Triangle Inscribed-Triangle Picking

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Abstract. Given a triangle ABC, the average area of an inscribed triangle RST whose vertices are uniformly distributed on BC, CA and AB, is proven to be one-fourth of the area of ABC. The average of the square of the area of RST is shown to be one-twelfth of the square of the area of ABC, and the average of the cube of the ratio of the areas is $5/144$. A Monte Carlo simulation confirms the theoretical results, as well as a Maxima program which computes the exact averages.

Keywords. Geometric probability, computational geometry, triangle triangle picking.

1 Introduction

In 1865, Professor James Joseph Sylvester proved [1] that the average area of a random triangle, whose vertices are picked inside of a given triangle with area A, is equal to $A/12$. This problem, originally proposed by S. Watson, and known as *Triangle Triangle Picking*, is one of the earliest examples of Geometric Probability. Many similar problems have been proposed [2,3], including Sylvester's own four-point problem which asks for the probability that the convex hull of four random points is a triangle. Here we study a sub-class of such problems, where the interior polygon has its vertices on the edges of the base polygon.

2 An Application of Barycentric Coordinates

Suppose the vertices of a triangle are denoted by the vectors $\vec{A}, \vec{B}, \vec{C}$. The barycentric coordinates [4] of a point $\vec{P}$, with respect to the triangle ABC, is (α, β, γ) if $\vec{P} = \alpha\vec{A} + \beta\vec{B} + \gamma\vec{C}$, and $\alpha + \beta + \gamma = 1$.

Bottema's Theorem [5]: Assume the vertices P_i of a triangle $P_1P_2P_3$ have barycentric coordinates (x_i, y_i, z_i), with respect to the triangle ABC then,

$$\mathrm{area(P_1P_2P_3)} = \det \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \mathrm{area(ABC)}.$$

Theorem: Given a triangle ABC, if three points R, S, and T are chosen randomly, on the faces AB, BC, CA then the average of area of RST is one-fourth of the area of ABC.

Proof: Consider an inscribed triangle whose vertices R, S, T, are defined by

$$\begin{cases} \vec{T} = \vec{A} + t\ \overrightarrow{AB} \\ \vec{R} = \vec{B} + r\ \overrightarrow{BC} \\ \vec{S} = \vec{C} + s\ \overrightarrow{CA} \end{cases} \tag{1}$$

where r, s, t are random numbers in $[0, 1]$.

In this case, the points R, S, T are respectively given by barycentric coordinates $(t, 1-t, 0), (0, r, 1-r)$, and $(1-s, 0, s)$. Therefore, the corresponding determinant is,

$$f(r, s, t) = \det \begin{bmatrix} r & 1-r & 0 \\ 0 & s & 1-s \\ 1-t & 0 & t \end{bmatrix} = rst + (1-r)(1-s)(1-t).$$

Hence, by Bottema's theorem, $\text{area}(RST) = f(r, s, t)\text{area}(ABC)$. The average value of rst, and $(1-r)(1-s)(1-t)$ can be represented as the product of the averages of r, s, t which is equivalent to $\left(\frac{1}{2}\right)^3$. Therefore,

$$\langle f(r, s, t) \rangle = \langle (rst + (1-r)(1-s)(1-t)) \rangle = \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 = \frac{1}{4}.$$

As a result $\langle \text{area}(RST) \rangle = \text{area}(ABC)/4$.

2.1 Computation of the 2nd and 3rd Moment

To calculate the second moment of $f(r, s, t)$, we expand f^2 as,

$$\begin{aligned} f^2(r, s, t) &= [rst + (1-r)(1-s)(1-t)]^2 \\ &= r^2s^2t^2 + (1-r)^2(1-s)^2(1-t)^2 + 2r(1-r)s(1-s)t(1-t). \end{aligned} \tag{2}$$

Using $\langle r^2 \rangle = \frac{1}{3}$, $\langle (1-r)^2 \rangle = \frac{1}{3}$, and $\langle r(1-r) \rangle = \frac{1}{6}$, the mean value of $f^2(r, s, t)$ is equal to,

$$\langle f^2(r, s, t) \rangle = \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^3 + 2\left(\frac{1}{6}\right)^3 = \frac{1}{12}.$$

Therefore, the average square of the area of triangle RST is equal to one-twelfth of the square of the area of ABC.

Similarly, to calculate the third moment of f,

$$\begin{aligned} f^3(r, s, t) =& r^3s^3t^3 + 3r(1-r)^2s(1-s)^2t(1-t)^2 + \\ & 3r^2(1-r)s^2(1-s)t^2(1-t) + (1-s)^3(1-r)^3(1-t)^3 \end{aligned} \tag{3}$$

$$\langle f^3(r,s,t)\rangle = \left(\frac{1}{4}\right)^3 + 3\left(\frac{1}{3}-\frac{1}{4}\right)^3 + 3\left(\frac{1}{3}-\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^3 = \frac{5}{144}.$$

Therefore, $\langle \text{area}^3(\text{RST})\rangle$ is equal to $\frac{5}{144}\text{area}^3(\text{ABC})$.

3 A Monte Carlo Simulation

A Monte Carlo simulation can be done to approximate the ratio of the average area of ABC and randomly generated inscribed triangles. The following program was written in Java to approximate this ratio.

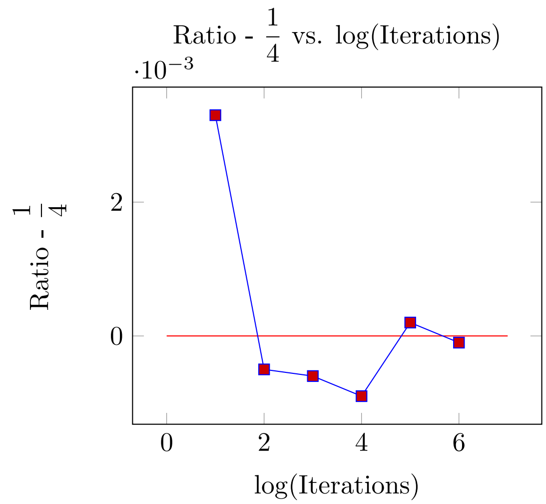
```java
public static void averageArea(int iterations, int power) {
	Random rand = new Random();
	double area = 0, totalArea = 0;
	double r, s, t;
	double Ax, Ay, Bx, By, Cx, Cy;
	Triangle baseTri = new Triangle(
		new Point(rand.nextInt(10), rand.nextInt(10)),
		new Point(rand.nextInt(10), rand.nextInt(10)),
		new Point(rand.nextInt(10), rand.nextInt(10)));
	Point A, B, C;
	for (int i = 0; i < iterations; i++) {
		r = rand.nextDouble();
		s = rand.nextDouble();
		t = rand.nextDouble();

		Ax = baseTri.getPoint(1).getX() + (s *
			(baseTri.getPoint(2).getX() - baseTri.getPoint(1).getX()));
		Ay = baseTri.getPoint(1).getY() + (s *
			(baseTri.getPoint(2).getY() - baseTri.getPoint(1).getY()));
		A = new Point(Ax, Ay);

		Bx = baseTri.getPoint(2).getX() + (t *
			(baseTri.getPoint(0).getX() - baseTri.getPoint(2).getX()));
		By = baseTri.getPoint(2).getY() + (t *
			(baseTri.getPoint(0).getY() - baseTri.getPoint(2).getY()));
		B = new Point(Bx, By);

		Cx = baseTri.getPoint(0).getX() + (r *
			(baseTri.getPoint(1).getX() - baseTri.getPoint(0).getX()));
		Cy = baseTri.getPoint(0).getY() + (r *
			(baseTri.getPoint(1).getY() - baseTri.getPoint(0).getY()));
		C = new Point(Cx, Cy);

		area = new Triangle(C, A, B).area();

		totalArea += Math.pow(area, power);
	}
```

```
    double averageArea = totalArea / iterations;
    System.out.println(averageArea);
    double ratio = averageArea / Math.pow(baseTri.area(), power); //The
        ratio of the random area^p to the base triangle's area^p.
    System.out.println(ratio);
}
```

The supporting classes are included below (Triangle.java, Point.java).

3.1 Monte Carlo Convergence

Iterations	10^2	10^3	10^4	10^5	10^6	10^7
Ratio - $\frac{1}{4}$	0.0033	-0.0005	-0.0006	-0.0009	0.0002	-0.0001



4 Exact Results from Maxima

Using the following Maxima program, the exact results were computed for, $\langle f^1(r,s,t)\rangle, \langle f^2(r,s,t)\rangle, ..., \langle f^{10}(r,s,t)\rangle$

```
for n from 1 thru 10 do(
f(r,s,t) := ((r*s*t) + (1-r)*(1-s)*(1-t))^n,
a : integrate(integrate(integrate(f(r,s,t), t, 0, 1), s, 0, 1), r, 0 ,1),
print(a));
```

Power(n):	1	2	3	4	5	6	7	8	9	10
$\langle f^n(r,s,t)\rangle$:	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{5}{144}$	$\frac{31}{1800}$	$\frac{7}{720}$	$\frac{1063}{176400}$	$\frac{403}{100800}$	$\frac{211}{75600}$	$\frac{143}{70560}$	$\frac{2593}{1707552}$

5 General Formula for $f^n(r,s,t)$

To be continued...

6 Asymptotic Behavior of $f^n(r,s,t)$

To be continued...

References

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