Proof of Global in Time Solvability of Incompressive NSIVP in Periodic Space Using Time Transformation Analysis

Naoya Isobe

abstract

Global in time solvability of incompressive NSIVP (Navier-Stokes initial value problem) in periodic space is proved using time transformation analysys.

Introduction

Navier-Stokes equations was derived as foundmental equations for hydromechanics by Navier[1] and Stokes[2]. Hereafter, its solvability is studied for a long time, but, due to the difficulty based on nonlinearity of the equations, even limited to incompressive fluid, not enough results for solvability have gotton over 170 years.

With regard to NSIVP for incompressible fluid, local in time solvability and global in time solvability for small initial value, including well-posedness, are known by means of arguments based on analysys stareted by Leray[3], Hopf[4] or Kiselev-Ladyzhenskaya[5], Ito[6], Kato-Fujita[7], and enhanced by Kato[8],Giga-Miyakawa[9]. But, global in time solvability for large initial values has not been known.

Although studies have been driven mainly by analizing problems in the whole space, results with regard to problems in periodic space are almost similar.

In the present paper, global in time solvability in periodic space for initial values that are not only small but also large is proved using time transformation analysis. Meanwhile, global in time solvability in the whole space for initial values that are not only small but also large is proved as well using time transformation analysis, in another paper[11].

This gives positive answer to the whole space version in the CMI millenium problem[10] related to Navier-Stokes equations, and similar results for more comprehensive initial values.

Overview

As main result, NSIVP in periodic space has global in time classical solution, for initial values that are not only small but also large. The global in time solution of NSIVP in periodic space under appropriate conditions is well-posed, which means this is unique, smooth and continuous to initial value.

Also, there is a decreasing upper limit function of initial value for norm of solution.

The proof consists of local in time analysis, and global in time analysis based on a priori estimation. Analysis based on a priori estimation is typical global in time analysis. Basic analysis has application limit and due to this limit, global in time solvability of incompressive Navier-Stokes initial value problem has not been proved. In the present argument, to overcome this limit, time transformation is used. By this time transformation analysis, effective area with regard to norm upper limit estimation is expanded to overcome the limit.

With regard to local in time analysis, as the first step, for initial value $\boldsymbol{a}_{\in \boldsymbol{L}^Q, Q\in(n,\infty)}$, the existence of existing time T_M and solution $\boldsymbol{u}_{\in \boldsymbol{L}^Q, t\in[0,T_M]}$ is proved, and for initial value $\boldsymbol{a}_{\in \boldsymbol{L}^2\cap \boldsymbol{L}^q, q\in(n,\infty]}$, the existence of existing time T_M and solution $\boldsymbol{u}_{\in \boldsymbol{L}^2\cap \boldsymbol{L}^q, t\in[0,T_M]}$ is proved.

And next, a priori energy nonincreasing is proved, and $\boldsymbol{L}^r{}_{,r\in[2,\infty]}$-norm of solution and derivertive of solution has nonincreasing or decreasing upper limit function. After that, based on local in time solvability and a priori estimation, global in time solvability is proved. Time transformation analysis are repeatedly used through these process not limited in the step of proving global in time solvability.

Although main results in sections 1-3 include known results, for the sake of consistency with following sections, these results are described with those proofs.

1 . preliminary

Difinition 1 (function space)

For a poritive constant a and basic vectors $\boldsymbol{e}_1,\dots,\boldsymbol{e}_n\,_{\in \boldsymbol{R}^n,\,n\in\boldsymbol{N}_{\ge 3}}$, periodical lattice $\boldsymbol{\Lambda}^n$ and unit periodical domain Ω and function space $\boldsymbol{L}^q(\boldsymbol{R}^n;\boldsymbol{\Lambda}^n)_{,q\in[1,\infty]}$ and mean value $\boldsymbol{\varphi}_\Omega$ of function $\boldsymbol{\varphi}\,_{\in \boldsymbol{L}^q_{loc}(\boldsymbol{R}^n)}$ are define as follows.

$$\boldsymbol{\Lambda}^n=\{\,\lambda_1 a\boldsymbol{e}_1+\dots+\lambda_n a\boldsymbol{e}_n \mid \lambda_1,\dots,\lambda_n\in\boldsymbol{Z}\,\}$$
$$\Omega=\{\,(x_1,\dots,x_n)_{\in\boldsymbol{R}^n}\mid x_1,\dots,x_n\in[0,a]\,\}$$
$$\boldsymbol{L}^q(\boldsymbol{R}^n;\boldsymbol{\Lambda}^n)=\{\,\boldsymbol{\varphi}\,_{\in\boldsymbol{L}^q_{loc}(\boldsymbol{R}^n)}\mid \boldsymbol{\varphi}(\boldsymbol{x}+\boldsymbol{l})=\boldsymbol{\varphi}(\boldsymbol{x})\,_{,\,\boldsymbol{x}\in\Omega\,,\,\boldsymbol{l}\in\boldsymbol{\Lambda}}\,,\ \|\boldsymbol{\varphi}\|_{\boldsymbol{L}^q(\boldsymbol{R}^n;\boldsymbol{\Lambda}^n)}<\infty\,\}$$
$$\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^q(\boldsymbol{R}^n;\boldsymbol{\Lambda}^n)}=\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^q(\Omega)}$$
$$\boldsymbol{\varphi}_\Omega=\frac{1}{|\Omega|}\int_\Omega dV\boldsymbol{\varphi}$$

Functions which belong to $\boldsymbol{L}^q(\boldsymbol{R}^n;\boldsymbol{\Lambda}^n)$ are periodical functions in the whole space $\boldsymbol{R}^n$, and these can be identified as functions in periodic space $\boldsymbol{R}^n/\boldsymbol{Z}^n$.

For time point $t\,_{\in[0,\infty)}$, time interval $TI\,_{\subseteq[0,\infty]}$ and function space $\boldsymbol{L}^q(\boldsymbol{R}^n;\boldsymbol{\Lambda}^n)$, function space $\boldsymbol{L}^q_t(\boldsymbol{R}^n;\boldsymbol{\Lambda}^n),\boldsymbol{L}^q_{TI}(\boldsymbol{R}^n;\boldsymbol{\Lambda}^n)$ are defined as follows.

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^q_t(\boldsymbol{R}^n;\boldsymbol{\Lambda}^n)}=\|\boldsymbol{\varphi}_t\|_{\boldsymbol{L}^q(\boldsymbol{R}^n;\boldsymbol{\Lambda}^n)}\ ,\ \boldsymbol{L}^q_t(\boldsymbol{R}^n;\boldsymbol{\Lambda}^n)=\left\{\boldsymbol{\varphi}\,\middle|\,\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^q_t(\boldsymbol{R}^n;\boldsymbol{\Lambda}^n)}<\infty\right\}$$
$$\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^q_{TI}(\boldsymbol{R}^n;\boldsymbol{\Lambda}^n)}=\sup_{t\in TI}\|\boldsymbol{\varphi}_t\|_{\boldsymbol{L}^q(\boldsymbol{R}^n;\boldsymbol{\Lambda}^n)}\ ,\ \boldsymbol{L}^q_{TI}(\boldsymbol{R}^n;\boldsymbol{\Lambda}^n)=\left\{\boldsymbol{\varphi}\,\middle|\,\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^q_{TI}(\boldsymbol{R}^n;\boldsymbol{\Lambda}^n)}<\infty\right\}$$

Here, function over spatial space $\boldsymbol{\varphi}_t$ is function over time and space $\boldsymbol{\varphi}$ with fixed time t.

These function space $\boldsymbol{L}^q(\boldsymbol{R}^n;\boldsymbol{\Lambda}^n),\boldsymbol{L}^q_t(\boldsymbol{R}^n;\boldsymbol{\Lambda}^n),\boldsymbol{L}^q_{TI}(\boldsymbol{R}^n;\boldsymbol{\Lambda}^n)$ are Banach space.

Hereafter, these are accordingly noted simply by $\boldsymbol{L}^q,\boldsymbol{L}^q_t,\boldsymbol{L}^q_{TI}$.

Moreover, function spaces $\boldsymbol{L}^{[p,q]},\boldsymbol{L}^{(p,q]}$ are defined as follows, providing $1\le p<q\le\infty$.

$$\boldsymbol{L}^{[p,q]}=\bigcap_{r\in[p,q]}\boldsymbol{L}^r\ ,\ \boldsymbol{L}^{(p,q]}=\bigcap_{r\in(p,q]}\boldsymbol{L}^r \qquad []$$

Difinition 2 (multiple index)

In the present paper, size of multiple index $\alpha=(\alpha_1,\dots,\alpha_n)_{\in(\{0\}\cup\boldsymbol{N})^n}$ is defined by $|\alpha|=\alpha_1+\dots+\alpha_n$, and product of powers of spatial variable corrrespond to α is defined by $x^\alpha=x_1^{\alpha_1}\dots x_n^{\alpha_n}$, derivative in regard as spatial variable correspond to α is defined by $\partial^\alpha=\partial_1^{\alpha_1}\dots\partial_n^{\alpha_n}$.

Difinition 3 (Helmholtz discomposition)

For a function $\boldsymbol{\varphi}$, its nondivergent component $\mathcal{P}\boldsymbol{\varphi}$ and nonrotational component $\overline{\mathcal{P}}\boldsymbol{\varphi}$ are defined as follows if these exist, and Helmholtz decomposition means decomposition of $\boldsymbol{\varphi}$ to $\mathcal{P}\boldsymbol{\varphi},\overline{\mathcal{P}}\boldsymbol{\varphi}$.

$$\boldsymbol{\varphi}=\mathcal{P}\boldsymbol{\varphi}+\overline{\mathcal{P}}\boldsymbol{\varphi}\ ,\ \boldsymbol{\partial}\cdot\mathcal{P}\boldsymbol{\varphi}=0\ ,\ (\mathcal{P}\boldsymbol{\varphi},\overline{\mathcal{P}}\boldsymbol{\varphi})_{\boldsymbol{L}^2}=0$$

If there is series of priodic functions, each of which has Helmholtz composition $\{\boldsymbol{\varphi}_k\}_{k\in\boldsymbol{N}},\boldsymbol{\varphi}_k=\mathcal{P}\boldsymbol{\varphi}_k+\overline{\mathcal{P}}\boldsymbol{\varphi}_k$ and serieses of Helmholtz components $\{(\mathcal{P}\boldsymbol{\varphi}_k,\overline{\mathcal{P}}\boldsymbol{\varphi}_k)\}_{k\in\boldsymbol{N}}$ have limits $(\mathcal{P}\boldsymbol{\varphi},\overline{\mathcal{P}}\boldsymbol{\varphi})$, nondivergent component of $\boldsymbol{\varphi}$ is defined by $\mathcal{P}\boldsymbol{\varphi}$ and nonrotational component of $\boldsymbol{\varphi}$ is defined by $\overline{\mathcal{P}}\boldsymbol{\varphi}$, and Helmholtz decomposition means decomposition of $\boldsymbol{\varphi}$ to $\mathcal{P}\boldsymbol{\varphi},\overline{\mathcal{P}}\boldsymbol{\varphi}$.

Generally for function space $\boldsymbol{X}$ (like $\boldsymbol{L}^q,\boldsymbol{L}^q_{TI}$ above), function space $\mathcal{P}\boldsymbol{X}$ is defined as follows.

$$\mathcal{P}\boldsymbol{X}=\{\,\boldsymbol{\varphi}\in\boldsymbol{X}\mid\boldsymbol{\varphi}=\mathcal{P}\boldsymbol{\varphi}\,\} \qquad []$$

Difinition 4 (heat kernel)

Heat kernel K corresponds to heat equation $\partial_t f - \nu\triangle f = 0$ is defined by following expression.

$$K(t,\boldsymbol{x}) = \frac{1}{\sqrt{4\pi\nu t}^{n}} \exp\Big(- \frac{\boldsymbol{x}^2}{4\nu t} \Big)$$

Difinition 5 (initial value)

Basic condition for initial value function $\boldsymbol{a}$ is defined as follows, providing $q \in (n,\infty]$; $m \geq 2$.

(1.1) $\boldsymbol{a} \in \mathcal{P}\boldsymbol{L}^{[2,q]}$, $\partial^\alpha \boldsymbol{a} \in \boldsymbol{L}^{[2,q]}$ $_{,\,|\alpha|\leq m}$

Difinition 6 (NSIVP)

NSIVP (Navier-Stokes initial value problem) is defined as follows.

(1.2) $\boldsymbol{u} \in \mathcal{P}\boldsymbol{L}^{[2,q]}_{(0,\infty)}$

$$\partial_t \boldsymbol{u} - \nu\triangle\boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\partial})\boldsymbol{u} + \frac{1}{\rho}\boldsymbol{\partial}p = \boldsymbol{0} \qquad {}_{,(t,\boldsymbol{x})\in(0,\infty)\times\boldsymbol{R}^n}$$

$$\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{a}(\boldsymbol{x}) \qquad {}_{,\boldsymbol{x}\in\boldsymbol{R}^n}$$

Following integral equation is equivalent to NSIVP(1.2) with regard to initial value $\boldsymbol{a}$ which satisfies $\boldsymbol{a}_\Omega = \boldsymbol{0}$, providing the solution $\boldsymbol{u}$ satisfies $\boldsymbol{u}_\Omega = \boldsymbol{0}$ and has derivertives.

(1.3) $$\boldsymbol{u}_t = K_t {*} \boldsymbol{a} - \int_0^t d\tau\, \mathcal{P}(\boldsymbol{\partial}K_{t-\tau} {*}{\cdot} \boldsymbol{u}_\tau \boldsymbol{u}_\tau) \qquad {}_{,(t,\boldsymbol{x})\in(0,\infty)\times\boldsymbol{R}^n}$$

For funtion $\Phi = \Phi(t)$ with property $\Phi(0) = 0, \Phi(t) > 0, \partial_t\Phi(t) = \varphi(t)$ and function $\boldsymbol{f} = \boldsymbol{f}(t,\boldsymbol{x})$, definding $\boldsymbol{f}^\Phi(t,\boldsymbol{x}) = \boldsymbol{f}(\Phi(t),\boldsymbol{x})$, time-transformed solution $\boldsymbol{u}^\Phi$ for original solution $\boldsymbol{u}$ is defined. Then, following time transformed integral equation for $\boldsymbol{u}^\Phi$ is equivarent to original equation.

(1.4) $$\boldsymbol{u}^\Phi_t = K^\Phi_t {*} \boldsymbol{a} - \int_0^t d\tau\, \mathcal{P}(\boldsymbol{\partial}K^\Phi_{t-\tau} {*} \varphi_\tau {\cdot} \boldsymbol{u}^\Phi_\tau \boldsymbol{u}^\Phi_\tau) \qquad {}_{,(t,\boldsymbol{x})\in(0,\infty)\times\boldsymbol{R}^n}$$

Difinition 7 (linear term and nonlinear term of solution)

In case of the solution $\boldsymbol{u}$ for NSIVP (1.2) is decomposed to 2 terms according to integral equation (1.3) or time transformed integral equation (1.4), term $\boldsymbol{u}^{(L)}$ is defined as linear term of the solution and term $\boldsymbol{u}^{(NL)}$ is defined as nonlinear term of the solution.

$$\boldsymbol{u}^{(L)}_t = K_t {*} \boldsymbol{a} \quad , \ \boldsymbol{u}^{(NL)}_t = -\int_0^t d\tau\, \mathcal{P}(\boldsymbol{\partial}K_{t-\tau} {*}{\cdot} \boldsymbol{u}_\tau \boldsymbol{u}_\tau)$$

$$\boldsymbol{u}^{\Phi(L)}_t = K^\Phi_t {*} \boldsymbol{a} \ , \ \boldsymbol{u}^{\Phi(NL)}_t = -\int_0^t d\tau\, \mathcal{P}(\boldsymbol{\partial}K^\Phi_{t-\tau} {*} \varphi_\tau {\cdot} \boldsymbol{u}^\Phi_\tau \boldsymbol{u}^\Phi_\tau) \qquad []$$

lemma(characteristics of convolution with kernel)

Related to convolution with kernel, following relations are confirmed, providing $K = K^{(\nu)}$; $\nu, \nu_0, \nu_1 > 0$; $p \in [1,q], q \in (n,\infty]$ and mean values φ_Ωs of φs are zeros.

(1.5) $K^{(\nu_0+\nu_1)} = K^{(\nu_0)} {*} K^{(\nu_1)}$

(1.6) $\|K{*}\varphi\|_{L^2(\Omega)} \leq e^{-C\nu t}\|\varphi\|_{L^2(\Omega)}$ $_{,\varphi\in L^2}$

(1.7) $\|K{*}\varphi\|_{L^q(\Omega)} \leq C(\nu t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|\varphi\|_{L^p(\Omega)}$ $_{,\varphi\in L^p}$

(1.8) $\|\partial_i K{*}\varphi\|_{L^q(\Omega)} \leq C(\nu t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}\|\varphi\|_{L^p(\Omega)}$ $_{,\varphi\in L^p}$ []

proof

Relation (1.5) is confirmed by means of expresssion $\mathcal{F}[K^{(\nu_0+\nu_1)}] = \mathcal{F}[K^{(\nu_0)}]\mathcal{F}[K^{(\nu_1)}]$ based on Fourier transformation analysis.

Relation (1.6) is confirmed by means of Poincaré-Wirtinger inequality as follows.

For $u = K * \varphi$, following expressions are confirmed.

$$\partial_t u^2 = 2u\partial_t u = 2u(\nu\triangle u + f) = 2u\nu\triangle u = 2\nu\boldsymbol{\partial}(u\boldsymbol{\partial}u) - 2\nu\boldsymbol{\partial}u\boldsymbol{\partial}u$$

$$\partial_t u^2 - 2\nu\boldsymbol{\partial}(u\boldsymbol{\partial}u) + 2\nu\boldsymbol{\partial}u\boldsymbol{\partial}u = 0$$

By integrating expression above, following expressions are confirmed using Gauss's theorem and periodicity of u.

$$d_t\|u\|^2_{L^2(\Omega)} - 2\nu\int_{\partial\Omega} d\boldsymbol{S}\, u\boldsymbol{\partial}u + 2\nu\|\boldsymbol{\partial}u\|^2_{\boldsymbol{L}^2(\Omega)} = 0$$

$$d_t\|u\|^2_{L^2(\Omega)} + 2\nu\|\boldsymbol{\partial}u\|^2_{\boldsymbol{L}^2(\Omega)} = 0$$

Using Poincaré-Wirtinger inequality $\|f - f_\Omega\|_{L^2(\Omega)} \le C\|\boldsymbol{\partial}f\|_{\boldsymbol{L}^2(\Omega)}$ and $u_\Omega = 0$, following inequality is confirmed.

$$d_t\|u\|^2_{L^2(\Omega)} + 2C^{-2}\nu\|u\|^2_{L^2(\Omega)} \le 0$$

Therefore, by means of Grönwall inequality, following expression is confirmed.

$$\|u\|_{L^2(\Omega)} \le \|a\|_{L^2(\Omega)} e^{-C^{-2}\nu t}$$

Relations (1.7)(1.8) are proved by means of following expressions in regard to estimation of norm of convolution, for which, on each devided periodic area, enhanced Young's inequality is used.

$$J * f(\boldsymbol{x}) = \int_{\boldsymbol{R}^n} dV(\boldsymbol{y})\, J(\boldsymbol{x}-\boldsymbol{y})f(\boldsymbol{y}) = \sum_{\boldsymbol{l}\in\boldsymbol{\Lambda}} \int_{\boldsymbol{l}+\Omega} dV(\boldsymbol{y})\, J(\boldsymbol{x}-\boldsymbol{y})f(\boldsymbol{y})$$

$$\|\, J * f\,\|_{L^q(\Omega)} \le \sum_{\boldsymbol{l}\in\boldsymbol{\Lambda}} M_{\boldsymbol{l}}^{\frac{1}{q}} N_{\boldsymbol{l}}^{1-\frac{1}{p}} \|f\|_{L^p(\Omega)}$$

$$M_{\boldsymbol{l}} = \sup_{\boldsymbol{y}\in\Omega} \int_\Omega dV(\boldsymbol{x})\, |\, J(\boldsymbol{x}-\boldsymbol{l}-\boldsymbol{y})\,|^r = \sup_{\boldsymbol{x}\in\Omega} \int_\Omega dV(\boldsymbol{y})\, |\, J(\boldsymbol{y}-\boldsymbol{l}-\boldsymbol{x})\,|^r$$

$$N_{\boldsymbol{l}} = \sup_{\boldsymbol{x}\in\Omega} \int_\Omega dV(\boldsymbol{y})\, |\, J(\boldsymbol{x}-\boldsymbol{l}-\boldsymbol{y})\,|^r$$

Especially for function $J(\boldsymbol{x})$ that is given by product of factors $J_1(x_1)\cdots J_n(x_n)$ each of which depends only on a spatial coordinate, following expressions are confirmed.

$$M_{\boldsymbol{l}} = \prod_{j=1}^n (M_j)_{l_j}\,, (M_j)_{l_j} = \sup_{x_j\in[0,a]} \int_0^a dy_j\, |\, J_j(y_j - l_j - x_j)\,|^r$$

$$N_{\boldsymbol{l}} = \prod_{j=1}^n (N_j)_{l_j}\,, (N_j)_{l_j} = \sup_{x_j\in[0,a]} \int_0^a dy_j\, |\, J_j(x_j - l_j - y_j)\,|^r$$

$$\begin{aligned}\|\, J * f\,\|_{L^q(\Omega)} &\le \prod_{j=1}^n \Big(\sum_{l_j\in\Lambda_j} (M_j)_{l_j}^{\frac{1}{q}} (N_j)_{l_j}^{1-\frac{1}{p}}\Big)\|f\|_{L^p(\Omega)} \\ &\le \prod_{j=1}^n \Big(\sum_{l_j\in\Lambda_j} (M_j)_{l_j}^{\frac{1}{r}}\Big)^{\frac{r}{q}} \Big(\sum_{l_j\in\Lambda_j} (N_j)_{l_j}^{\frac{1}{r}}\Big)^{r(1-\frac{1}{p})} \|f\|_{L^p(\Omega)} \\ &\le \prod_{j=1}^n \big(2\|J_j\|_{L^r(\boldsymbol{R})}\big)^{\frac{r}{q}} \big(2\|J_j\|_{L^r(\boldsymbol{R})}\big)^{r(1-\frac{1}{p})} \|f\|_{L^p(\Omega)} = \prod_{j=1}^n \big(2\|J_j\|_{L^r(\boldsymbol{R})}\big)\|f\|_{L^p(\Omega)} \\ &= 2^n\|J\|_{L^r(\boldsymbol{R}^n)}\|f\|_{L^p(\Omega)}\end{aligned}$$

Therefore, by setting $J = K$ or $J = \partial_i K$, following expressions are confirmed.

$$\|\, K * f\,\|_{L^q(\Omega)} \le c\|K\|_{L^r(\boldsymbol{R}^n)}\|f\|_{L^p(\Omega)} = C(\nu t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\Omega)}$$

$$\|\, \partial_i K * f\,\|_{L^q(\Omega)} \le c\|\partial_i K\|_{L^r(\boldsymbol{R}^n)}\|f\|_{L^p(\Omega)} = C(\nu t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}\|f\|_{L^p(\Omega)}$$

$\square$

2 .local in time solvability analysis

In this chapter, local in time solvability of NSIVPs is proved. First local in time solvability of integral equation (1.4) is proved, and second smoothness of its solution is proved, then local in time solvability of NSIVPs is proved. Using strict contracting map based fixed point theorem, local in time solvability of integral equation is proved.

proposition 1 (local in time solvability of integral equaion)

There exist decreasing function $TM = TM(\xi)$ of poritive number ξ, for arbitrary initial value $\boldsymbol{a}_{\in \mathcal{P}\boldsymbol{L}^q,\, q\in(n,\infty]}$ which satisfies $\boldsymbol{a}_\Omega = \boldsymbol{0}$, integral equation (1.3) has a solution $\boldsymbol{u}_{\in \boldsymbol{L}^q_{[0,T_M]}, T_M = TM(\|\boldsymbol{a}\|_{\boldsymbol{L}^q})}$ which satisfies $\boldsymbol{u}_\Omega = \boldsymbol{0}$.$\square$

proof

Map Ψ and set S_λ are defined as follws, for $T \in (0,\infty]\,, t \in [0,T]$ and $\lambda \in (0,\infty)$.

$$\Psi \boldsymbol{f}_t = K_t * \boldsymbol{a} - \int_0^t d\tau\, \mathcal{P}(\boldsymbol{\partial} K_{t-\tau} * \cdot \boldsymbol{f}_\tau \boldsymbol{f}_\tau)$$

$$S_\lambda = \left\{ \boldsymbol{f}_{\in \boldsymbol{L}^q_{[0,T]}} \,\middle|\, \|\boldsymbol{f}\|_{\boldsymbol{L}^q_{[0,T]}} \le \lambda, \boldsymbol{f}_\Omega = \boldsymbol{0} \right\}$$

Then following estimation relations are derived for $\boldsymbol{f}, \boldsymbol{g}_{\in S_\lambda}$, providing $\chi(t) = \nu^{-1}(\nu t)^\beta{}_{,\, \beta = \frac{1}{2}(1-\frac{n}{q})}$.

$$\|\Psi \boldsymbol{f}\|_{\boldsymbol{L}^q_t} \le C_1 \|\boldsymbol{a}\|_{\boldsymbol{L}^q} + C_2 \chi(t) \|\boldsymbol{f}\|^2_{\boldsymbol{L}^q_{[0,T]}}$$

$$\|\Psi \boldsymbol{f} - \Psi \boldsymbol{g}\|_{\boldsymbol{L}^q_t} \le C_2 \chi(t) (\|\boldsymbol{f}\|_{\boldsymbol{L}^q_{[0,T]}} + \|\boldsymbol{g}\|_{\boldsymbol{L}^q_{[0,T]}}) \|\boldsymbol{f} - \boldsymbol{g}\|_{\boldsymbol{L}^q_{[0,T]}}$$

Constant $\Theta_{\in(0,1)}$ and $TM(\xi), T_M, \lambda$ are defined as follows.

$$TM(\xi) = \Theta \frac{1}{\nu} \Big(\frac{\nu}{4C_1C_2\xi} \Big)^{\frac{1}{\beta}} \ , \ T_M = TM(\|\boldsymbol{a}\|_{\boldsymbol{L}^q})$$

$$\lambda = \frac{1}{2C_2\chi(T_M)} \big(1 - \sqrt{1 - 4C_1C_2 \|\boldsymbol{a}\|_{\boldsymbol{L}^q} \chi(T_M)}\ \big)$$

Then, following relations are derived.

$$\chi(T_M) = \Theta^\beta (4C_1C_2\|\boldsymbol{a}\|_{\boldsymbol{L}^q})^{-1}$$

$$\lambda \le 2C_1 \|\boldsymbol{a}\|_{\boldsymbol{L}^q}$$

Also, there exist constant $C_{\in(0,1)}$, and following relations are derived for each time $t_{\in[0,T_M]}$.

$$C_1 \|\boldsymbol{a}\|_{\boldsymbol{L}^q} + C_2 \chi(t) \lambda^2 < \lambda$$

$$2C_2 \chi(t) \lambda \le C$$

Therefore, based on estimation relations above, following relations are derived for $\boldsymbol{f}, \boldsymbol{g}_{\in S_\lambda}$.

$$\Psi \boldsymbol{f} \in S_\lambda$$

$$\|\Psi \boldsymbol{f} - \Psi \boldsymbol{g}\|_{\boldsymbol{L}^q_{[0,T_M]}} \le C \|\boldsymbol{f} - \boldsymbol{g}\|_{\boldsymbol{L}^q_{[0,T_M]}}$$

These mean that map Ψ gives strict contradicting map over set S_λ.

Therefore, by fixed point theorem, map Ψ has a fixed point $\boldsymbol{u}$ in set S_λ.

This gives $\Psi \boldsymbol{u}_t = \boldsymbol{u}_t{}_{\ ,t\in[0,T_M]}$ and therefore $\boldsymbol{u}$ is a solution of integral equation (1.3) over time interval $[0, T_M]$.

Moreover, based on aforestated arguement, following relations are derived, so $\boldsymbol{u} \in \boldsymbol{L}^q_{[0,T_M]}$ and $\boldsymbol{u}_\Omega = \boldsymbol{0}$.

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^q_{[0,T_M]}} \le 2C_1 \|\boldsymbol{a}\|_{\boldsymbol{L}^q}$$

$$\|\boldsymbol{u}^{(NL)}\|_{\boldsymbol{L}^q_{[0,T_M]}} \le C_1 \|\boldsymbol{a}\|_{\boldsymbol{L}^q} \qquad \square$$

Note (estimation of upper limit for nonlinear term)

Estimation of upper limit of $\|\mathcal{P}(\boldsymbol{\partial} K_{t-\tau} * \cdot \boldsymbol{f}_\tau \boldsymbol{f}_\tau)\|_{\boldsymbol{L}^q, q\in(n,\infty]}$ in the proof above is based on following relations.

$$\|\mathcal{P}_{ij}\partial_k K_{t-\tau} * f_{k\,\tau} f_{j\,\tau}\|_{L^q(\boldsymbol{R}^n)} \le \|\mathcal{P}_{ij}\partial_k K_{t-\tau}\|_{L^Q(\boldsymbol{R}^n)} \|f_{k\,\tau}\|_{L^q(\boldsymbol{R}^n)} \|f_{j\,\tau}\|_{L^q(\boldsymbol{R}^n)}$$

$$\|\mathcal{P}_{ij}\partial_k K\|_{L^Q(\boldsymbol{R}^n)} \le c_Q \Big\| \Big(1 - \frac{\xi_i\xi_j}{\boldsymbol{\xi}^2}\Big)\xi_k \hat{K} \Big\|_{L^q(\boldsymbol{R}^n)}$$

$$\le 2c_Q \|\xi_k \hat{K}\|_{L^q(\boldsymbol{R}^n)} = C(\nu t)^{-\frac{n}{2}\frac{1}{q}-\frac{1}{2}}{}_{;\,1=\frac{1}{q}+\frac{1}{Q}\ ,\, Q\in[1,2]}$$

Here $\mathcal{P}_{ij}$s are components correspond to spatial coordinates of $\mathcal{P}$s, $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ is n-dimentional Fourlier variable, $\hat{K}$ is Fourlier transformed function of heat kernel K; the former expression is confirmed by means of Young's inequality and the latter expression is confirmed by means of Hausdolf-Young inequality.[]

proposition 2 (regularity of local in time solution of integral equation)

Local in time solution $\boldsymbol{u}_{\in\mathcal{P}\boldsymbol{L}^q_{[0,T_M]}}$ of integral equation (1.3) based on proposition 1 for initial value $\boldsymbol{a}_{\,\in\mathcal{P}\boldsymbol{L}^q, q\in(n,\infty]}$ has following characteristics (2.1-4) and is regular, providing $|\alpha| \ge 1$ and, as for (2.2)(2.4), $q < \infty$.

(2.1) $\boldsymbol{u} \in \boldsymbol{L}^r_{[0,T_M], r\in[2,q]}$

(2.2) $\boldsymbol{u} \in \boldsymbol{L}^r_{(0,T_M], r\in(q,\infty]}$; $\sup_{t\in(0,T_M]} t^{\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\|\boldsymbol{u}\|_{\boldsymbol{L}^r_t} < \infty$

(2.3) $\partial^\alpha\boldsymbol{u} \in \boldsymbol{L}^r_{(0,T_M], r\in[2,q]}$; $\sup_{t\in(0,T_M]} t^{\frac{|\alpha|}{2}}\|\partial^\alpha\boldsymbol{u}\|_{\boldsymbol{L}^r_t} < \infty$

(2.4) $\partial^\alpha\boldsymbol{u} \in \boldsymbol{L}^r_{(0,T_M], r\in(q,\infty]}$; $\sup_{t\in(0,T_M]} t^{\frac{n}{2}(\frac{1}{q}-\frac{1}{r})+\frac{|\alpha|}{2}}\|\partial^\alpha\boldsymbol{u}\|_{\boldsymbol{L}^r_t} < \infty$

Moreover, local in time solution $\boldsymbol{u}_{\in\mathcal{P}\boldsymbol{L}^q_{[0,T_M]}}$ of integral equation (1.3) based on proposition 1 for initial value $\boldsymbol{a}_{\,\in\mathcal{P}\boldsymbol{L}^q, q\in(n,\infty]}, \partial^\alpha\boldsymbol{a}_{\,\in\mathcal{P}\boldsymbol{L}^q, q\in(n,\infty], |\alpha|\le m}$ has following charactristics (2.5) adding to (2.1-4) above.

(2.5) $\partial^\alpha\boldsymbol{u} \in \boldsymbol{L}^r_{[0,T_M], r\in[2,q], |\alpha|\le m}$ []

proof

(1) First, charactricitcs (2.1) are proved as follows.

Following expression is confirmed for time $t_{\,\in[0,T_M]}$, providing $r \in [2,q]$.

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^r_t} \le C_1\|\boldsymbol{a}\|_{\boldsymbol{L}^r} + C_2\|\boldsymbol{u}\|_{\boldsymbol{L}^q_{[0,T_M]}} \int_0^t d\tau\, (\nu(t-\tau))^{-\frac{n}{2}\frac{1}{q}-\frac{1}{2}} \|\boldsymbol{u}\|_{\boldsymbol{L}^r_\tau}$$

Here, set A, X and operator $\mathcal{K}$ as follows, providing $t \in (0, T_M]$.

$$A = C_1\|\boldsymbol{a}\|_{\boldsymbol{L}^r}$$

$$X_t = \|\boldsymbol{u}\|_{\boldsymbol{L}^r_t}$$

$$\mathcal{K}f_t = C_2\|\boldsymbol{u}\|_{\boldsymbol{L}^q_{[0,T_M]}} \int_0^t d\tau\, (\nu(t-\tau))^{-\frac{n}{2}\frac{1}{q}-\frac{1}{2}} f_\tau$$

Then, the expression above can be expressed by following relation.

$$X \le A + \mathcal{K}X$$

Iterating use of this relation, following expression is confirmed for $k_{\,\in\boldsymbol{N}}$.

$$X \le \sum_{j=0}^{k} \mathcal{K}^j A + \mathcal{K}^{k+1} X$$

Then, for $j_{\,\in\boldsymbol{N}}$ and $t_{\,\in[0,T_M]}$, following expression is derived, providing $B(x,y)$s are beta functions, $\Gamma(x)$s are gamma functions and $\beta = \dfrac{1}{2}\Big(1 - \dfrac{n}{q}\Big)$.

$$\mathcal{K}^j A \le A\big(c_2\|\boldsymbol{u}\|_{\boldsymbol{L}^q_{[0,T_M]}}\nu^{-1}(\nu t)^{\beta}\big)^j\prod_{k=0}^{j-1}B(\beta,1+k\beta)$$
$$\le A\frac{1}{\Gamma(1+j\beta)}\big(C_2\Gamma(\beta)\|\boldsymbol{a}\|_{\boldsymbol{L}^q}\nu^{-1}(\nu t)^{\beta}\big)^j$$
$$\mathcal{K}^j X \le \|\boldsymbol{u}\|_{\boldsymbol{L}^q_{[0,T_M]}}\frac{1}{\Gamma(1+j\beta)}\big(C_2\Gamma(\beta)\|\boldsymbol{a}\|_{\boldsymbol{L}^q}\nu^{-1}(\nu t)^{\beta}\big)^j$$

Therefore, limit of $\mathcal{K}^k X$ as k approaches infinity equals 0, and following relations are confirmed, provided $t\in[0,T_M]$.

$$X \le U^{(r)}(\|\boldsymbol{a}\|_{\boldsymbol{L}^r},t) < \infty$$
$$U^{(r)}(\|\boldsymbol{a}\|_{\boldsymbol{L}^r},t) = C_1\|\boldsymbol{a}\|_{\boldsymbol{L}^r}\sum_{j=0}^{\infty}\frac{1}{\Gamma(1+j\beta)}\big(C_2\Gamma(\beta)\|\boldsymbol{a}\|_{\boldsymbol{L}^q}\nu^{-1}(\nu t)^{\beta}\big)^j$$

This concludes following result.

$$\sup_{t\in[0,T_M]}\|\boldsymbol{u}\|_{\boldsymbol{L}^r_t} < \infty$$

This is what is to be proved.

(2) Next, charactricitcs (2.2) are proved as follows.

In case of $r\in(q,\infty]$, by setting $\Phi(t) = t^{\varepsilon}{}_{,\varepsilon\in(0,(\frac{n}{2}(\frac{2}{q}-\frac{1}{r})+\frac{1}{2})^{-1})}$, following characteristics is confirmed under condition $\Phi_t\in(0,T_M]$.

$$\|\boldsymbol{u}^{\Phi}\|_{\boldsymbol{L}^r_t} \le C_1(\nu\Phi_t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\|\boldsymbol{a}\|_{\boldsymbol{L}^q} + c_2\|\boldsymbol{u}\|^2_{\boldsymbol{L}^q_{[0,T_M]}}\int_0^t d\tau\,(\nu\Phi_{t-\tau})^{-\frac{n}{2}(\frac{2}{q}-\frac{1}{r})-\frac{1}{2}}\varphi_\tau$$
$$\le C_1(\nu\Phi_t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\|\boldsymbol{a}\|_{\boldsymbol{L}^q} + C_2\nu^{-1}(\nu\Phi_t)^{\frac{1}{2}-\frac{n}{2}(\frac{2}{q}-\frac{1}{r})}\|\boldsymbol{a}\|^2_{\boldsymbol{L}^q}$$

Therefore following expressions are confirmed.

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^r_t} \le C_1(\nu t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\|\boldsymbol{a}\|_{\boldsymbol{L}^q} + C_2\nu^{-1}(\nu t)^{\frac{1}{2}-\frac{n}{2}(\frac{2}{q}-\frac{1}{r})}\|\boldsymbol{a}\|^2_{\boldsymbol{L}^q}$$
$$(\nu t)^{\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\|\boldsymbol{u}\|_{\boldsymbol{L}^r_t} \le C_1\|\boldsymbol{a}\|_{\boldsymbol{L}^q} + C_2\nu^{-1}(\nu t)^{\frac{1}{2}(1-\frac{n}{q})}\|\boldsymbol{a}\|^2_{\boldsymbol{L}^q}$$
$$(\nu t)^{\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\|\boldsymbol{u}\|_{\boldsymbol{L}^r_t} \le U^{(q)}(\|\boldsymbol{a}\|_{\boldsymbol{L}^q})$$

This is what is to be proved.

(3) Next, charactricitcs (2.3) are proved as follows.

In case of $r\in[2,q]$, by setting $Q=\min\{q,2r\}, \Phi(t) = t^{\varepsilon}{}_{,\varepsilon\in(0,(\frac{n}{2}(\frac{2}{Q}-\frac{1}{r})+\frac{1}{2}+\frac{|\alpha|}{2})^{-1})}$, following expression is confirmed under condition $\Phi_t\in(0,T_M]$.

$$\|\partial^{\alpha}\boldsymbol{u}^{\Phi}\|_{\boldsymbol{L}^r_t} \le C_1(\nu\Phi_t)^{-\frac{|\alpha|}{2}}\|\boldsymbol{a}\|_{\boldsymbol{L}^r} + C_2\|\boldsymbol{u}\|^2_{\boldsymbol{L}^Q_{[0,T_M]}}\int_0^t d\tau\,(\nu\Phi_{t-\tau})^{-\frac{n}{2}(\frac{2}{Q}-\frac{1}{r})-\frac{1}{2}-\frac{|\alpha|}{2}}\varphi_\tau$$
$$= C_1(\nu\Phi_t)^{-\frac{|\alpha|}{2}}\|\boldsymbol{a}\|_{\boldsymbol{L}^r} + C_2\nu^{-1}(\nu\Phi_t)^{\frac{1}{2}-\frac{n}{2}(\frac{2}{Q}-\frac{1}{r})-\frac{|\alpha|}{2}}b_\varepsilon\|\boldsymbol{u}\|^2_{\boldsymbol{L}^Q_{[0,T_M]}}$$
$$b_\varepsilon = \varepsilon B\Big(1-\varepsilon\Big(\frac{n}{2}\Big(\frac{2}{Q}-\frac{1}{r}\Big)+\frac{1}{2}+\frac{|\alpha|}{2}\Big),\varepsilon\Big)$$
$$\|\partial^{\alpha}\boldsymbol{u}\|_{\boldsymbol{L}^r_t} \le C_1(\nu t)^{-\frac{|\alpha|}{2}}\|\boldsymbol{a}\|_{\boldsymbol{L}^r} + C_2\nu^{-1}(\nu t)^{\frac{1}{2}-\frac{n}{2}(\frac{2}{Q}-\frac{1}{r})-\frac{|\alpha|}{2}}b_\varepsilon\|\boldsymbol{u}\|^2_{\boldsymbol{L}^Q_{[0,T_M]}}$$
$$(\nu t)^{\frac{|\alpha|}{2}}\|\partial^{\alpha}\boldsymbol{u}\|_{\boldsymbol{L}^r_t} \le U^{(r,\alpha)}(\|\boldsymbol{a}\|_{\boldsymbol{L}^r})$$

This is what is to be proved.

(4) Next, charactricitcs (2.4) are proved as follows.

In case of $r\in(q,\infty]$, by setting $\Phi(t) = t^{\varepsilon}{}_{,\varepsilon\in(0,(\frac{n}{2}(\frac{2}{q}-\frac{1}{r})+\frac{1}{2}+\frac{|\alpha|}{2})^{-1})}$, following expressin is confirmed under condition $\Phi_t \in (0, T_M]$.

$$\|\partial^\alpha \boldsymbol{u}^\Phi\|_{\boldsymbol{L}^r_t} \le C_1(\nu\Phi_t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-\frac{|\alpha|}{2}}\|\boldsymbol{a}\|_{\boldsymbol{L}^q} + c_2\|\boldsymbol{u}\|^2_{\boldsymbol{L}^q_{[0,T_M]}}\int_0^t d\tau\,(\nu\Phi_{t-\tau})^{-\frac{n}{2}(\frac{2}{q}-\frac{1}{r})-\frac{1}{2}-\frac{|\alpha|}{2}}\varphi_\tau$$

$$\le C_1(\nu\Phi_t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-\frac{|\alpha|}{2}}\|\boldsymbol{a}\|_{\boldsymbol{L}^q} + C_2\nu^{-1}(\nu\Phi_t)^{\frac{1}{2}-\frac{n}{2}(\frac{2}{q}-\frac{1}{r})-\frac{|\alpha|}{2}}b_\varepsilon\|\boldsymbol{a}\|^2_{\boldsymbol{L}^q}$$

$$b_\varepsilon = \varepsilon B\Big(1-\varepsilon\Big(\frac{n}{2}(\frac{2}{q}-\frac{1}{r})+\frac{1}{2}+\frac{|\alpha|}{2}\Big),\varepsilon\Big)$$

$$\|\partial^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^r_t} \le C_1(\nu t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-\frac{|\alpha|}{2}}\|\boldsymbol{a}\|_{\boldsymbol{L}^q} + C_2\nu^{-1}(\nu t)^{\frac{1}{2}-\frac{n}{2}(\frac{2}{q}-\frac{1}{r})-\frac{|\alpha|}{2}}b_\varepsilon\|\boldsymbol{a}\|^2_{\boldsymbol{L}^q}$$

$$(\nu t)^{\frac{n}{2}(\frac{1}{q}-\frac{1}{r})+\frac{|\alpha|}{2}}\|\partial^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^r_t} \le U^{(q,\alpha)}(\|\boldsymbol{a}\|_{\boldsymbol{L}^q})$$

This is what is to be proved.

(5) At last, charactricitcs (2.5) are proved as follows.

Here, $(2.5)_{|\alpha|\le k}$ for $k = 1, \dots, m$ are confirmed by mathematical induction. In case of $k = 1$, for α s that satisfy $|\alpha| = 1$, following expression is confirmed.

$$\|\partial^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^r_t} \le C_1\|\partial^\alpha \boldsymbol{a}\|_{\boldsymbol{L}^r} + C_2\|\boldsymbol{u}\|_{\boldsymbol{L}^q_{[0,T_M]}}\int_0^t d\tau\,(\nu(t-\tau))^{-\frac{n}{2}\frac{1}{q}-\frac{1}{2}}\|\partial^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^r_\tau}$$

Here, set A, X and operator $\mathcal{K}$ as follows for time $t_{\in(0,T_M]}$.

$$A = C_1\|\partial^\alpha \boldsymbol{a}\|_{\boldsymbol{L}^r}$$

$$X_t = \|\partial^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^r_t}$$

$$\mathcal{K}f_t = C_2\|\boldsymbol{u}\|_{\boldsymbol{L}^q_{[0,T_M]}}\int_0^t d\tau\,(\nu(t-\tau))^{-\frac{n}{2}\frac{1}{q}-\frac{1}{2}}f_\tau$$

Then, like arguement in (1), following results are confirmed for α s that satisfy $|\alpha| = 1$.

$$\sup_{t\in[0,T_M]}\|\partial^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^r_t} < \infty$$

Therefore $(2.5)_{|\alpha|=1}$ is confirmed.

In case of $(2.5)_{|\alpha|\le k\le m-1}$ is valid, for α s that satisfy $|\alpha| = k+1$, following relations are confirmed, providing $\beta+\gamma = \alpha$; $|\beta|, |\gamma| \le k$ for β, γ as regards to summention.

$$\|\partial^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^r_t} \le C_1\|\partial^\alpha \boldsymbol{a}\|_{\boldsymbol{L}^r} + C_2\sum_{\beta,\gamma}c_{\beta,\gamma}\|\partial^\beta \boldsymbol{u}\|_{\boldsymbol{L}^q_{[0,T_M]}}\int_0^t d\tau\,(\nu(t-\tau))^{-\frac{n}{2}\frac{1}{q}-\frac{1}{2}}\|\partial^\gamma \boldsymbol{u}\|_{\boldsymbol{L}^r_\tau}$$

$$+C_2\|\boldsymbol{u}\|_{\boldsymbol{L}^q_{[0,T_M]}}\int_0^t d\tau\,(\nu(t-\tau))^{-\frac{n}{2}\frac{1}{q}-\frac{1}{2}}\|\partial^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^r_\tau}$$

Here, A, X and operator $\mathcal{K}$ are defined as follows for time $t_{\in(0,T_M]}$.

$$A = C_1\|\partial^\alpha \boldsymbol{a}\|_{\boldsymbol{L}^r} + C_2\sum_{\beta,\gamma}c_{\beta,\gamma}\|\partial^\beta \boldsymbol{u}\|_{\boldsymbol{L}^q_{[0,T_M]}}\int_0^t d\tau\,(\nu(t-\tau))^{-\frac{n}{2}\frac{1}{q}-\frac{1}{2}}\|\partial^\gamma \boldsymbol{u}\|_{\boldsymbol{L}^r_\tau}$$

$$X_t = \|\partial^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^r_t}$$

$$\mathcal{K}f_t = C_2\|\boldsymbol{u}\|_{\boldsymbol{L}^q_{[0,T_M]}}\int_0^t d\tau\,(\nu(t-\tau))^{-\frac{n}{2}\frac{1}{q}-\frac{1}{2}}f_\tau$$

Then, like argument in (1), following results are comfirmed for α s that satisfy $|\alpha| = k+1$.

$$\sup_{t\in[0,T_M]}\|\partial^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^r_t} < \infty$$

Therefore, $(2.5)_{|\alpha|=k+1}$ is confirmed.

These above are what are to be proved. □

Theorem 1(Local in time solvability of NSIVP)

NSIVP(1.2) has a local in time solution.

[]

proof

According to **proposition 2**, integral equation (1.3) has a local in time solution that can be partially differentiated arbitraly times. Then integral equation (1.3) is equivallent to following integral equation.

$$\boldsymbol{u}_t = K_t * \boldsymbol{a} - \int_0^t d\tau\, K_{t-\tau} * \mathcal{P}((\boldsymbol{u}_\tau \cdot \boldsymbol{\partial})\boldsymbol{u}_\tau)$$

Therefore, this integral equation has local in time solution which can be partially derivated arbitraly times. Then, this integral equation is equivallent to NSIVP(1.2). Therefore, NSIVP(1.2) has a local in time solution. []

3. A priori estimation

Here, a priori estimation for NSIVP are confirmed. This corresponds to energy decreasing.

proposition 3(energy decreasing)

Following a priori estimation is confirmed for local in time solution $\boldsymbol{u}_{\in \mathcal{P}\boldsymbol{L}^q_{[0,T_M]}}$ of NSIVP for initial value $\boldsymbol{a}_{\in \mathcal{P}\boldsymbol{L}^q, q\in(n,\infty]}$.

(3.1) $\|\boldsymbol{u}\|_{\boldsymbol{L}^2} \le e^{-C\nu t}\|\boldsymbol{a}\|_{\boldsymbol{L}^2}$

proof

Solution $\boldsymbol{u}$ of NSIVP with initial value $\boldsymbol{a}$ satisfies following equation, providing $t \in [0, T_M]$.

$$\partial_t \boldsymbol{u}^2 = -2\nu \boldsymbol{\partial u \partial u} + 2\nu\boldsymbol{\partial}(\boldsymbol{u}\ \cdot\boldsymbol{\partial u}) - \boldsymbol{\partial}\Big(\cdot \boldsymbol{u}\Big(\boldsymbol{u}^2 + \frac{2}{\rho}p\Big)\Big) \quad , \ \boldsymbol{\partial u \partial u} = \sum_{i,j} \partial_i u_j \partial_i u_j$$

$$d_t\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2} + 2\nu\|\boldsymbol{\partial u}\|^2_{\boldsymbol{L}^2} = 0$$

$$d_t\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2} + 2C\nu\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2} \le 0$$

Therefore, by means of Grownwall inequality, (3.1) is confirmed. []

proposition 4(estimation of upper limit of solution)

Following a priori estimation for $\boldsymbol{L}^r_{t\,,\,r\in[2,\infty]}$-norm of local in time solution $\boldsymbol{u}_{\in \mathcal{P}\boldsymbol{L}^q_{[0,T_M]}}$ for NSIVP (1.2) with initial value $\boldsymbol{a}_{\in \mathcal{P}\boldsymbol{L}^q, q\in(n,\infty]}$ is confirmed, providing $\nu = \nu_0 + \nu_{1\,,\,\nu_0,\nu_1>0}$.

(3.2) $\|\boldsymbol{u}^{(L)}\|_{\boldsymbol{L}^r} \le C_0(\nu_0 t)^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{r})} e^{-C_2\nu_1 t}\|\boldsymbol{a}\|_{\boldsymbol{L}^2}$

$\|\boldsymbol{u}^{(NL)}\|_{\boldsymbol{L}^r} \le C_1\nu_0^{-1}(\nu_0 t)^{-\frac{n}{2}(1-\frac{1}{r})+\frac{1}{2}} e^{-C_2\nu_1 t}\|\boldsymbol{a}\|^2_{\boldsymbol{L}^2}$

(3.3) $\|\partial^\alpha \boldsymbol{u}^{(L)}\|_{\boldsymbol{L}^r} \le C_0(\nu_0 t)^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{r})-\frac{|\alpha|}{2}} e^{-C_2\nu_1 t}\|\boldsymbol{a}\|_{\boldsymbol{L}^2}$

$\|\partial^\alpha \boldsymbol{u}^{(NL)}\|_{\boldsymbol{L}^r} \le C_1\nu_0^{-1}(\nu_0 t)^{-\frac{n}{2}(1-\frac{1}{r})+\frac{1}{2}-\frac{|\alpha|}{2}} e^{-C_2\nu_1 t}\|\boldsymbol{a}\|^2_{\boldsymbol{L}^2}$ []

proof

Following expressions are confirmed by means of **lemma**.

$$\|\,K^{(\nu)} * \boldsymbol{a}\|_{\boldsymbol{L}^r} = \|\,K^{(\nu_0)} * K^{(\nu_1)} * \boldsymbol{a}\|_{\boldsymbol{L}^r}$$
$$\le c_0(\nu_0 t)^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{r})}\|K^{(\nu_1)}*\boldsymbol{a}\|_{\boldsymbol{L}^2} \le c_0(\nu_0 t)^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{r})} e^{-C_2\nu_1 t}\|\boldsymbol{a}\|_{\boldsymbol{L}^2}$$
$$\|\,\partial^\alpha K^{(\nu)} * \boldsymbol{a}\|_{\boldsymbol{L}^r} = \|\,\partial^\alpha K^{(\nu_0)} * K^{(\nu_1)} * \boldsymbol{a}\|_{\boldsymbol{L}^r}$$
$$\le c_0(\nu_0 t)^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{r})-\frac{|\alpha|}{2}}\|K^{(\nu_1)}*\boldsymbol{a}\|_{\boldsymbol{L}^2} \le c_0(\nu_0 t)^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{r})-\frac{|\alpha|}{2}} e^{-C_2\nu_1 t}\|\boldsymbol{a}\|_{\boldsymbol{L}^2}$$

Aiso, following expressions are confirmed by means of **lemma** and **proposition 3**, providing $\nu_2 = \dfrac{1}{2}\nu_0$.

$$\|\mathcal{P}(\boldsymbol{\partial} K^{(\nu)\,\Phi}_{t-\tau} * \cdot \boldsymbol{u}^\Phi_\tau \boldsymbol{u}^\Phi_\tau)\|_{\boldsymbol{L}^r} \le c(\nu_2 \Phi_{t-\tau})^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}}\|K^{(\nu-\nu_2)\,\Phi}_{t-\tau} * \cdot \boldsymbol{u}^\Phi_\tau \boldsymbol{u}^\Phi_\tau\|_{\boldsymbol{L}^2}$$
$$\le c(\nu_2 \Phi_{t-\tau})^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}} e^{-C_2\nu_1\Phi_{t-\tau}}\|K^{(\nu_2)\,\Phi}_{t-\tau} * \cdot \boldsymbol{u}^\Phi_\tau \boldsymbol{u}^\Phi_\tau\|_{\boldsymbol{L}^2}$$

$$\le c(\nu_2 \varPhi_{t-\tau})^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}} e^{-C_2\nu_1\varPhi_{t-\tau}} (\nu_2\varPhi_{t-\tau})^{-\frac{n}{2}(1-\frac{1}{2})} \|\cdot \boldsymbol{u}_\tau^{\varPhi} \boldsymbol{u}_\tau^{\varPhi}\|_{\boldsymbol{L}^1}$$
$$= c_1(\nu_0\varPhi_{t-\tau})^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}} e^{-C_2\nu_1\varPhi_{t-\tau}} \|\boldsymbol{u}_\tau^{\varPhi}\|_{\boldsymbol{L}^2}^2$$
$$\le c_1(\nu_0\varPhi_{t-\tau})^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}} e^{-C_2\nu_1\varPhi_{t-\tau}} e^{-2C_2\nu\varPhi_\tau} \|\boldsymbol{a}\|_{\boldsymbol{L}^2}^2$$
$$\le c_1(\nu_0\varPhi_{t-\tau})^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}} e^{-C_2\nu_1\varPhi_{t}} e^{-C_2(\nu+\nu_0)\varPhi_\tau} \|\boldsymbol{a}\|_{\boldsymbol{L}^2}^2$$
$$\|\boldsymbol{u}_t^{(NL)\varPhi}\|_{\boldsymbol{L}^r} \le c_1 e^{-C_2\nu_1\varPhi_t} \|\boldsymbol{a}\|_{\boldsymbol{L}^2}^2 \int_0^t d\tau\, (\nu_0\varPhi_{t-\tau})^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}} e^{-C_2(\nu+\nu_0)\varPhi_\tau}$$
$$\le C_1\nu_0^{-1}(\nu_0\varPhi_t)^{-\frac{n}{2}(1-\frac{1}{r})+\frac{1}{2}} e^{-C_2\nu_1\varPhi_t} \|\boldsymbol{a}\|_{\boldsymbol{L}^2}^2$$
$$\|\mathcal{P}(\partial^\alpha \boldsymbol{\partial} K_{t-\tau}^{(\nu)\,\varPhi} *\cdot \boldsymbol{u}_\tau^{\varPhi}\boldsymbol{u}_\tau^{\varPhi})\|_{\boldsymbol{L}^r} \le c(\nu_2\varPhi_{t-\tau})^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}-\frac{|\alpha|}{2}} \|K_{t-\tau}^{(\nu-\nu_2)\,\varPhi} *\cdot \boldsymbol{u}_\tau^{\varPhi}\boldsymbol{u}_\tau^{\varPhi}\|_{\boldsymbol{L}^2}$$
$$\le c(\nu_2\varPhi_{t-\tau})^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}-\frac{|\alpha|}{2}} e^{-C_2\nu_1\varPhi_{t-\tau}} \|K_{t-\tau}^{(\nu_2)\,\varPhi} *\cdot \boldsymbol{u}_\tau^{\varPhi}\boldsymbol{u}_\tau^{\varPhi}\|_{\boldsymbol{L}^2}$$
$$\le c(\nu_2\varPhi_{t-\tau})^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}-\frac{|\alpha|}{2}} e^{-C_2\nu_1\varPhi_{t-\tau}} (\nu_2\varPhi_{t-\tau})^{-\frac{n}{2}(1-\frac{1}{2})} \|\cdot \boldsymbol{u}_\tau^{\varPhi}\boldsymbol{u}_\tau^{\varPhi}\|_{\boldsymbol{L}^1}$$
$$= c_1(\nu_0\varPhi_{t-\tau})^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}-\frac{|\alpha|}{2}} e^{-C_2\nu_1\varPhi_{t-\tau}} \|\boldsymbol{u}_\tau^{\varPhi}\|_{\boldsymbol{L}^2}^2$$
$$\le c_1(\nu_0\varPhi_{t-\tau})^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}-\frac{|\alpha|}{2}} e^{-C_2\nu_1\varPhi_{t-\tau}} e^{-2C_2\nu\varPhi_\tau} \|\boldsymbol{a}\|_{\boldsymbol{L}^2}^2$$
$$\le c_1(\nu_0\varPhi_{t-\tau})^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}} e^{-C_2\nu_1\varPhi_{t}} e^{-C_2(\nu+\nu_0)\varPhi_\tau} \|\boldsymbol{a}\|_{\boldsymbol{L}^2}^2$$
$$\|\partial^\alpha \boldsymbol{u}_t^{(NL)\varPhi}\|_{\boldsymbol{L}^r} \le c_1 e^{-C_2\nu_1\varPhi_t} \|\boldsymbol{a}\|_{\boldsymbol{L}^2}^2 \int_0^t d\tau\, (\nu_0\varPhi_{t-\tau})^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}-\frac{|\alpha|}{2}} e^{-C_2(\nu+\nu_0)\varPhi_\tau}$$
$$\le C_1\nu_0^{-1}(\nu_0\varPhi_t)^{-\frac{n}{2}(1-\frac{1}{r})+\frac{1}{2}-\frac{|\alpha|}{2}} e^{-C_2\nu_1\varPhi_t} \|\boldsymbol{a}\|_{\boldsymbol{L}^2}^2 \qquad []$$

4. analysys of global in time solvability

proposition 5 (global in time solvability of NSIVP)

In regard to NSIVP with initial value $\boldsymbol{a}_{\in\mathcal{P}\boldsymbol{L}^q, q\in(n,\infty]}$, following results are confirmed.

(1) For a local in time solution $\boldsymbol{u}_{\in\mathcal{P}\boldsymbol{L}^q_{[0,T]}}$, there exist expanded time $\varDelta T = \varDelta T(\|\boldsymbol{a}\|_{\boldsymbol{L}^2}, T)$ and expanded local in time solution $\boldsymbol{u}_{\in\mathcal{P}\boldsymbol{L}^q_{[0,T+\varDelta T]}}$. Moreover, $\varDelta T(\|\boldsymbol{a}\|_{\boldsymbol{L}^2}, T)$ is increasing function of T.

(2) There exists global in time solution $\boldsymbol{u}$. This satisfies $\boldsymbol{u} \in \mathcal{P}\boldsymbol{L}^q_{[0,\infty)} \cap \mathcal{P}\boldsymbol{L}^{(q,\infty]}_{(0,\infty)}, \partial^\alpha \boldsymbol{u} \in \mathcal{P}\boldsymbol{L}^{[2,\infty]}_{(0,\infty)}$. []

proof

(1) is confirmed as follows.

Resetting the endpoint time of finite time interval of a solution $\boldsymbol{u}_{t, t\in(0,T], T\in(0,\infty)}$ as initial time resetting the solution at endpoint time as initial value, and using solvability analisys (**proposition 1**), expanded solution is constructed. Based on a priori estimation of expanded solution at time $t = T$: $\|\boldsymbol{u}_T\|_{\boldsymbol{L}^q} \le U_2^{(q)}(\|\boldsymbol{a}\|_{\boldsymbol{L}^2}, T)_{, q\in(n,\infty]}$(**proposition 4**), for existance time $\varDelta T$ of the expanded solution, $\varDelta T \ge TM(U_2^{(q)}(\|\boldsymbol{a}\|_{\boldsymbol{L}^2}, T))$ is confirmed. Because $U_2^{(q)}(\|\boldsymbol{a}\|_{\boldsymbol{L}^2}, t)$ is a decreasing function of t and $TM(\xi)$ is discreasing function of ξ, $TM(U_2^{(q)}(\|\boldsymbol{a}\|_{\boldsymbol{L}^2}, T))$ is increasing function of T.

(2) is confirmed as follows.

Based on local in time solvability of NSIVP (**proposition 1**), solution of NSIVP in finite time interval is expandable ((1)abobe) and its expanded existing time increase to the length of existing time interval ((1)above), by iterating expansion of solution, for arbitrary length of interval, solution can be constructed. Therefore NSIVP is global in time solvable.

Moreover, by means of **proposition 2**, $\boldsymbol{u} \in \mathcal{P}\boldsymbol{L}^{[2,q]}_{[0,\infty)} \cap \mathcal{P}\boldsymbol{L}^{(q,\infty]}_{(0,\infty)}, \partial^\alpha \boldsymbol{u} \in \mathcal{P}\boldsymbol{L}^{[2,\infty]}_{(0,\infty)}$ are comfirmed. []

proposition 6 (asymptotic attnueation characteristic and global in time boundedness of solution of NSIVP)

For global in time solution $\boldsymbol{u}$ of NSIVP(1.2) with initial value $\boldsymbol{a}_{\in\mathcal{P}\boldsymbol{L}^{[2,q]},q\in(n,\infty]}$, following results are confirmed.

(1) asymptotic attnueation characteristic

Solution $\boldsymbol{u}$ and its partial derivertives in regard to spatial cordinates have asymptotic attnueation characteristic for $t\to\infty$ as follows.

(4.1) $\|\boldsymbol{u}^{(L)}\|_{\boldsymbol{L}_t^r} = O(t^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{r})})$

$\|\boldsymbol{u}^{(NL)}\|_{\boldsymbol{L}_t^r} = O(t^{-\frac{n}{2}(1-\frac{1}{r})+\frac{1}{2}})$

(4.2) $\|\partial^\alpha\boldsymbol{u}^{(L)}\|_{\boldsymbol{L}_t^r} = O(t^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{r})-\frac{|\alpha|}{2}})$

$\|\partial^\alpha\boldsymbol{u}^{(NL)}\|_{\boldsymbol{L}_t^r} = O(t^{-\frac{n}{2}(1-\frac{1}{r})+\frac{1}{2}-\frac{|\alpha|}{2}})$

(2) global in time boundedness

Solution $\boldsymbol{u}$ is global in time bounded as follows.

(4.3) $\|\boldsymbol{u}\|_{\boldsymbol{L}^r_{[0,\infty)}} < \infty \qquad {}_{,r\in[2,q]}$ []

proof

(1) asymptotic attnueation characteristic

Based on **proposition 5** and, (3.2) in regard to (4.1), (3.3) in regard to (4.2), these are confirmed.

(2) global in time boundedness

Based on **proposition 1**, $\|\boldsymbol{u}^{(NL)}\|_{\boldsymbol{L}^q} \le C\|\boldsymbol{a}\|_{\boldsymbol{L}^q,t\in[0,T_M]}$ is confirmed, and based on **proposition 3** $\|\boldsymbol{u}^{(NL)}\|_{\boldsymbol{L}^2} = \|\boldsymbol{u}-\boldsymbol{u}^{(L)}\|_{\boldsymbol{L}^2}$ is confirmed; and interpolating these, following relation is comfirmed.

$$\|\boldsymbol{u}^{(NL)}\|_{\boldsymbol{L}^r} \le C_1^{(r)}\|\boldsymbol{a}\|^{\theta_q}_{\boldsymbol{L}^q}\|\boldsymbol{a}\|^{\theta_2}_{\boldsymbol{L}^2}\,{}_{,\theta_q=\frac{\frac{1}{2}-\frac{1}{r}}{\frac{1}{2}-\frac{1}{q}},\theta_2=\frac{\frac{1}{r}-\frac{1}{q}}{\frac{1}{2}-\frac{1}{q}},t\in[0,T_M],r\in[2,q]}$$

On the other hand, following relation is confirmed based on **proposition 5** and **proposition 4**.

$$\|\boldsymbol{u}^{(NL)}\|_{\boldsymbol{L}^r} \le c_2^{(r)}\nu^{-1}(\nu t)^{-\gamma}\|\boldsymbol{a}\|^2_{\boldsymbol{L}^2} \le c_2^{(r)}\nu^{-1}(\nu T_M)^{-\gamma}\|\boldsymbol{a}\|^2_{\boldsymbol{L}^2,t\in[T_M,\infty),r\in[2,q]}$$

Therefore, with expression of T_M in the proof of **proposition 1**, following relation that gives (4.3) is confirmed.

$$\begin{aligned}\|\boldsymbol{u}^{(NL)}\|_{\boldsymbol{L}^r} &\le \max\{C_1^{(r)}\|\boldsymbol{a}\|^{\theta_q}_{\boldsymbol{L}^q}\|\boldsymbol{a}\|^{\theta_2}_{\boldsymbol{L}^2}, c_2^{(r)}\nu^{-1}(\nu T_M)^{-\gamma}\|\boldsymbol{a}\|^2_{\boldsymbol{L}^2}\}\\ &\le \max\{C_1^{(r)}\|\boldsymbol{a}\|^{\theta_q}_{\boldsymbol{L}^q}\|\boldsymbol{a}\|^{\theta_2}_{\boldsymbol{L}^2}, C_2^{(r)}\nu^{-1-\gamma_\beta}\|\boldsymbol{a}\|^{\gamma_\beta}_{\boldsymbol{L}^q}\|\boldsymbol{a}\|^2_{\boldsymbol{L}^2}\}_{,\gamma_\beta=\frac{\gamma}{\beta}}\end{aligned}$$

proposition 7 (equable continuous on initial value and uniqueness of solution of NSIVP)

Global in time solution of NSIVP(1.2) correspond to initial value $\boldsymbol{a}_{\in\mathcal{P}\boldsymbol{L}^{[2,q]}}$ depends equably and continuously on initial value $\boldsymbol{a}$, moreover and is unique. []

proof

For 2 solutions $\boldsymbol{u}^{(1)},\boldsymbol{u}^{(2)}$ of NSIVP which correspond to 2 initial values $\boldsymbol{a}^{(1)},\boldsymbol{a}^{(2)}{}_{\in\mathcal{P}\boldsymbol{L}^q}$ respectively, following expressions are confirmed, providing $r\in[2,q]$ and $t\in[0,T_M]$.

$$\boldsymbol{u}_t^{(1)}-\boldsymbol{u}_t^{(2)} = K_t*(\boldsymbol{a}^{(1)}-\boldsymbol{a}^{(2)}) - \int_0^t d\tau\,\mathcal{P}\boldsymbol{\partial}K_{t-\tau}*(\cdot\boldsymbol{u}_\tau^{(1)}\boldsymbol{u}_\tau^{(1)} - \cdot\boldsymbol{u}_\tau^{(2)}\boldsymbol{u}_\tau^{(2)})$$

$$\|\boldsymbol{u}^{(1)}-\boldsymbol{u}^{(2)}\|_{\boldsymbol{L}_t^r} \le C_1\|\boldsymbol{a}^{(1)}-\boldsymbol{a}^{(2)}\|_{\boldsymbol{L}^r}$$

$$+C_2(\|\boldsymbol{u}^{(1)}\|_{\boldsymbol{L}^q_{[0,T_M]}}+\|\boldsymbol{u}^{(2)}\|_{\boldsymbol{L}^q_{[0,T_M]}})\int_0^t d\tau\,(\nu(t-\tau))^{-\frac{n}{2}\frac{1}{q}-\frac{1}{2}}\|\boldsymbol{u}^{(1)}-\boldsymbol{u}^{(2)}\|_{\boldsymbol{L}^r_\tau}$$

Therefore, similarly as in proof of **proposition 2** (1), following expressions are confirmed, providing $t\in[0,T_M]$.

$$\|\boldsymbol{u}^{(1)}-\boldsymbol{u}^{(2)}\|_{\boldsymbol{L}^r_t}\le C_1\|\boldsymbol{a}^{(1)}-\boldsymbol{a}^{(2)}\|_{\boldsymbol{L}^r}U^{(r)}(t)$$

$$\sup_{t\in[0,T_M]}\|\boldsymbol{u}^{(1)}-\boldsymbol{u}^{(2)}\|_{\boldsymbol{L}^r_t}\le C_1\|\boldsymbol{a}^{(1)}-\boldsymbol{a}^{(2)}\|_{\boldsymbol{L}^r}U^{(r)}(T_M)$$

$$U^{(r)}(t)=\sum_{j=0}^{\infty}\frac{1}{\Gamma(1+j\beta)}\big(C_2\Gamma(\beta)(\|\boldsymbol{u}^{(1)}\|_{\boldsymbol{L}^q_{[0,T_M]}}+\|\boldsymbol{u}^{(2)}\|_{\boldsymbol{L}^q_{[0,T_M]}})\nu^{-1}(\nu t)^{\beta}\big)^j$$

On the other hand, similarly as in proof of **proposition 4**, following expressions are confirmed, providing $r\in[2,q]$ and $t>0$.

$$\|\boldsymbol{u}^{(1)\Phi}_t-\boldsymbol{u}^{(2)\Phi}_t\|_{\boldsymbol{L}^r}\le C_1(\nu\Phi_t)^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{r})}\|\boldsymbol{a}^{(1)}-\boldsymbol{a}^{(2)}\|_{\boldsymbol{L}^2}$$
$$+c_2(\|\boldsymbol{a}^{(1)}\|_{\boldsymbol{L}^2}+\|\boldsymbol{a}^{(2)}\|_{\boldsymbol{L}^2})\int_0^t d\tau\,(\nu\Phi_{t-\tau})^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}}\varphi_\tau\|\boldsymbol{a}^{(1)}-\boldsymbol{a}^{(2)}\|_{\boldsymbol{L}^2}$$
$$\le C_1(\nu\Phi_t)^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{r})}\|\boldsymbol{a}^{(1)}-\boldsymbol{a}^{(2)}\|_{\boldsymbol{L}^2}$$
$$+C_2\nu^{-1}(\nu\Phi_t)^{-\frac{n}{2}(1-\frac{1}{r})+\frac{1}{2}}(\|\boldsymbol{a}^{(1)}\|_{\boldsymbol{L}^2}+\|\boldsymbol{a}^{(2)}\|_{\boldsymbol{L}^2})\|\boldsymbol{a}^{(1)}-\boldsymbol{a}^{(2)}\|_{\boldsymbol{L}^2}$$
$$\|\boldsymbol{u}^{(1)}-\boldsymbol{u}^{(2)}\|_{\boldsymbol{L}^r_t}\le\|\boldsymbol{a}^{(1)}-\boldsymbol{a}^{(2)}\|_{\boldsymbol{L}^2}U^{(r)}_2(t)$$

Therefore, following expression is confirmed, providing $t\in[T_M,\infty)$.

$$\sup_{t\in[T_M,\infty)}\|\boldsymbol{u}^{(1)}-\boldsymbol{u}^{(2)}\|_{\boldsymbol{L}^r_t}\le\|\boldsymbol{a}^{(1)}-\boldsymbol{a}^{(2)}\|_{\boldsymbol{L}^2}U^{(r)}_2(T_M)$$

Therefore, following expression is confirmed.

$$\sup_{t\in[0,\infty)}\|\boldsymbol{u}^{(1)}-\boldsymbol{u}^{(2)}\|_{\boldsymbol{L}^r_t}\le\|\boldsymbol{a}^{(1)}-\boldsymbol{a}^{(2)}\|_{\boldsymbol{L}^r}\max\{U^{(r)}(T_M),U^{(r)}_2(T_M)\}$$

Therefore providing $\|\boldsymbol{a}^{(1)}-\boldsymbol{a}^{(2)}\|_{\boldsymbol{L}^r}\to 0$, $\sup_{t\in[0,\infty)}\|\boldsymbol{u}^{(1)}-\boldsymbol{u}^{(2)}\|_{\boldsymbol{L}^r_t}\to 0$ is confirmed. Namely, solutions equably and continuously depends on initial values.

By setting $\boldsymbol{a}^{(1)}=\boldsymbol{a}^{(2)}$ in arguement above, $\boldsymbol{u}^{(1)}=\boldsymbol{u}^{(2)}$ is confirmed. Namely, solution is unique.[]

5. general mean value problem

Whereas, in main part within the preceding sections, problems with initial value functions and solutions of which mean values are zeros are considered, in this section, problems with initial value functions and solutions of which mean values are general constant values $\boldsymbol{c}_{\in\boldsymbol{R}^n}$s are considered.

Namely, initial value problem of which initial value function $\boldsymbol{a}^{\#}$ and solution $\boldsymbol{u}^{\#}$ satisfy following relations (5.1)(5.2) is analized.

(5.1) $\boldsymbol{a}^{\#}\in\mathcal{P}\boldsymbol{L}^q$, $\partial^\alpha\boldsymbol{a}^{\#}\in\boldsymbol{L}^q$ $_{;\,q\in(n,\infty];\,|\alpha|=1,\dots,m}$

$\boldsymbol{a}^{\#}_{\Omega}=\boldsymbol{c}$

(5.2) $\boldsymbol{u}^{\#}\in\mathcal{P}\boldsymbol{L}^q_{(0,\infty)}$

$$\partial_t\boldsymbol{u}^{\#}-\nu\triangle\boldsymbol{u}^{\#}+(\boldsymbol{u}^{\#}\cdot\boldsymbol{\partial})\boldsymbol{u}^{\#}+\frac{1}{\rho}\boldsymbol{\partial}p=\boldsymbol{0}\qquad{}_{,(t,\boldsymbol{x})\in(0,\infty)\times\boldsymbol{R}^n}$$

$$\boldsymbol{u}^{\#}(0,\boldsymbol{x})=\boldsymbol{a}^{\#}(\boldsymbol{x})\qquad{}_{,\boldsymbol{x}\in\boldsymbol{R}^n}$$

$$\boldsymbol{u}^{\#}_{\Omega}=\boldsymbol{c}\qquad{}_{,t\in(0,\infty)}$$

Global in time existence and property of solution of this problem can be analized by means of function transformation, and by similar arguments as in main part. Namely, by setting $\widetilde{\boldsymbol{u}} = \boldsymbol{u}^{\#} - \boldsymbol{c}, \widetilde{\boldsymbol{a}} = \boldsymbol{a}^{\#} - \boldsymbol{c}$, transformed initial value problem that have initial value function $\widetilde{\boldsymbol{a}}$ and solution $\widetilde{\boldsymbol{u}}$ with following conditions $(5.1)\tilde{}(5.2)\tilde{}$ is derived.

$$(5.1)\tilde{}\quad \widetilde{\boldsymbol{a}} \in \mathcal{P}\boldsymbol{L}^q \ , \ \partial^\alpha \widetilde{\boldsymbol{a}} \in \boldsymbol{L}^q \ {}_{;\, q\in(n,\infty];\, |\alpha|=1,\ldots,m}$$
$$\widetilde{\boldsymbol{a}}_\Omega = \boldsymbol{0}$$

$$(5.2)\tilde{}\quad \widetilde{\boldsymbol{u}} \in \mathcal{P}\boldsymbol{L}^q_{(0,\infty)}$$
$$\partial_t \widetilde{\boldsymbol{u}} - \nu\triangle\widetilde{\boldsymbol{u}} + (\boldsymbol{c}\cdot\boldsymbol{\partial})\widetilde{\boldsymbol{u}} + (\widetilde{\boldsymbol{u}}\cdot\boldsymbol{\partial})\widetilde{\boldsymbol{u}} + \frac{1}{\rho}\boldsymbol{\partial} p = \boldsymbol{0} \qquad {}_{,(t,\boldsymbol{x})\in(0,\infty)\times\boldsymbol{R}^n}$$
$$\widetilde{\boldsymbol{u}}(0,\boldsymbol{x}) = \widetilde{\boldsymbol{a}}(\boldsymbol{x}) \qquad {}_{,\boldsymbol{x}\in\boldsymbol{R}^n}$$
$$\widetilde{\boldsymbol{u}}_\Omega = \boldsymbol{0} \qquad {}_{,t\in(0,\infty)}$$

Partial differential equation in $(5.2)\tilde{}$ differs from Navier-Stokes equation in that this has linear first order spatial partial differential term. To analize this equation, following kernel $\widetilde{K}^{(\nu)}$ that corresponds to the initial value problem for linear equation $\partial_t f - \nu\triangle f + (\boldsymbol{c}\cdot\boldsymbol{\partial})f = 0$ in the whole space $\boldsymbol{R}^n$ is introduced, instead of heat kernel $K^{(\nu)}$.

$$\widetilde{K}^{(\nu)}(t,\boldsymbol{x}) = \frac{1}{\sqrt{4\pi\nu t}^{\,n}} \exp\Big(- \frac{(\boldsymbol{x}-\boldsymbol{c}t)^2}{4\nu t}\Big)$$

Following integral equation (5.3) is equivalent to initial value problem $(5.1)\tilde{}(5.2)\tilde{}$, providing existence of derivertives of solution.

$$(5.3)\quad \widetilde{\boldsymbol{u}}_t = \widetilde{K}_t * \widetilde{\boldsymbol{a}} - \int_0^t d\tau\, \mathcal{P}(\boldsymbol{\partial}\widetilde{K}_{t-\tau} *\cdot \widetilde{\boldsymbol{u}}_\tau \widetilde{\boldsymbol{u}}_\tau) \qquad {}_{,(t,\boldsymbol{x})\in(0,\infty)\times\boldsymbol{R}^n}$$

Kernel $\widetilde{K}$ has similar charasteristic as heat kernel K, especially expressions $\|\widetilde{K}\|_{L^q_t(\boldsymbol{R}^n)} = \|K\|_{L^q_t(\boldsymbol{R}^n)}, \|\partial^\alpha\widetilde{K}\|_{L^q_t(\boldsymbol{R}^n)} = \|\partial^\alpha K\|_{L^q_t(\boldsymbol{R}^n)}\,{}_{,q\in[1,\infty]}$ are comfirmed. Based on these expressions, almost similar results in previous section are confirmed and it is confirmed that this problem has global in time solution with following form.

$$\boldsymbol{u}^{\#} = \boldsymbol{c} + \widetilde{\boldsymbol{u}}^{(L)} + \widetilde{\boldsymbol{u}}^{(NL)}$$

On the other hand, solution $\boldsymbol{u}$ of problem in previous sections and solution $\boldsymbol{u}^{\#}$ of problem in this section have following relation.

$$(5.4)\quad \boldsymbol{u}^{\#}(t,\boldsymbol{x}) = \boldsymbol{c} + \boldsymbol{u}(t,\boldsymbol{x}-\boldsymbol{c}t)$$

This is proved as follows.

$$\begin{aligned}(\partial_t - \nu\triangle)\boldsymbol{u}^{\#}(t,\boldsymbol{x}) &= (\partial_t - \nu\triangle)\boldsymbol{u}(t,\boldsymbol{x}-\boldsymbol{c}t) - (\boldsymbol{c}\cdot\boldsymbol{\partial})\boldsymbol{u}(t,\boldsymbol{x}-\boldsymbol{c}t)\\ &= -((\boldsymbol{u}(t,\boldsymbol{x}-\boldsymbol{c}t)+\boldsymbol{c})\cdot\boldsymbol{\partial})\boldsymbol{u}(t,\boldsymbol{x}-\boldsymbol{c}t) - \frac{1}{\rho}\boldsymbol{\partial} p\\ &= -(\boldsymbol{u}^{\#}(t,\boldsymbol{x})\cdot\boldsymbol{\partial})\boldsymbol{u}^{\#}(t,\boldsymbol{x}) - \frac{1}{\rho}\boldsymbol{\partial} p\end{aligned}$$

Using relation (5.4), based on extistence of solution of problem in previous sections, existence of solution of problem in this section is automatically confirmed.

6. initial value and boundry value problem

Whereas, in the previous section, problem with initial value $\boldsymbol{a}$ with mean value $\boldsymbol{a}_\Omega$ and solution $\boldsymbol{u}$ that converges to $\boldsymbol{a}_\Omega$, i.e. for $t\to\infty$ $\boldsymbol{u}_t \to \boldsymbol{a}_\Omega$, is argued, in this section, problem of which solution $\boldsymbol{u}$ has initial value $\boldsymbol{a}$ and constant boundry balue $\boldsymbol{b}$ on boundry $\partial\Omega$ is argued. Namaly, initial value boundry value problem that initial value $\boldsymbol{a}$ and solution $\boldsymbol{u}$ satisfy following conditions (6.1)(6.2) is analized.

$$(6.1)\quad \boldsymbol{a} \in \mathcal{P}\boldsymbol{L}^q \ , \ \partial^\alpha \boldsymbol{a} \in \boldsymbol{L}^q \ {}_{;\, q\in(n,\infty];\, |\alpha|=1,\ldots,m}$$

$$\boldsymbol{a}(\boldsymbol{x}) = \boldsymbol{b} \qquad {}_{,\boldsymbol{x}\in\partial\Omega}$$

(6.2) $$\boldsymbol{u} \in \mathcal{P}\boldsymbol{L}^q_{(0,\infty)}$$

$$\partial_t \boldsymbol{u} - \nu\triangle\boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\partial})\boldsymbol{u} + \frac{1}{\rho}\boldsymbol{\partial}p = \boldsymbol{0} \qquad {}_{,(t,\boldsymbol{x})\in(0,\infty)\times\boldsymbol{R}^n}$$

$$\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{a}(\boldsymbol{x}) \qquad {}_{,\boldsymbol{x}\in\boldsymbol{R}^n}$$

$$\boldsymbol{u}(t,\boldsymbol{x}) = \boldsymbol{b} \qquad {}_{,t\in(0,\infty),\boldsymbol{x}\in\partial\Omega}$$

This problem is global in time solvable and its solution converges to $\boldsymbol{b}$. Analysis with regard to this problem can be performed by means of following kernel K that corresponds to heat equation $\partial_t f - \nu\triangle f = 0$ with periodic boundry which differs from main part (sections 1-4).

$$K(t,\boldsymbol{x}) = \sum_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} \frac{1}{\sqrt{4\pi\nu t}^n} \exp\Big(-\frac{(\boldsymbol{x}-\boldsymbol{\lambda})^2}{4\nu t}\Big)$$

Following integral equation is equivallent to the initial value boundry value problem here, providing existence of derivatives of solution.

(6.3) $$\boldsymbol{u}_t = K_t{*}\boldsymbol{a} - \int_0^t d\tau \int_{\partial\Omega} d\boldsymbol{S}\cdot\boldsymbol{\partial}K_{t-\tau}\,\boldsymbol{b} - \int_0^t d\tau\,\mathcal{P}(\boldsymbol{\partial}K_{t-\tau}{*}\cdot\boldsymbol{u}_\tau\boldsymbol{u}_\tau) \qquad {}_{,(t,\boldsymbol{x})\in(0,\infty)\times\boldsymbol{R}^n}$$

Therefore, it is understood that as for $\boldsymbol{a}$ with mean value $\boldsymbol{a}_\Omega = \boldsymbol{c}$ and constant boundry value $\boldsymbol{b}$ on boundry $\partial\Omega$ and $\boldsymbol{c} \neq \boldsymbol{b}$, problem in previous section (secsion 5) and problem in this section (section 6) have solutions that converges different value, namely these solutions are different.

This means that as for problem with regard to periodic equation without additive condition like mean value condition or boundry value condition doesn't have unique solution.

note (relation to CMI problem)

Relation between results of this paper and CMI (Clay Mathematical Institute) problem[8] is as follows.

In the CMI problem, 4 candidate propositions (A)(B)(C)(D) are given, 2 of which (B)(D) correspond to initial value problem over 3 dimensional periodic space $\boldsymbol{R}^3$. In these 2 propositions (B)(D), regard to initial value, conditions that are consisted from (1)nondivergence, and (2) existence of arbitraly times derivatives on spatial variables and its spatial discreasing.

$$|\,\partial^\alpha \boldsymbol{a}\,| \leq C_{\alpha,K}(1+x)^{-K} \qquad {}_{,\boldsymbol{x}\in\boldsymbol{R}^3,\alpha\in(\{0\}\cup\boldsymbol{N})^3,K\geq 0}$$

These 2 conditions are sufficient condition for initial value condition $\boldsymbol{a} \in \mathcal{P}\boldsymbol{L}^2 \cap \boldsymbol{L}^\infty$ in the present paper, therefore, results of the present paper conclude for the CMI problem, viz. proposition (B) is comfirmed and proposition (D) is denied.

Results of the present paper give not only existence of global in time solutions, but also various characteristic regard to solutions like uniqueness, partial derivativability, equable continuous on initial value, etc. Also results of the present paper weaken conditions of propositions from these 2 propositions (A)(B), and give similar results for comprehensive initial values.

Nevertheless, as mentioned in section 6, CMI problem has uncertainty, that is, uniqueness of solution is not garanteed and this is based on lack of contitions. For problems that have appropriate contitions like in main part (sections 1-4), in section 5 and in section 6, uniqueness of solution is garanteed.

On the other hand, results related to 2 propositions (A)(C) are given in other paper[9], which are like as results related to 2 propositions (B)(D). Although, as for uniqueness, attention should be required.

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