we have that the position, $x' \equiv 0$ (on K') has arrived at location $x = vt$. Inserting this value for x in the Lorentz transformation, (3), we get the following relation between t and t' at this location:

$$t' = t\sqrt{1 - (\frac{v}{c})^2} \tag{4}$$

This equals the 'standard' time dilation formula.

- *Principle B*. Now there is a single clock at $x \equiv 0$ on *K*, and at this position we make comparisons with clocks on *K′* as they pass along: inserting this value, x=0 in the Lorentz transformation, it directly gives that at time *t* on *K* the time observed on *K′* at this location are related by

$$t' = \frac{1}{\sqrt{1-(\frac{v}{c})^2}} t \tag{5}$$

In conclusion, both $t' = t\sqrt{1-(\frac{v}{c})^2}$ and $t' = t\,/\sqrt{1-(\frac{v}{c})^2}$ are valid results for the 'simultaneous' times observed on *K′* and *K*. The first expression being valid at position, $x = vt$, ($x' = 0'$); the second for position $x = 0, x' = -vt\,/\sqrt{1-(\frac{v}{c})^2} = -vt'$. As observed, both results being a direct consequences of the Lorentz transformation.

Actually, the result could rather be presented as follows. Let A be the reference frame where there are used two clocks for time comparison, letting x_A and t_A be the position and time measurements on this system. Thus, there (at least) two clocks on A, located at $x_A = 0$ and $x_A = vt_A$. Further, the reference frame, B has time, t_B, and utilizes just one clock, located at $x_B = 0$. Then, identical time measurements, (*i.e.* clock readings at same location and same time), will be related by the formula:

$$t_B = t_A\sqrt{1-(\frac{v}{c})^2} \tag{6}$$

This result combines *eqs.* (4) and (5). Two comments are relevant here.

First we stress that observers on both reference frames agree on this result (6). Thus, I find it rather misleading here to apply the phrase 'as seen', (regarding the clock reading on 'the other' system); which is an expression used by several authors. The time readings are objective, and all observers (observational equipment) on the location in question will 'see' the same thing. Actually, the point is that observers at *different locations* will not agree with respect to time observed on the two systems.

Secondly, we have the formulation 'moving clock goes slower'. It is true that an observer on a reference frame (A), observing a *specific clock* passing by (on B), will see this clock going slower, when it is compared to his own clocks. So in one sense one could perhaps say that this confirms the standard phrase 'moving clock goes slower'. However, we could equally well take the perspective of the single clock (B), considering this to be at rest, implying that the clocks on A are moving. The point is definitely not that clocks on B are moving and clocks on A are not. Rather: We start out with two clocks at origin moving relative to each other; and the decision to either compare the clock at $x_A = 0$ or the clock at $x_B = 0$ with a clock on the other system, (a decision that can be interchanged at random), will decide which clock comes out as the fast one, and which is the slow! So, none of the clocks are more moving than the others. It is definitely the observational principle that decides. So I find the talk about the 'moving clock' rather misleading.

3.2 Unidirectional flashes. A third observational principle based on light rays

Now consider the situation that at time $t = t'$=0 there is emitted a flash of light at location $x = 0$ (and/or $x' = 0'$) along the positive *x*-axis. At a later time, *t* on *K* we know that position, $x = ct$ coincides with position, $x' = ct'$ (due to constancy of speed of light). Inserting these values of x and x' in the Lorentz transformation we directly get:

$$t' = \frac{1-v/c}{\sqrt{1-(\frac{v}{c})^2}} t = \frac{\sqrt{1-v/c}}{\sqrt{1+v/c}} t; \quad (\text{for } x = ct) \tag{7}$$

So here we apply (at least) two clocks on both system: one at $x = 0$ and one at $x = ct$ on *K*; and similarly, one at $x' = 0'$ and one at $x' = ct'$ on *K′*. We may refer to this approach for time comparison as observational principle C. This utilizes the constancy of speed of light to give the relation between time, t' on K' and time, t on K at a specific position along the positive x-axis, (*i.e.* at locations, $x = ct$).

So eq. (7) is valid when the light ray is emitted in the positive direction ($x > 0$); *i.e.* c having the same direction as the velocity v, as seen from K. In the negative direction, (choosing $x = -ct$) we similarly get another well-known result

$$t' = \frac{1+v/c}{\sqrt{1-(\frac{v}{c})^2}} t = \frac{\sqrt{1+v/c}}{\sqrt{1-v/c}} t; \quad (\text{for } x = -ct) \tag{8}$$

We refer to this result as being obtained using observational principle C*. So the difference between the observational principles C and C* is not so much the different use of clocks, but rather the direction of the light flash. We note that *eqs.* (7)-(8) give quite other time dilation formulas than those obtained by observational principles A and B discussed above. Use of bidirectional rays is also relevant, but does not give essential new insight and is not discussed here, see [1].

4 A generalization

We have seen that the observational principles, A, B and C give different relations between t and t'. At an instant when the time all over K is found to equal t, one will at different locations on K observe different times, t' on K'.

Now consider a generalization of the principles A, B and C. Taking the perspective of K, we may at time t choose an 'observational position' equal to $x = wt$, (for an arbitrarily chosen w). By inserting $x = wt$ in (3) we directly get that time on K' at this position equals:

$$t' = \frac{1-\frac{vw}{c^2}}{\sqrt{1-(\frac{v}{c})^2}} t \tag{9}$$

Further, we specify a w' so that position $x' = w't'$, at this time corresponds to (has the same location as) $x = wt$. Now, also inserting $x' = w't'$ in (2), we will after some manipulations obtain

$$w' = \frac{w-v}{1-\frac{w}{c}\frac{v}{c}} \tag{10}$$

So equations (9), (10) represent the version of Lorentz transformation, expressed by (t, w) rather than (t, x). Here we see that the results for the observational principles A, B and C directly follows from (9) by choosing $w = 0$, $w = v$ and $w = c$, respectively. Actually, (see (9), we could consider

$$\gamma_{v,w} = (1 - \frac{vw}{c^2}) / \sqrt{1 - (\frac{v}{c})^2} \tag{11}$$

to be the generalized time dilation factor, valid for any observational principle, (any $w=x/t$). Note that we should not think of w as a velocity, rather a way to specify a position $x=wt$, representing the fixed location of the clock applied at time t.

In addition, now having these general expressions, (9), (10), we could ask which value of w and w' would results in $t = t'$. It is easily derived that this equality is obtained by choosing

$$w = w_0 = \frac{c^2}{v}\left(1 - \sqrt{1 - \left(\frac{v}{c}\right)^2}\right) = \frac{v}{1+\sqrt{1-(\frac{v}{c})^2}} \tag{12}$$

also $w' = -w_0$. This means that if we consistently consider the positions where simultaneously $x = w_0 t$ and $x' = -w_0 t'$, then no time dilation will be observed in these positions. Since $x' = -x$, we consider it to be the midpoint between the origins of the two reference frames; in total providing a nice symmetry.

We refer to this choice, $x = w_0 t$, as observational principle D. Observe that when we choose this observational principle - which is symmetric with respect to the two reference frames - then absolutely everything is symmetric, and so we get $t' = t$. Thus, it is tempting to state that in cases where $t' \neq t$ this is caused by applying a non-symmetric observational principle, everything, except the observational principle being symmetric.

Figure 1 below illustrates eq. (3), (or equivalently (9)), giving t' as a linear, decreasing function of x, when t is fixed. It presents a total picture of the relation between t and t', as obtained by the Lorentz transformation. Thus, time, t' is uniquely given by the position, x on K where it is observed, and we might thus refer to *positional* (*i.e.* location specific) time.

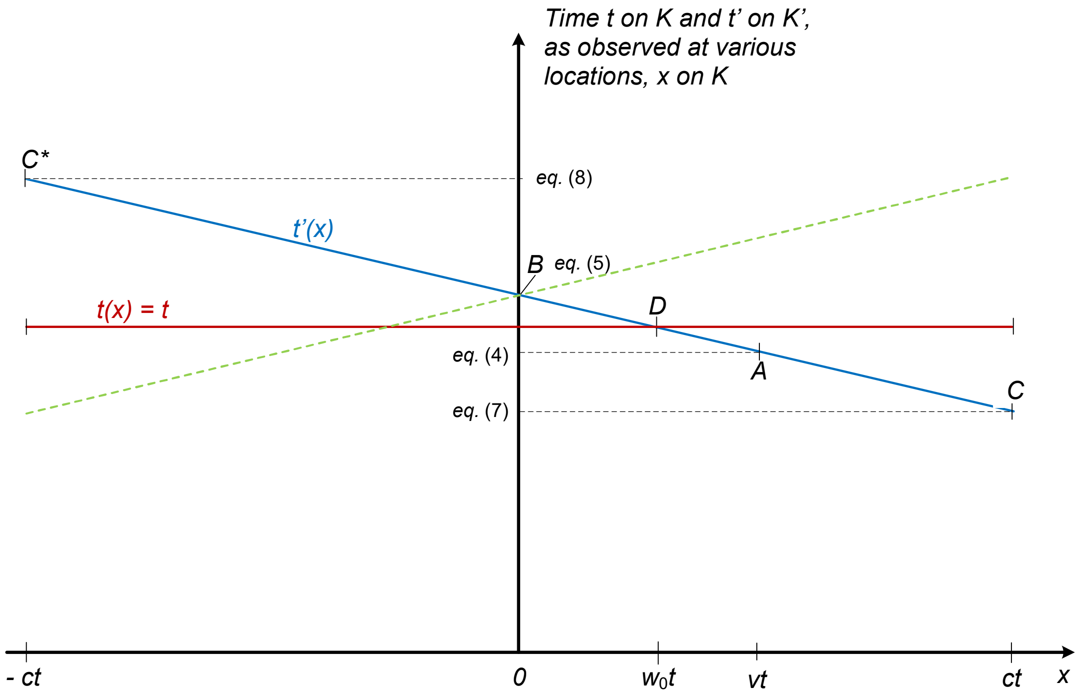

Figure 1 Time, $t' = t'(x)$, (blue) on K' at different locations, $x = wt$, on K, when the time on K equals t, (red); *i.e.* perspective of K. Various observational principles specified along the blue line, $t'(x)$. The dotted green line gives values of, $t' = t'(x)$, when v is replaced by $-v$.

As a further illustration we have in the figure included a dotted green line, representing time, t' - as measured from position x on K - on a system moving at speed, $-v$. We see that the blue and green lines corresponding to v and $-v$, respectively, form a 'bow tie' with a knot in observational principle B. This midpoint of the 'bow tie' is shifted upwards relative to the red line, t, with a factor $1/\sqrt{1-\left(\frac{v}{c}\right)^2}$, *cf. eq* (5). Actually, if we from this figure should suggest an 'overall' (average) time dilation factor for the 'moving system', then this factor could seem quite appropriate; but this would rather tell that *on the average* 'time on the moving system goes *faster'*. As previously noted we would actually choose not to use the term 'moving system', but rather the term 'secondary system', (as opposed to the 'primary system' having $t(x) \equiv t$).

In summary the relations presented in Figure 1 seem rather fundamental for the interpretation of relative time. Being a direct consequence of the Lorentz transformation. This is of course well-known. In spite of this, many authors focus on the time dilation as given by equation (4); representing a rather narrow description of the phenomenon of time dilation.

5 Transformations of the state variables

The above discussion demonstrates the problem we have in describing time by the single parameter, t (the 'clock time'). All clocks on both reference frames are synchronized to give this time. But when we compare clocks across reference frames, there is just one location where the clock readings are equal. We consider this a paradox, and will now investigate whether the introduction of an additional time dimension could alleviate this problem. Or rather, we investigate transformations of the (time, space) vector, but having a focus on time.

5.1 The Lorentz transformation expressed by time variables

First introduce

$$t_c = x/c, \tag{13}$$

and similarly $t_c'=x'/c$. This is the time required for a light flash to go from position 0 to x on K. We now will formulate the Lorentz transformation in terms of t and t_c (rather than t and x). At the same time we replace v with an angle, θ, given by $\sin\theta = v/c$; implying that also

$$\cos\theta = \sqrt{1-(\frac{v}{c})^2}$$

Now the Lorentz transformation (2), (3), can be formulated as follows:

$$t_c' \cdot \cos\theta = t_c - t \cdot \sin\theta \tag{14}$$

$$t' \cdot \cos\theta = t - t_c \cdot \sin\theta \tag{15}$$

A first observation is that this relation is completely symmetric in the two variables, t, t_c. And this version of the Lorentz transformation formulates relations between two aspects of time. Actually we could postulate that t and t_c represents two dimensions of what we can refer to as 'time'. The argument for investigating this concept of a multidimensional time is that we find it hard to give a consistent description of time by a single parameter, when reference frames are moving relative to each other.

How could we interpret these two dimensions of time? Here t represents the 'clock time' of a specific reference frame (K). In this sense it is a 'local' time. All clocks on K are synchronized, and locally on K show this time in a consistent way. To achieve this synchronization - and internal simultaneity on K - one utilizes two-ways beams of light.

The time dimension, t_c, however, is based on one-way beams of light. This dimension seems essential in order to relate the time on different reference frames. It will secures the features that one way beams of light have speed, c, seen from all reference frames.

So now we suggest – in order to give a richer description of time – that one should utilize a time *vector*, (t, t_c). This will give precise conditions for defining simultaneity, as this will correspond to having equality of both clock time, t and time t_c, where t_c corresponds to location, x. So this dependence on location is a view utilized also in the first part of the paper.

In this time space, we could draw any 'trace of time', corresponding to a specific event. Both dimensions, (t, t_c) will increase as the event evolves. However, such a multidimensional time will not give a strict ordering of time. For two different time vectors it is no always possible to state that one is strictly 'greater' than the other.

In the version (14), (15) of the Lorentz transformation it is quite easy to realize that this includes a rotation of the (t, t_c)-plane with the angle, θ around the origin, in addition to some stretching and compression. We do not follow up on this aspect here.

5.2 A transformation of the time vector

Now observe that the above version of the Lorentz transformation is simplified by performing a transformation. We introduce the time parameters, t_1, t_2,

$$t_1 = t - t_c = t(1 - w/c) \tag{16}$$

$$t_2 = t + t_c = t(1 + w/c) \tag{17}$$

Here alternative expressions are obtained by also utilizing the identities *x = wt, x'=w't'* (as defined in Chapter 4). The times t_1' and t_2' are defined analogously. The Lorentz transformation (14), (15) with respect to these variables takes the simple form

$$t_1' \cdot \cos\theta = t_1(1 + \sin\theta) \tag{18}$$

$$t_2' \cdot \cos\theta = t_2(1 - \sin\theta) \tag{19}$$

We easily see that a key feature of the transformed variables is that $t_1't_2' = t_1t_2$, and also using (16), (17) we further have:

$$t_1t_2 = t^2(1 - \left(\frac{w}{c}\right)^2) = t_1't_2' = t'^2(1 - \left(\frac{w'}{c}\right)^2) \tag{20}$$

That is, the geometric mean of these time coordinates, $t_m = \sqrt{t_1t_2}$, is invariant under the Lorentz transformation, and is given as a very simple expression of clock time, *t* and 'location', *w = x/t*. So the time vector itself is of course not invariant, but the product of its coordinates is. We try to utilize this further in the next section.

5.3 The state expressed as a complex number

Now we presents the final step regarding the (time, space) variable. We could have gone directly to this formulation, but have chosen to present the above line of arguments leading to it.

The result just obtained in Section 5.2 is (*cf.* (20)) that $t_m = \sqrt{t_1t_2} = t\sqrt{1-(w/c)^2}$ is invariant under the Lorentz transformation. There is a rather obvious alternative way to express this. Letting the angle φ be defined by

$$\sin\varphi = \mathrm{w}/\mathrm{c}\,,$$

then the above identity is written

$$t_m = \sqrt{t_1t_2} = t\cos\varphi \tag{21}$$

This actually suggests that we could define a two-dimensional (time, space) vector as a complex number:

$$\bar{\boldsymbol{t}}_{\boldsymbol{x}} = te^{i\varphi} \tag{22}$$

So in this formulation, $\bar{\boldsymbol{t}}_{\boldsymbol{x}}$ is a complex variable in *t* and *φ*, (or equivalently in *t* and *x*, or in *t* and *w*).The absolute value of this complex number equals the clock time, *t,* for the synchronized clocks on the reference frame in question:

$$|\bar{\boldsymbol{t}}_{\boldsymbol{x}}| = t \tag{23}$$

Further, its real part equals

$$Re(\bar{\boldsymbol{t}}_{\boldsymbol{x}}) = t_x = t\cos\varphi = t\sqrt{1-(w/c)^2} \tag{24}$$

This equals the clock time, *t*, modified by the location factor,

$$\gamma_w = \sqrt{1-(w/c)^2}.$$

We should mention that if *w = v* this factor is identical to the 'generalized time dilation factor' of *eq.* (11). As this real part, t_x depends on both position and, clock time, *t,* we could refer to it as 'positional time'; *cf. eq.* (24).

The concept 'positional time' was also used in Chapter 4, but there referring to the time observed on 'the other reference frame' at a position, *x* on *K*.

The imaginary part of $\bar{\boldsymbol{t}}_{\boldsymbol{x}}$ equals

$$Im(\bar{\boldsymbol{t}}_{\boldsymbol{x}}) = t\sin\varphi = tw/c \tag{25}$$

By referring back to the variables of Sections 5.1, 5.2 we also have the alternative expressions

$$Re(\bar{\boldsymbol{t}}_{\boldsymbol{x}}) = \sqrt{t^2 - t_c^2} = \sqrt{t_1 t_2}$$

$$Im(\bar{\boldsymbol{t}}_{\boldsymbol{x}}) = tw/c = x/c = t_c$$

Now an alternative expression for the (time, space) vector for instance equals

$$\bar{\boldsymbol{t}}_{\boldsymbol{x}} = \sqrt{t^2 - t_c^2} + i \cdot t_c \tag{26}$$

giving the link to time parameters of Section 5.1. Or, alternatively, we have

$$\bar{\boldsymbol{t}}_{\boldsymbol{x}} = t\sqrt{1 - (w/c)^2} + i \cdot x/c \tag{27}$$

where the real part equals positional time and the imaginary part represents position.

Now consider the Lorentz transformation for this (time, space) vector. Expressed by the parameters (*t, w*) we have, as already stated in *eqs* (9), (10) of Chapter 4 that the Lorentz transformation becomes

$$t' = \frac{1 - \frac{w}{c}\cdot\frac{v}{c}}{\sqrt{1 - (\frac{v}{c})^2}}t = \frac{1 - \sin\varphi\sin\theta}{\cos\theta}t$$

$$w' = \frac{w - v}{1 - \frac{w}{c}\cdot\frac{v}{c}}$$

The last relation is equivalent to

$$\sin\varphi' = \frac{\sin\varphi - \sin\theta}{1 - \sin\varphi\sin\theta}$$

This provides both parameters of the transformed (time, space) vector,

$$\bar{\boldsymbol{t}}'_{\boldsymbol{x}} = t' e^{i\varphi'}$$

And quite interestingly, its real part is found to equal

$$Re(\bar{\boldsymbol{t}}'_{\boldsymbol{x}}) = t'\cos\varphi' = \; t\cos\varphi$$

i.e., cf. eq. (24)

$$Re\left(\bar{\boldsymbol{t}}'_{\boldsymbol{x}}\right) = Re(\bar{\boldsymbol{t}}_{\boldsymbol{x}}) \tag{28}$$

So the real parts of the two (time, space) vectors are identical. This means that we on any reference frame can start out from a 'clock time', *t*, and a location $x = wt$; and by postulating a (time, space) vector equal to $\bar{\boldsymbol{t}}_{\boldsymbol{x}} = te^{i\varphi}$, where φ = Arcsin(*w/c*), then the real part of time at this location can be defined to be positional time, $t_x = t\cos\varphi = t\sqrt{1 - (w/c)^2}$. It follows that this is equal to the positional time *for any reference frame*; as long as we consider systems moving at constant speed, (and that time of the moving reference frames are synchronized at origin at time *t*=0).

So the rather surprising result is that if we have clock time, *t* on a reference frame, and then assign the positional time, $t\sqrt{1 - (w/c)^2}$ to any location $x = wt$, and if all reference frames apply this rule, then at a given location we assign the *same* position time to all frames!

We note that this result of course differs from what might seem as a 'similar' result given in Chapter 4 (observational principle D). There we identified a specific location for the clock comparison, *i.e.* $x = w_0 t$, (the midpoint between $x = 0$ and $x' = 0$) in order to obtain identical clock readings. But in the present approach – now using complex numbers – the positional time gives identical results at *all* locations, (all x and the associated x').

5.4 Summary, Chapter 5

This chapter has offered a rather long journey, involving a large number of variables. The good thing is that we can give a very short summary of the main results. I guess the simplest version of the summary is the following:

As usual we start out with the two state variables, x = position, and t = clock time (*i.e.* the time given by all synchronized clocks on the reference frame). Now we replace these with the following two variables:

$w = x/t$, ('velocity' at which the point of observation has moved away from the origin).

$t_x = t\sqrt{1-(w/c)^2}$, (positional time).

(An alternative to the first variable could be w/c rather than w.) The Lorentz transformation, expressed in terms of these two new variables, now takes the form

$$w' = \frac{w - v}{1 - (w/c)\cdot(v/c)}$$

$$t'_x = t_x$$

There are two reasons for this to be an attractive result. First, we get two independent relations: one in w, only, and one in t_x, only. Secondly, and most important, the positional time, t_x, is invariant under this transformation. So at any given position (common for the reference frames), both systems have the same positional time.

6 Conclusions

The Lorentz transformation is first utilized to discuss a number of standard results on time dilation between two reference frames moving relative to each other at speed v. We assume that the conditions of the special theory of relativity (STR) hold. There is a complete *symmetry* between the two reference frames, and all clocks on the same reference frame are synchronized. Further:

- We do not utilize any definition of *simultaneity* across systems. The approach restricts to explore direct comparisons of clocks being at the same location at the same time.
- We do not use the expression '*as seen'* (from the other reference frame). Observers (observational equipment) on both reference frames agree on the time readings; as they are carried out 'on location'.
- We specify the applied *observational principle*, *i.e.* the location of the clocks that are used for time comparisons. Thus, we focus on how observed time, t', on the 'other' '(secondary') system depends on the position, x on the 'primary' reference frame. It is argued that one should look at the total picture, taking all information into account, as provided by the Lorentz transformation
- We do not describe the phenomenon as 'moving clock goes slower', as it seems irrelevant which of the two reference frames we consider to be moving, as it is rather the observational principle that matters.

Thus, we stress the fact that at a given time, t on K, the time, t' observed on K' will depend on the position, x on K. The usual special cases are treated, *e.g.* the standard result, $t' = t\sqrt{1-(v/c)^2}$. Further, we discuss the result that if we at a time t choose the location at the midpoint between $x = 0$ and $x' = 0'$ for time comparison, then we will at this location observe $t' = t$. This choice represents an observational principle being symmetric with respect to the two reference frames. So if we observe $t \neq t'$, in an otherwise symmetric situation, we could claim that this is caused by the asymmetry of the chosen observational principle.

The discussions suggest that under the strict symmetry conditions there is hardly any 'true' time dilation that could cause different ageing on the two systems. So it should be interesting to identify the conditions – in particular departures from symmetry - that would cause time dilation to represent a physical reality.

An additional conclusion – within this given framework - might be that an observer moving relative to the reference frame where the event takes place, is a rather 'unreliable' observer regarding time. The various observational principles will give different results; so one should be careful to let such an observer define the phenomenon.

The above considerations may suggest that the clock readings in isolation may not be 'rich' enough to capture all aspects of time. So in the second part of the paper we investigate transformations of the the (time, space) variable, looking for alternative presentations. One solution is obtained by defining the (time, space) variable as a complex number. Its absolute value is equal to the clock readings of the reference frame in question. Further, the real part of this complex number is interpreted as *positional time*, and this is actually invariant under the Lorentz transformation; *i.e*. at any position it has the same value on all reference frames moving at constant speed relative to each other.

This offers an alternative narrative, to the one commonly presented in the STR. We postulates that 'time' evolves at a different speed, depending on the distance from the 'point of initiation' (origin). Now from the discussions of the first part of this paper, *e.g*. see Fig. 1, this may not be an unreasonable assumption. Further, I find no obvious inconsistencies in the above formulation. So, hopefully it could prove fruitful also for further investigations.

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