*Title of the article*

# On the dimensional characteristics and interpretation of vectors

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**Abstract**

The paper proposes a generalization of geometric notion of vectors concerning dimensionality of the configuration space. In certain dimensional spaces, certain types of ordered directions exist along which elements of vector spaces can be interpreted. Scalars along the ordered directions form Banach spaces. Different types of geometrical vectors are algebraically identical, the difference arises in the configuration space geometrically. In the universe four types of vectors exists. Thus any physical quantity in the universe comes in four types of vectors. Though All the types of vectors belong to different Banach spaces (& their directions can't be compared), their magnitudes can be compared. A gross comparison between the magnitudes of the different typed geometric vectors is obtained at end of the paper.

**Keywords:** Vectors, Banach spaces, spacetime

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## 1. Introduction

The objective of this paper is to provide generally possible interpretations of the vectors and corresponding vector spaces. The geometrical generalization of vectors based on dimensionality of the configuration space is discussed. Actually the starting point of the proposed theory is dimensional characteristics associated with the configuration space. The intrinsic characteristics of a specific numbered dimensional space have intrinsic analytical importance. The intrinsic dimensional characteristics those come with every number of dimension are conventionally called as n-volume and n-plane. Both these characteristics imply classes of length, area, volume and point, line, plane respectively. There is another important dimensional characteristic-*geometrical relation* meaning class of distance, angle & solid angle. We found such characteristics useful for defining notion of vector. In the initial section we make mathematical propositions and prove the useful theorems amenable to provide a general interpretation of vectors & vector spaces. It will be proved that every number of dimensions comes with a type of ordered direction, facilitating definition of corresponding dimensional vector. Hence different types of vectors can be constructed or identified based on the dimension of underlying ordered direction. It is also that different types of vectors can be interpreted to be elements of arbitrary vector space. The theorems in section 2 regard general n-dimensional spaces. Then in the successive section 3, special case of the universe as 4-dimensional space locally having time evolution & three spatial dimensions is considered. In that section, the types of vectors existing in the universe and their properties are discussed. The theory model is developed in section 2 and applied to a case in section 3. Results from the paper are to be used for a theory with consolidation about physics, proposing in [1].

In the paper, dimension is to be referred as Euclidean dimension. In the article, n-dimensional space or geometrical space means the Euclidean space unless specified.

## 2. Dimensional Characteristics

In geometrical sense a dimension is a linearly independent direction. Thus the dimension has associated geometrical characteristics those have certain realization in certain number of dimensions. An m-dimensional space embedded in and n-dimensional space with n>m leads to specific realization for each m; point, line, plane are the examples. Further, the Lebesgue measures on such embedded spaces also serve as a geometrical characteristic with respect to the number of dimensions concerned. Let's denote set of all points in m-dimensional space which may be embedded in higher spaces by $I_m$ i.e. $I_m = \{(x_1,x_2,\dots,x_m) | x_i \in \mathbb{R}\}$. With this notation, points are identical with $I_0$s, lines with $I_1$s and planes are with $I_2$s. In this paper, the highest dimensional space concerned for analysis (in which different $I_m$s can be identified) will be denoted by $X_n$ n being the highest number of dimensions. For specific choice of m except m=n, there are infinitely many $I_m$s existing in $X_n$. $I_m$ & $I_{m'}$ such that $m \neq m'$ are different types of $I_m$s; in general the $I_m$s can be classified depending on number of dimensions spanned by them. In other words, $I_m$s can be classified on different values of m.

**Definition 1:** *An observer in $X_n$ is defined as any entity in $X_n$ that can identify open neighborhoods of all the points in $X_n$ along all the n dimensions for purpose of analysis.*

The observer can identify points in $X_n$ and is amenable to do the mathematical analysis.

**Lemma 1:** *An observer in n-dimensional space can manifest n+ 1 types of $I_m$s.*

**Proof:** An observer in n-dimensional space $X_n$ can draw at most n number of mutually perpendicular lines at a point identifying the n dimensions. Also, she can consider m lines only out of the m; the subspace of $X_n$ consisting the m mutually perpendicular lines is nothing but the $I_m$. Hence she can manifest $I_m$ such that $0< m \ngtr n$. In this way she can manifest n types of $I_m$s differing by number of dimensions. Additionally the one can manifest points as $I_0$s; thus in total manifests n+ 1 types of $I_m$s in $X_n$.

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For fruitful analysis on the spaces, quantification of subsets of the space is required. In any space $X_n$, we can identify different $I_m$s in it. Quantification of subsets of the $I_m$s then would provide us a useful tool for quantitative analysis. Any quantification in $X_n$ is possible through quantifications of subsets of $I_m$s only. We can quantify subsets of $I_m$s by defining appropriate measures on them. Lebesgue measure provides trivial quantification of subsets of the $I_m$s.

Let's denote the quantification of a proper subset E of $I_m$ given by Lebesgue measure on it by $L_m$ (E). For ease of expression, we can omit the E in bracket as long as possible i.e. $L_m(E)$ can be written hereafter as just $L_m$.

Thus length, area and volume are $L_1$, $L_2$ & $L_3$ respectively. $I_0$s being just a point don't have any proper subset. Hence we can't define Lebesgue measure on $I_0$; hence there is no existence $L_0$.

Going a step forward with the lemma 1, an n-dimensional geometrical object i.e. proper subset of $X_n$ will have n types of $L_m$s obtained by Lebesgue measures on all the types of corresponding $I_m$s except on $I_0$. For instant, a 3-dimensional object has length (or perimeter), area (or surface area) and volume. We can regard the $L_m$s as trivial geometrical properties (or quantifications); in $X_n$ any subset would have at most n types of geometrical properties.

**Definition 2:** *In $X_n$ $n \geq m$, m+1 points as relative positions of m points with respect to a point can be specified by single real valued function defined as m-dimensional Geometrical Relation* ($R_m$) *of the m points about the point .i.e.* $R_m : X_{n\geq m} \to \mathbb{R}$

For instance, let's assume that such function exists for each m. Soon we will make a conjecture about detail of the function. But such functions do exist in Euclidean geometry; we can check that distance and angle are the functions which fit in definition 2.

Distance is $R_1$ which specifies positions of two points i.e. relative position of a point with respect to another point yields distance. Angle is $R_2$ obtained by relative positions of three points- as of two points about a point. In similar fashion solid angle is $R_3$ obtained from four points (relative positions of three points about a point).

Distance, angle & solid angle are defined by using concept of the dimensional spheres. Hence spheres seem to be useful for defining $R_m$s. Topology can be induced on $X_n$ by considering collection of all the open subsets of $X_n$. Spheres exist in general topological space. Let's denote an m-sphere in $X_n$ by $S_m$ i.e. $S_m \equiv \left\{ x \in \mathbb{R}^m : \|x\| = r \right\}$. By a sphere about a point we will mean the sphere having centre at the point.

$R_m$s are most important dimensional characteristics for dynamical analysis as they facilitate specification of relative positions. Here we propose a useful conjecture.

***Conjecture:*** *The m-dimensional geometrical relation ($R_m$) of m points about a point is given by*

$$R_m = \frac{L_{m-1}(E)}{r^{m-1}} \tag{1}$$

*Where, E is the m vertex open set formed by projections of the m points on a $S_{m-1}$ having the point (about which $R_m$ is defined) at centre. And r is the radius of the $S_{m-1}$ on which E is realized.*

m-dimensional geometrical relation of an open set (E) formed by the m points of interest on the sphere can be signified as: $R_m(E)$ with respect to centre point of the sphere. Further, for consideration of E and quantification $R_m$ a frame in $X_n$ is essential. The frame should facilitate $S_{m-1}$ with the point at centre.

**Lemma 2:** *$R_m$ defined by the conjecture is a measure in $X_n$ such that $n \geq m$*

**Proof:** In n-dimensional space $X_n$ $n \geq m$, embedding of $S_{m-1}$ is possible. Hence there exists $S_{m-1}$ about each point (i.e. considering it to be centre). Further, any point can be projected on a $S_{m-1}$ about a point along the radial direction.

Thus any m points can be projected on a $S_{m-1}$ about a point, so that on the spherical surface they lead to an open set E (analogous curved polygon) fixed by the projections as vertices. Let $\sum$ be a σ-ring of open sets over the $S_{m-1}$; then the $R_m$ given by (1) is function from $\sum$ to $\mathbb{R}$. $L_{m-1}$ of any E is non-negative and therefore $R_m$ is non-negative as r too is non-negative. i.e for all sets E on the $S_{m-1}$,

$$R_m(E) \geq 0 \qquad (2)$$

As we are considering open sets E, an empty set would be that which has no point excluding the boundary points. For the empty set ∅ containing no points $L_{m-1}(\emptyset)=0$, thus by (1) $R_m$ of empty set is zero

$$\text{i.e. } R_m(\emptyset) = 0 \qquad (3)$$

For all countable collections $\{E_i\}_{i\in N}$ of pair wise disjoint sets in Σ, by the conjecture:

$$\sum_{i=1} R_m(E_i) = \sum_{i=1} \frac{L_{m-1}(E_i)}{r^{m-1}}$$

As the sets in $\{E_i\}_{i\in N}$ are disjoint & $L_{m-1}$ is a measure, $\sum_{i=1} \frac{L_{m-1}(E_i)}{r^{m-1}} = \frac{L_{m-1}\left(\bigcup_{i=1} E_i\right)}{r^{m-1}}$

Hence rewriting the RHS by using the conjecture,

$$\sum_{i=1} R_m(E_i) = R_m\left(\bigcup_{i=1} E_i\right) \qquad (4)$$

Essential conditions for a function to be measure are non-negativity, null empty set and countable additivity (or σ-additivity) which are proved by (2), (3) and (4) respectively. Hence the conjecture is a measure on $S_{m-1}$ embedded in $X_n$.

$S_{n-1}$ about the centre point exists in $X_n$. $R_m$ is defined for m points about the centre point (the centre point is fixed by the frame). And any m points in $X_n$ can be radially projected on a $S_{m-1}$ about the point. $S_{m-1}$ is subset of same centered $S_{n-1}$; hence any $S_{m-1}$ needed to realize radial projections of the m points exists on the $S_{n-1}$. Thus $R_m$ can be used for any m+1 points in $X_n$ by proper choice of the $S_{m-1}$; hence it is measure in $X_n$.

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For every value of an $R_m$, because of continuity of $S_{m-1}$ & $L_{m-1}$ we can find at least one corresponding point in $X_n$ in fixed frame (i.e. given centre point of $S_{m-1}$ and the fixed m points). Hence $R_m$ is surjective map from $S_{m-1}$ to real numbers $R_m$: $S_{m-1} \to \mathbb{R}$.

For m=1, the conjecture is meaningless due to geometry of $S_0$. It is fact that end points of a diameter (arbitrary line segment) represent $S_0$; but there is no existence of proper subsets of $S_0$. This makes $L_{m-1}(E)$ in (1) meaningless. Hence the conjecture is meaningless for m=1. However, we can indeed identify $R_1$ by using $S_0$ & obeying definition 2. The $R_1$ should be amenable to specify relative positions of two points. We can conjecture $R_0$ to be diameter of $S_0$ on which the two points lie. A $S_0$ lies on a line i.e. $I_1$ thus it is embedded in higher dimensional spaces; and any two points can be considered to lie on a $S_0$. Thus what we conventionally know as distance is nothing but $R_1$. $R_1$ too is a measure in $X_n$.

$R_m$ and $L_m$, both are measures in $X_n$. $L_m$ is measure of proper subset in the space and $R_m$ is measure of relative positions of points about a point. For a dynamical analysis where changes happen with time, essential characteristic of a measure to be parameter is that continuous variation in its magnitude is possible in certain reference frame. Existence of Cauchy sequences is essential for this. $R_m$ is better measure for studying dynamics where out of m+1 points, m can be fixed as the references frame and variation in positions of a point object can be analyzed as variation in its $R_m$ in the frame.

As n types of spheres exist in $X_n$, the n types of geometrical relations such that $m \le n$ are evident. Variation in position of a point object with respect to certain reference frame can be measured in form of its varying $R_m$s. Thus in n-dimensional space, a motion can be characterized by any of n types of $R_m$s as suitable. In 3-dimensional space a motion can be described in terms of variation in distance or that in angle or even in solid angle whichever is suitable. Here we can make difference between general direction and ordered direction. Direction is the manifestation of variation in positions of a point object in its neighborhood in a reference frame. It can be configured by variation of $R_m$s in the frame. An ordered direction is special in a sense that it is realized in ordered pattern and can be configured by single type of $R_m$.

**Definition 3:** *In $X_n$, a continuous path $\Gamma$ is defined as an m-dimensional ordered direction ($\mathcal{D}_m$) if in a frame, there exists an isomorphism $R_m : x_\Gamma \to \mathbb{R}$ for every point $x_\Gamma \in \Gamma$ such that $m \le n$.*

When all points on a path are described by values of single typed geometrical relation in a frame, then the direction described by the path is to be called as ordered direction. Rectilinear path is set of points that can be analyzed by concerning only distances in a frame. Curvilinear path is set of points that can be analyzed by concerning distances and angles in a frame. While angular path is the set that can be analyzed by concerning only angles in a frame. Thus rectilinear and angular are ordered directions, while curvilinear isn't. It is easy to identify rectilinear direction as $\mathcal{D}_1$ & angular direction as $\mathcal{D}_2$.

**Lemma 3:** *In a frame in $X_n$, Cauchy sequence along a $\mathcal{D}_m$ exists converging to a point along the $\mathcal{D}_m$.*

**Proof:** Consider a sequence of points $\{x_i\} = x_1, x_2, x_3, \ldots$ along an m-dimensional ordered direction $\mathcal{D}_m$ in $X_n$. Then the sequence $\{x_i\}$ is identified by varying values of $R_m$ in a constant frame. The points are identified by values of $R_m$ in the frame i.e. $x_i = \frac{L_{m-1}(E_i)}{r^{m-1}}$ where, $E_i$ is the set defined by the point $x_i$ & the reference points on the $S_{m-1}$ of the frame. As the m points are fixed due to frame, only $x_i$ determines $E_i$. As range of $L_{m-1}(E_i)$ is $\mathbb{R}$, for any positive real number ε and N < i, j, $N \in \mathbb{N}$ we can obtain $| L_{m-1}(E_i) - L_{m-1}(E_j) | \le \varepsilon$. This ensures existence of the Cauchy sequence $\{L_{m-1}(E_i)\}$. And as $R_m$ is division of $L_{m-1}(E_i)$ by just a positive number $r^{m-1}$, for any positive real number ε and N < i, j, $N \in \mathbb{N}$ we have $\left|\frac{L_{m-1}(E_i)}{r^{m-1}} - \frac{L_{m-1}(E_j)}{r^{m-1}}\right| \leq \varepsilon$, equivalently we have $|R_m(x_i) - R_m(x_j)| \leq \varepsilon$. This proves that Cauchy sequence on the range of $R_m$ exists. And as $R_m$ is surjective map, Cauchy sequence for $R_m$ in $X_n$ exists.

As all the points along a $\mathcal{D}_m$ are described by single type of geometrical relation i.e. $R_m$, such direction can be parameterized by the $R_m$ in the frame. As Cauchy sequences for the $R_m$s exist, their continuous variations are possible. In fact $\mathcal{D}_m$ is manifestation of varying $R_m$ in $X_n$, thus the Cauchy sequence along $\mathcal{D}_m$ exists.

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If a point object is taking different positions $x_\Gamma \in \Gamma$ varying with time, then the path $\Gamma$ describes the motion. Thus the motions of point objects along the $\mathcal{D}_m$s can simply be defined as ordered motions. Then according to definition 3, an observer in n-dimensional space can manifest m-dimensional ordered motions such that $m \leq n$. Hence in n-dimensional space, one can manifest at most n types of ordered motions (& directions). Thus in 3-dimensions one can manifest 3 types viz. rectilinear ($\mathcal{D}_1$), angular ($\mathcal{D}_2$) & solid angular ($\mathcal{D}_3$) of ordered motions.

There is no unique geometric interpretation of vectors. But it is clear that a vector has magnitude & direction. Here we will generalize notion of vector while preserving all the algebraic properties. The directions $\mathcal{D}_m$s would be useful for interpreting/identifying vectors in $X_n$. Let's proceed with primary theorems.

**Theorem 1:** *In a frame in $X_n$ $m \leq n$, continuous variation in $R_m$ signifies direction along the. $I_m$.*

**Proof:** In a frame in $X_n$ the $R_m$ is map from a $S_{m-1}$, defined for all points on the $S_{m-1}$. Recognize that the reference points needed for $R_m$ are fixed by the frame.

$R_m = \frac{L_{m-1}(E)}{r^{m-1}}$ E is the set formed by the m points on the $S_{m-1}$. $L_{m-1}(E_i)$ is conventionally called (m-1)-surface area. It for entire $S_{m-1}$ is given [2] as

$$L_{m-1}(S_{m-1}) = \frac{2\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)} r^{m-1} \tag{5}$$

Where, $\Gamma$ denotes gamma function and r is the radius. For a set formed on the sphere, the $L_{m-1}(E_i)$ will be fraction of (5).

i.e. $L_{m-1}(E_i) = \frac{2f_i.\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)} r^{m-1}$ , $0 \leq f_i \leq 1$

Putting this in the conjecture (1) we get $R_m = \frac{2f_i.\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)}$ (6)

This new expression (6) of the $R_m$ indicates that in a frame, $R_m$ is defined irrespective of radius of the sphere. As $S_{m-1}$ exists in m-dimensional space (and concentric $S_{m-1}$s cover the m-dimensional space), now $R_m$ can be thought as a function on whole m-dimensional space spanned by the $S_{m-1}$s. But m-dimensional space embedded in higher dimensional space is nothing but an $I_m$. Thus $R_m$ is morphism from the $I_m$ to $\mathbb{R}$.

Thus any point x in an $I_m$ can be identified by a value of $R_m$ as the $R_m(x)$ in the frame. Due to existence of Cauchy sequence for $R_m$ in a neighborhood of x, for each neighboring point the $R_m$ will either increase or decrease (or may be unchanged). We can assign directions to such variations, suppose we assign direction $\mathcal{D}_m$ to manifestation of increasing $R_m$, then $-\mathcal{D}_m$ will be manifestation of decreasing $R_m$. No change in $R_m$ of neighboring point will not lead to manifestation of the direction $\mathcal{D}_m$ as on the ordered path $R_m$ is isomorphism according to definition 3. Conclusively, any change in $R_m$ manifests single direction $\mathcal{D}_m$ in (or along) the $I_m$. And, no change in $R_m$ manifests no $\mathcal{D}_m$.

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$\mathcal{D}_m$ is actually algebraic notion of direction realized by varying $R_m$. For manifestation of direction along the $I_m$, there should be continuous variation in $R_m$ so that $\mathcal{D}_m$ is continuously manifested. If $R_m$ value of neighboring points remain same, then no $\mathcal{D}_m$ is realized during the variation.

$I_m$ is collection of points that is equivalent to m-dimensional space. Thus co-ordinate chart on an $I_m$ is possible by identifying points in $I_m$ with elements of $\mathbb{R}^m$ as $C : I_m \to \mathbb{R}^m$. But Theorem 1 & (6) suggest that points in $I_m$ can be identified by $R_m$ with elements of $\mathbb{R}$ i.e. $C_m \equiv R_m : I_m \to \mathbb{R}$. Thus $R_m$ may be thought as the 1-dimensional co-ordinate system for m-dimensional space; but it has non empty kernel, all points along a radius of the $S_{m-1}$ (frame) are mapped to same element of $\mathbb{R}$. Further, all the points having same $R_m$ (those don't manifesting the $\mathcal{D}_m$) are too mapped to same element of $\mathbb{R}$. However, we get a useful corollary from theorem 1.

**Corollary 1.1**: *Any m-dimensional space can be identified with set of real numbers by $R_m$ as the chart $C_m \equiv R_m : I_m \to \mathbb{R}$. Hence the geometrical relations provide trivial real numbered chart for corresponding dimensional space.*

All points in neighborhood of a point $x \in X_n$ having same $R_m$s in the frame constitute to kernel of the chart $C_m$. As $R_m$ is same for all points along a radial direction, it is inevitably non-injective surjective map. A good coordinate chart is needed to be injective & surjective. In order to achieve this, extra components should be considered in the chart amenable to distinguish the kernel points. This can be done by considering extra components from lower dimensional geometrical relations i.e. $R_{m'}$s such that m'<m in the chart. For example, points along same radial direction in the frame having same $R_m$ can be distinguished by considering the radial distance (i.e. $R_1$) as a component of the chart. Two points having same $R_m$ in a frame can be distinguished by values of $R_{m'}$ in a subframe. By subframe we mean subset of the frame amenable to provide m' fixed reference points in order to quantify $R_{m'}$ of a point. By adopting lower dimensional geometrical relations in the chart in order to make it bijective, we are needed to consider all the m types of $R_m$ m=1,2,3..m. Thus eventually we get map of $I_m$ to $\mathbb{R}^m$. In other words, set of the geometrical relations provide a potential co-ordinate chart for $C : I_m \to \mathbb{R}^m$.

Realization of an infinitesimal path is nothing but the direction defined by the path. The directions realized in $X_n$ are useful for interpretation of directional quantities i.e. vectors. Before exploring characteristics of the directions, let's clarify two concepts.

**Definition 4:** *A set of directions S= {$D_i$} near a point is to be called as mutually exclusive directions if realization of a direction $D_j \in S$ along a path in $X_n$ implies non-realization of all other directions $D_{i\neq j} \in S$ along same path.*

**Definition 5:** *A set of directions S= {$D_i$} near a point is to be called as collectively exhaustive directions if no direction other than elements of S can be realized along any path in neighborhood of a point in $X_n$.*

Definitions of mutually exclusive and collectively exhaustive directions can be used for ordered directions. This is clear as ordered directions are special type (subsets with respect to underlying paths) of general directions: as paths configured by $R_m$s are examples of general paths.

**Theorem 2:** *In n-dimensional space, continuous variation in position of a point object can lead to manifestation of n types of mutually exclusive ordered directions.*

**Proof:** In an n-dimensional space $X_n$, $S_{m-1}$ m being at most n exists. Thus highest dimensional spherical path would exist on $S_{n-1}$. The direction along $S_{n-1}$ configured by continuously varying $R_n$ in a frame is $\mathcal{D}_n$. As implied by definition 3, $\mathcal{D}_n$ isn't manifested on the continuous path defined by the non varying $R_n$ value because of conditional isomorphism $R_n$ in definition of $\mathcal{D}_n$.

If in neighborhood $N_x$ of a point x in $X_n$, $R_n$ values of all the points in a frame are same, then $N_x$ constitutes kernel of $R_n$. The n reference points being constant in the frame, the set E is identified by point x only. Hence it is fair to call the $L_m(E)$ be $L_m$ of x i.e. Lebesgue measure of the point. From (1) we infer that same $R_n$ implies same $L_{n-1}$ of the points in the frame. If a Lebesgue measure of continuous (neighboring) points is same, then we can find a subframe wherein an ordinate (in same dimension) of all the points is same. That is- all those points lie in a lesser dimensional cross section of the space. The cross sectional space accommodating all those point has number of dimensions one lesser than that of the prior space. In short, if $L_m$ of continuous points is same, then all those points lie in single $I_{m-1}$ (i.e. a lesser dimensional section of the $I_m$). Thus points in $N_x$ having same $R_n$ should lie on cross section of the $S_{n-1}$with the $I_{n-1}$ containing $N_x$. Cross section of the $S_{n-1}$with $I_{n-1}$is nothing but the $S_{n-2}$. Hence $N_x$ lies on a $S_{n-2}$ which is subset of $S_{n-1}$. Frames for $S_{n-2}$ are subsets of frames for $S_{n-1}$; thus in the same frame we can obtain map $R_{n-1} : N_x \to \mathbb{R}$ for the points which do not lead to manifestation of $\mathcal{D}_n$. Continuous varying $R_{n-1}$ signifies direction $\mathcal{D}_{n-1}$ along the $S_{n-2}$. The general $R_m$s aren't injective (or bijective) but the $\mathcal{D}_m$s are defined by the isomorphism i.e. $\mathcal{D}_m$s pick up the subsets on which corresponding $R_m$s are bijective. Hence on the $S_{n-2}$ (equivalently in $N_x$), there will be some continuous points (let's identify their set be $N_x$') leading to a path for which $R_{n-1}$ is constant and not manifesting of $\mathcal{D}_{n-1}$. This is possible only when $N_x' \in S_{n-3} \subset S_{n-2}$. Then paths on the $S_{n-3}$ for which $R_{n-2}$ uniquely identifies the points, are manifested as $\mathcal{D}_{n-2}$. But yet there would be continuous points having same $R_{n-2}$. Such points must lie on $S_{n-4}$ leading to $\mathcal{D}_{n-3}$. Following this scheme, on the most general sphere i.e. $S_{n-1}$, different ordered directions are manifested as $\mathcal{D}_n$, $\mathcal{D}_{n-1}$, $\mathcal{D}_{n-2}$,…,$\mathcal{D}_3$, $\mathcal{D}_2$. Direction $\mathcal{D}_2$ is manifested on $S_1$, and on $S_1$, there are no two points having same $R_2$ i.e. angle in a frame.

In addition to these ordered directions, a type of ordered directions is possible along paths that change radius of the spheres considered so far. This is manifestation of direction along a straight line $\ell$, in terms of distances as $R_1: \ell \rightarrow \mathbb{R}$. Straight line is nothing but $I_1$. Such rectilinear path is manifested as primary ordered direction $\mathcal{D}_1$. Hence there are n types of ordered directions $\mathcal{D}_i$, $1 \le i < n, i \in \mathbb{N}$ in $X_n$.

When $R_m$ doesn't lead to identification of difference in points along a path, then we adopt $R_{m-1}$ to identify the points. Equivalently when $\mathcal{D}_m$ is not manifested along a path, then $\mathcal{D}_{m-1}$ can be manifested; and sequentially when $\mathcal{D}_{m-1}$ isn't manifested, we may manifest $\mathcal{D}_{m-2}$ by employing $R_{m-2}$. This sequence is followed till manifestation of $\mathcal{D}_1$. Further, any two neighboring points having varying $R_m$ don't lie on same $S_{m-2}$ (or lower spheres), thus they can't be distinguished by $R_{m-1}$(or lower dimensional geometrical relations). That is when $\mathcal{D}_m$ is manifested, then no lower dimensional ordered direction is manifested. Hence no two ordered directions $\mathcal{D}_i$ are manifested on same path in the frame. In other words, the n types of ordered directions $\mathcal{D}_i$s existing in $X_n$ are mutually exclusive.

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In $X_n$, there exist infinitely many $\mathcal{D}_m$s such that $m < n$. This is because with this condition, infinitely many $S_{m-1}$s exist about a point in $X_n$. While there only one $S_{n-1}$ exists at a point; thus single $\mathcal{D}_n$ is manifested. This is a useful corollary.

**Corollary 2.1**: *In $X_n$ there exists infinitely many $\mathcal{D}_m$s such that* $1 \le m < n, m \in \mathbb{N}$ *, but only one $\mathcal{D}_n$.*

Ordered directions are manifested by paths on spheres or along straight lines. But there are general infinitesimal paths which are neither along any sphere nor along lines. Such paths manifest directions different from ordered directions. Therefore different directions can be manifested in $X_n$ which aren't ordered direction. This leads to following proposition.

**Corollary 2.2**: *The n types of ordered directions manifested in $X_n$ aren't collectively exhaustive.*

**Theorem 3:** *Different $\mathcal{D}_m$s obey triangle law of addition in* $X_n$ $m < n$. *i.e. if points* $A, B, C \in X_n$ *and $\mathcal{D}_m$ for specific m are manifested along the paths joining any two of these three points, then*

*i.e.* $\mathcal{D}_m(AB) + \mathcal{D}_m(BC) = \mathcal{D}_m(AC)$
*Where, $\mathcal{D}_m(ij)$ implies the direction along the path going from i to j manifested as $\mathcal{D}_m$.*

**Proof:** In $X_n$, $S_{m-1}$ exist m being at most n. A $S_{n-1}$ having centre at point x accommodates many $S_{m-1}$ for every $m < n$. The cross section of $S_{n-1}$ made by a $I_m$ is set of all points in the $I_m$ equidistant from x. Set of all points in $I_m$ equidistant from a point is nothing but a $S_{m-1}$. If the cross section contains x, then radius of $S_{m-1}$ is same as radius of the $S_{n-1}$. Otherwise $S_{m-1}$ has smaller radius and centre at projection of x on the $I_m$. Thus every cross section of $S_{n-1}$ made by an $I_m$ is a $S_{m-1}$. As $\mathcal{D}_m$ is manifestation of path along $S_{m-1}$ (continuously varying $R_m$), the path along arbitrary section of $S_{n-1}$ made by a $I_m$ leads to manifestation of $\mathcal{D}_m$. Different cross sections of a $S_{n-1}$ made by different $I_m$s in $X_n$ lead to manifestation of different $\mathcal{D}_m$s. $S_{n-1}$ has infinitesimally $S_{m-1}$ structure in the cross section with $I_m$.

Consider left hand side of the equality as $\mathcal{D}_m(AB) + \mathcal{D}_m(BC)$. It implies that in $X_n$, $\mathcal{D}_m$ along paths AB & BC exists. Thus existence of the isomorphisms $R_m$s from the paths in a frame is evident. According to the conjecture (which is used for defining $\mathcal{D}$s), all the points along path AB should lie on a $S_{m-1}$ of radius r. Similarly all points along path BC too lie on a $S_{m-1}$ of same radius r as it goes through common point B. Thus points A & C lie on same sphere of radius r. As arbitrary cross section of $S_{n-1}$ made by an $I_m$ leads to manifestation of $\mathcal{D}_m$, for any two points A & C in the frame we can get a cross section to manifest $\mathcal{D}_m$ along AC. We get a path on $S_{m-1}$ of radius r going from A to C the points along which can be isomorphically identified by the $R_m$ in the frame.

Conclusively, we have $\mathcal{D}_m(AB) + \mathcal{D}_m(BC) = \mathcal{D}_m(AC)$ for any $A, B, C \in X_n$.

■

**Theorem 4:** *Set of the $\mathcal{D}_m$s with consideration of specific path length forms vector space over field of numbers.*

**Proof:** $\mathcal{D}_m$s are manifestations of paths along $S_{m-1}$s. Consider in a frame in $X_n$, a set $V_m$ of all the $\mathcal{D}_m$s having associated specific length of the path on the $S_{m-1}$s.

i.e. $V_m = \{\mathbf{v^a} = v \otimes \mathcal{D}_m^a : v \in \mathbb{R}, \,\&\, \mathcal{D}_m^a$ is identification of a $\mathcal{D}_m$ out of various other $\mathcal{D}_m s\}$ (7)

Above v is assumed to be field of real numbers. One can form the complex field by direct product of two real number fields. The formalism with real numbers would be similar to that with complex numbers. Elements of $V_m$ are m-dimensional ordered directions $\mathcal{D}_m$s having certain path length v. The path length can be quantified in terms of $R_m$ values of path extremities in a frame; this is due to the continuous variation in $R_m$ leads to manifestation of $\mathcal{D}_m$. Many $\mathcal{D}_m$s are possible depending on number n of dimensions of the space in which the set $V_m$ is considered (as stated by corollary 2.1).

On a path, if $\mathcal{D}_m^a$ is manifestation of increasing $R_m$, then $-\mathcal{D}_m^a$ is manifestation of decreasing $R_m$ on same path. If a point object goes path length v along a $\mathcal{D}_m^a$, then further going same v along $-\mathcal{D}_m^a$ (or –v along $\mathcal{D}_m^a$) will bring it to the initial point. Thus $\mathbf{v^a}$ & $-\mathbf{v^a}$ are inverses of each other under addition. Here addition of elements of $V_m$ is meant to successively following paths (along with lengths) described by the elements.

Denote elements of $V_m$ having either v=0 or absence of $\mathcal{D}_m$ by $\mathbf{0}$. Then $\mathbf{0}$ followed with a $\mathbf{v^a}$ implies no variation in $\mathbf{v^a}$ i.e. the initial point. Thus for any $\mathbf{v^a} \in V_m$ we have $\mathbf{v^a} + \mathbf{0} = \mathbf{0} + \mathbf{v^a} = \mathbf{v^a}$ i.e. $\mathbf{0}$ is identity element of $V_m$ under addition.

The $S_{m-1}$ are obtained as arbitrary cross sections of higher sphere $S_{i>m}$ made by $I_m$s. As an $S_{m-1}$ represents the $I_m$ in which it exists, two $S_{m-1}$s are transverse or parallel (or inclined at specific angle) only if corresponding $I_m$s are so. Therefore such spheres can be adopted to facilitate projections of $\mathbf{v^a} \in V_m$ at desired points in $S_{n-1}$ thereby in $X_n$. Thus projection of a $\mathbf{v^a}$ on every other $\mathbf{v^b}$ is defined due to existence of unique $I_m$ transverse to the $\mathcal{D}_m^b$ (i.e. to $\mathbf{v^b}$) & going through the extremity of $\mathbf{v^a}$ .

As projections of the $\mathcal{D}_m^a$s on every other $\mathcal{D}_m^b$s are defined and all $\mathcal{D}_m$s span higher spheres, we can transfer any $\mathbf{v^a} \in V_m$ to any point on the sphere. Thus at every point on the sphere we have all the $\mathcal{D}_m$s. Thus effectively can transfer every element of $V_m$ to any point on the sphere. Consider a point O on the $S_{n-1}$ relative to which path length v of all $\mathbf{v^a}$s $\in V_m$ is defined. Then let points A & B are described by $\mathbf{v}_A$ & $\mathbf{v_B} \in V_m$ respectively i.e. there is equivalence in the frame $OA \equiv \mathbf{v}_A = v_A \otimes \mathcal{D}_m^a$ & $OB \equiv \mathbf{v_B} = v_B \otimes \mathcal{D}_m^b$. Note that paths like OA, OB etc. are along $\mathcal{D}_m$s and not just one dimensional curves. $\mathbf{v}_A + \mathbf{v_B}$ means going from O to A by $\mathbf{v}_A$ and then further from A by $\mathbf{v_B}$. An element of $V_m$ must preserve its intrinsic direction $\mathcal{D}_m^a$ while transferred to any point in $X_n$ i.e. any element should be transported at any point of $X_n$ as parallel to its original position in the frame. Hence when we say going from point A by $\mathbf{v_B}$, it means to cover path length of $v_B$ along the path specified by $\mathcal{D}_m^b$ parallel to OB. Let the resultant position due to $\mathbf{v}_A + \mathbf{v_B}$ from O be C, thus according to theorem 3 we can write $OC \equiv \mathbf{v}_A + \mathbf{v_B}$. Transfer of $\mathbf{v_B}$ to A means equivalence $AC \equiv \mathbf{v_B}$. As two paths are manifested along same direction if they are parallel, OB & AC are parallel with same path length $v_B$. Now consider $\mathbf{v_B} + \mathbf{v}_A$, it means going from O to B by $\mathbf{v_B}$ and then from B by $\mathbf{v}_A$. Let the resultant position due to $\mathbf{v_B} + \mathbf{v}_A$ from O be D i.e. $OD \equiv \mathbf{v_B} + \mathbf{v}_A$. Here too we have equivalence $BD \equiv \mathbf{v}_A$ and OA & BD are parallel with same path length $v_A$. Thus OA & OB are parallel to BD &

AC respectively having same path lengths, so are the $I_m$s specified by them. Two pairs of parallel $I_m$s, each pair having same path length form a closed four vertex set (very primarily, two pairs of parallel lines having same parallel lengths form parallelogram). The three vertices being fixed by O, A & B, positions of points C & D are identical, thus OC=OD; this directly means $\mathbf{v}_A + \mathbf{v_B} = \mathbf{v_B} + \mathbf{v}_A$ what is commutativity of addition of elements of $V_m$.

As transfer (& projections) of all elements of $V_m$ at any desired point on the $S_{n-1}$ is possible and geometry of neighborhoods of all points on $S_{n-1}$ is identical, addition of the elements should be associative under addition.

Elements of $V_m$ are defined in (7) as direct products of numbers (scalars) field & directions, thus arithmetic characteristics of scalar multiplication are obvious. The characteristics are Compatibility of scalar multiplication with field multiplication, identity element of scalar multiplication, distributivity of scalar multiplication with respect to addition, distributivity of scalar multiplication with respect to field addition. Thus all the axioms for a set to be vector space are satisfied and we can conclude that $V_m$ is a vector space.

Any point x in the $X_n$ can be identified with an element **x** of $V_m$ in certain frame.

■

There several $\mathcal{D}_m$s are possible depending on dimensionality n of the space. Different vector spaces $V_m$s having different value of m lead to different realizations of the vector elements. For instance, elements of $V_1$ have rectilinear direction while those of $V_2$ have angular direction. Dimensionality of elements of vector spaces is inherently intrinsic due to directions $\mathcal{D}_m$ in their definition. Thus we can explicitly define dimensional vectors.

***Definition 6*:** The m-dimensional vector is defined as element of $V_m$ having direction along a $\mathcal{D}_m$.

According to Theorem 2, in n-dimensional space we manifest n types of ordered directions. Hence in n-dimensional space we have n-types of vectors viz. m-dimensional vectors with $m \le n, m \in \mathbb{N}$. Also according to Theorem 1, $R_m$ signifies direction along the $I_m$. There exists only one $I_n$, hence only one $\mathcal{D}_n$ in $X_n$. Therefore $V_n$ has only one direction for all elements. Thus $V_n$ is not much useful for analysis in $X_n$ as the $V_m$s $m < n, m \in \mathbb{N}$ are.

We can identify the scalar number field v in definition (7) of vector space $V_m$ with range of $R_m$ in same frame. For utilization of $V_m$s for analysis on $X_n$, every $x \in X_n$ should be identified with single element of $V_m$. We have a simple scheme to do so. Any $x \in X_n$ can be considered on a path defined by single $\mathcal{D}_m$. For this we need a frame having fixed centre for all the $S_{m-1}$s to be considered and having fixed points on the $S_{n-1}$ (this implies fixed rays for varying radii of the spheres) for quantification of $R_m$s. Then any point x will lie along specific $\mathcal{D}_m$ specified by cross section of corresponding $I_m$ with the $S_{n-1}$. The reference points fixed by the frame & x define the cross sectional $I_m$. In this way specific direction $\mathcal{D}_m^{\mathbf{a}}$ for every x is identified. The scalar value corresponding to x can be simply identified with path length given by $R_m$ of the x from the reference point on the sphere. As $R_m$ is isomorphsm on the $\mathcal{D}_m$, no two points have same path length in same direction. Hence in order to identify points in $X_n$ with elements of $V_m$, we should use the configuration $\mathbf{v}$: $X_n \rightarrow V_m$ given by

$$\mathbf{v}(x) = R_m(x) \otimes \mathcal{D}_m^{a} \qquad (8)$$

Where $R_m(x)$ is the path length of x along the $\mathcal{D}_m^{a}$ w.r.t. the reference point of the frame (x along with the m-1reference points defines the E needed for $R_m$). Thus a vector space is direct product of the directions $\mathcal{D}_m^{a}$s and range of $R_m$ for points along corresponding $\mathcal{D}_m^{a}$s.

Further, every element of $V_m$ configured in $X_n$ can be identified with a scalar as $R_m$: $V_m \to \mathbb{R}$ with reference to same frame in which the configuration done.

**Theorem 5:** *In n-dimensional space,* $V_m$*s* $m \le n, m \in \mathbb{N}$ *are topological Banach spaces.*

**Proof:** Theorem 4 clearly concludes existence of vector space $V_m$ in $X_n$ and definition 3 defines it for all $m \le n, m \in \mathbb{N}$. Also from the definition & above configuration, map $R_m$ for every element of $V_m$ is evident. Now, consider $R_m$: $V_m \to \mathbb{R}$ along specific $\mathcal{D}_m^{\mathbf{a}}$. In a specific frame, all the elements of $V_m$ can be identified with corresponding value of $R_m$ irrespective of $\mathcal{D}_m^{\mathbf{a}}$. Specifically elements of $V_m$ have directions $\mathcal{D}_m^{a}$s along different $S_{m-1}$s on the $S_{n-1}$; and Theorem 1 implies that $R_m$ is measure from $S_{m-1}$ in the frame quantifying a $\mathcal{D}_m$. Hence all the elements of $V_m$ can be identified with corresponding $R_m$s.

As by lemma 2 $R_m$ is a measure, for any $\mathbf{v} \in V_m$, always $R_m(\mathbf{v}) \ge 0$ i.e. $R_m$ is non negative. Further, when for a $\mathbf{v} \in V_m$, $R_m(\mathbf{v})=0$ it means that the $L_{m-1}(E)$ concerned by the conjecture is zero. In such case, no separation of the point x (which is identified with **v**) from the reference point occurs; thus no manifestation of any path by x & hence of any $\mathcal{D}_m$. Thus in such case the element **v** has no direction i.e. $\mathbf{v} = \mathbf{0}$. Conclusively we get non degeneracy of $R_m$ i.e. $R_m(\mathbf{v})=0 \leftrightarrow \mathbf{v}=\mathbf{0}$. By linearity of the conjecture, scalar multiplicativity is obvious i.e. $R_m(\lambda\mathbf{v})= \lambda\, R_m(\mathbf{v})$. Further for $\mathbf{v},\mathbf{w} \in V_m$, let $P_{\mathbf{v}}$ & $P_{\mathbf{w}}$ be corresponding $L_{m-1}(E_x)$ as considered in (1). Let the $L_{m-1}$ corresponding to the vector element $\mathbf{v} + \mathbf{w}$ be $P_{\mathbf{v+w}}$. As $L_{m-1}$ is Lebesgue measure from the open sets on spheres to $\mathbb{R}$, using property of sum of the sets $P_{\mathbf{v+w}} = P_{\mathbf{v}} + P_{\mathbf{w}} - P_{|\mathbf{v}-\mathbf{w}|}$. Using this relation in the conjecture we get $R_m(\mathbf{v}+\mathbf{w}) \le R_m(\mathbf{v}) + R_m(\mathbf{w})$ what is the triangle inequality. As $R_m$ has essential properties of non negativity, non degeneracy, multiplicativity and triangle inequality on $V_m$, $R_m$ is norm on $V_m$. $R_m$ makes $V_m$ a normed vector space.

Lemma 3 implies that every Cauchy sequence in $V_m$ with respect to the norm converges to points (elements) in $V_m$. Alternatively, the spheres are complete. Therefore $V_m$ is a complete normed space i.e. Banach space. Further, a norm always gives raise to metric and thus induces the topology on same space. Thus $V_m$s are topological Banach spaces.

■

From this point, one can derive all the aspects of conventional vectors spaces such as geometrical, topological, algebraic, functional etc. for $V_m$.

**Theorem 6:** *If an entity exists as a vector quantity in n-dimensional space then it essentially exists in all the n types of vectors as elements of* $V_m$*s* $m \le n, m \in \mathbb{N}$*; and induces same dynamics with all the types.*

**Proof:** An entity existing in n-dimensional space can be considered as a point object in corresponding n-dimensional configuration space $X_n$. If the entity exists as a vector quantity, then it intrinsically has magnitude & direction. That is the point object is to be considered along with a direction (& a path length being its magnitude) in $X_n$ with respect to the frame. Image of the entity in $X_n$ can have any direction; however an arbitrary direction can be considered as resultant of several simultaneous ordered directions. Importantly, theorem 2 states that continuous variation in position of a point object leads to manifestation of n types of mutually exclusive ordered directions. Hence the vector quantity should be intrinsically along all the possible ordered directions $\mathcal{D}_m$ $m \le n$ $m \in \mathbb{N}$ in order quantify any infinitesimal change in it. The entity should be configured along the $\mathcal{D}_m$s. Scalars (magnitudes of the entity) along these directions according to Theorem 4 form vector spaces $V_m$ $m \le n$. Thus overall n types of vectors spaces $V_m$ exist. Thus the entity exists as n types of vector as elements of corresponding $V_m$s.

Further, Theorem 2 implies existence of mutually exclusive ordered directions in $X_n$. Therefore variation in point object is along any of the n types of ordered directions independently. An infinitesimal variation results in change in magnitude of any one type of vector (along any $\mathcal{D}_m$ $m \le n$) and not of other. Therefore in order to configure any change in the configuration, the entity essentially comes in all the vector versions as elements of $V_m$s $m < n, m \in \mathbb{N}$.

Further, any other vector quantity too should come in all the versions (i.e. the n types). Thus in every type of vector space, all the quantities exists. Additionally field of scalars exists for all the vector spaces. If **x** & **y** are two vector quantities, then in all types of vector spaces $V_m$ $m \le n$ their versions exist as $\mathbf{x}_m$ & $\mathbf{y}_m$ correspondingly. Then all the mathematical operations of arithmetic, geometry & calculus are possible with them. Thus their relation is preserved in every version out of m. Thus a vector quantity induces or derives same dynamics in all typed vectors spaces $V_m$s $m < n, m \in \mathbb{N}$.

■

Now we have reached to the point from which the mathematics developed can be applied to special objective. The trivial case of the application is of our physical universe. In upcoming section we will discuss the universe with respect to the work done in this section.

Elements of any algebraic vector space can be interpreted in $X_n$ as discussed. Conventionally they are interpreted to be straight line segments ($\mathcal{D}_1$), while now we can interpret them to be segments along any of $\mathcal{D}_m$s. For the new interpretation, dimensionality n of $X_n$ & m of the $\mathcal{D}_m$ is important. In same $X_n$, dimensionality of $V_m$ varies with m due to limitation on number of mutually perpendicular $I_m$s.

## 3. Case of the universe

Our universe can be identified with a 4-dimensional general manifold. Out of the four dimensions, locally 3 are spatial & 1 is temporal. Such space having 3 spatial dimensions and a parameter of evolution will be written as 3+1-dimensional space. More precisely, the universe U is globally 4-dimensional while locally it is 3+1-dimensional. Theorem 5 clearly implies that for n=4, $V_m$s $m \le 4, m \in \mathbb{N}$ form topological Banach spaces i.e. there would exist 4 types of vectors as elements of $V_1$, $V_2$, $V_3$ & $V_4$. But out of them, 4-dimensional vectors i.e. elements of $V_4$ are useless for analysis. This is because in U, single $\mathcal{D}_4$ exists i.e. $V_4$ (configured) in U is 1-dimensional Banach space; 1-dimensional vector space has least analytical value since it can be considered as scalar space. If linearly independent directions of vectors exist, then the vectors are useful for analysis. In this sense in U there are three types of analytical vectors viz. 1-dimensional, 2-dimensional & 3-dimensional (4-dimensional being dormant for analysis).

1-dimensional vectors are the conventional vectors having directions along straight lines; let's call them rectilinear vectors. 2-dimensional vectors have directions along $S_1$ i.e. circular path; let's call them angular vectors. While 3-dimensional vectors are having directions along $S_2$; let's call them sangular vectors. In the immediate subsection, we will elaborate on the 2-dimensional vectors.

The case study of our universe is presented here purposefully. A theory in physics to be proposed in [1] concerns the universe as configuration space accommodating four types of vectors.

*3.1 Angular Vectors*

It is well accepted that the infinitesimal angular rotations can be represented as vectors [3]. As a special case of vectors in curvilinear coordinates, the angular vectors are already explored. Special spherical vectors **r**, **θ** & **φ** are useful for analysis of conventional (rectilinear) vectors. If one ignores **r**, then the space can accommodate angular vectors only (and no rectilinear vector). In such angular vector space, directed angles can be identified with elements of any algebraic vector space. For convention we will consider anticlockwise or right handed angular direction to be positive and the clockwise or left handed to be negative.

Angle is measure of arc of circle in plane. And as every section of the sphere made by a plane is a circle, every infinitesimal curve on circle can be measured in terms of angle (i.e. $R_2$). In general $R_m$ is measure on a $S_{m-1}$, and every cross section of $I_m$ & higher sphere is $S_{m-1}$. Thus the higher spheres have infinitesimally piecewise $\mathcal{D}_m$ structure to accommodate m-dimensional vectors.

***Definition 7*:** Elements of $V_1$ having direction along $\mathcal{D}_1$ are defined as rectilinear vectors.

***Definition 8*:** Elements of $V_2$ having direction along $\mathcal{D}_2$ are defined as angular vectors.

For configuration/identification of rectilinear vectors, in the frame, origin in form of a point is needed. For angular vectors, origin in form of a ray (giving centre and a point on every radius sphere) is needed. Origin ray for angular vectors is the line starting from the centre of the frame $S_n$ and propagating in a direction. The angular magnitudes are measured with respect to this ray. (In general for m-dimensional vectors, origin in form of $I_{m-1}$ is needed for fixing of all the m points for $R_m$. And the norm of a vector point is measured relative to such origin.) The angular vectors can exist on higher spheres or 4-balls.

Algebraic expressions for all types of vectors are same such as linear combination of components, identities of dot product & cross product etc. This is valid if the magnitude in terms of $R_m$ is considered for m-dimensional vectors. As discussed in proof of theorem 5, in the universe trivial norms for vectors are $R_m$s i.e. distance, angle & solid angle correspondingly. But comparison of different typed vector magnitudes can be done by fixing all the quantifications ($R_m$) in terms of distances. For this, we can exploit the conjecture. Angle can be written as ratio of arc and radius. Basis can be easily identified for the vector spaces, wherein an arbitrary vector can be expanded in terms of basis vectors. Suppose an angular vector $\boldsymbol{a}$ is written as

$$\boldsymbol{a} = \mathrm{M}\boldsymbol{m} + \mathrm{N}\boldsymbol{n} \tag{9}$$

Where, M & N are quantified in angles and $\boldsymbol{m}$ & $\boldsymbol{n}$ are basis angular vectors in $X_3$. Then same can be written as

$$\boldsymbol{a} = \frac{M}{\mathrm{r}}\boldsymbol{m} + \frac{N}{\mathrm{r}}\boldsymbol{n}$$

Where, $M$ & $N$ are quantified in distances (or lengths) on sphere of radius r. In $X_{n\geq 3}$, the resultant vector and its components form spherical triangle on $S_2$. We have equality from spherical trigonometry [4] as

$$\cos(a) = \cos(\mathrm{M})\cos(\mathrm{N}) + \sin(\mathrm{M})\sin(\mathrm{N}).\cos \acute{\upsilon} \tag{10}$$

Where $a$, M and N are sides of spherical triangle formed on a sphere. $\acute{\upsilon}$ is angle opposite to side $a$. The spherical triangle formed by resultant angular vector and its components is right angled, i.e. if $a$ is resultant of M

& N, then $\acute{\upsilon} = \frac{\pi}{2}^{c}$ . Hence second term in RHS of (10) vanishes. Thus using (9) & (10) we get magnitude of angular vector as

$$a = \arccos\left(\cos\left(\mathrm{M}\right)\cos\left(\mathrm{N}\right)\right) \tag{11}$$

Further, we obtain unit angular vector as

$$\boldsymbol{u} = \frac{\boldsymbol{a}}{a} = \frac{\mathrm{M}\boldsymbol{m} + \mathrm{N}\boldsymbol{n}}{\arccos\left(\cos\left(\mathrm{M}\right)\cos\left(\mathrm{N}\right)\right)} \tag{12}$$

Let two angular vectors in spatial universe $X_3$, $\boldsymbol{a}$ =M$\boldsymbol{m}$+N$\boldsymbol{n}$ and $\boldsymbol{b}$ =M’$\boldsymbol{m}$+N’$\boldsymbol{n}$ then we get magnitude of the vector obtained by their addition as

$$|\boldsymbol{a} + \boldsymbol{b}| = \arccos\,[\cos(\mathrm{M+M'}).\cos(\mathrm{N+N'})] \tag{13}$$

The essential triangle inequality $|\boldsymbol{a} + \boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}|$ holds for angular vectors as used in Theorem 5.

The scalar product of two vectors is obtained as product of their projections on each other. Consider projections of two angular vectors on each other in their space i.e. $S_2$ as shown in fig.1. Let GA=$a$ & GB=b. From spherical law of sine [5] for triangle GAA’

$$\frac{\sin a}{\sin\frac{\pi}{2}} = \frac{\sin \mathrm{x}}{\sin\theta} \tag{14}$$

i.e. $\mathrm{x} = \arcsin\left(\sin a . \sin\theta\right)$

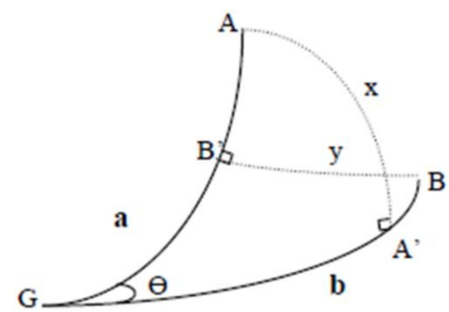


**Fig.1:** Projections of two angular vectors **a** & ***b*** on each other

Then using general formula (10) for same triangle,

$$\mathrm{GA'} = a' = \arccos\left(\frac{\cos a}{\cos\left(\arcsin\left(\sin a.\ \sin\theta\right)\right)}\right) \tag{15}$$

Repeating same procedure for triangle GBB’,

$$\mathrm{GB'} = \mathrm{b'} = \arccos\left(\frac{\cos b}{\cos\left(\arcsin\ \left(\sin b.\ \sin\theta\right)\right)}\right) \tag{16}$$

Combining (15) & (16) we get scalar product of angular vectors in terms of their magnitudes and angle between them

$$\boldsymbol{a}.\boldsymbol{b} = a'b' = \arccos\left(\frac{\cos a}{\cos\left(\arcsin\left(\sin a.\ \sin\theta\right)\right)}\right).\arccos\left(\frac{\cos b}{\cos\left(\arcsin\ \left(\sin b.\ \sin\theta\right)\right)}\right) \tag{17}$$

To verify- it is commutative and fulfills desired properties of scalar product such as $\boldsymbol{a}.\boldsymbol{a} = a^2$, and for basis units $\boldsymbol{m}.\boldsymbol{m}$ =1, $\boldsymbol{n}.\boldsymbol{n}$ =1 and $\boldsymbol{m}.\boldsymbol{n}$ =$\boldsymbol{n}.\ \boldsymbol{m}$ =0. Using these relations for basis vectors, it is easy to get the scalar product in terms of components (equivalent to general expression) as

$$\boldsymbol{a}.\boldsymbol{b} = (\mathrm{MM'})+(\mathrm{NN'}) \tag{18}$$

Vector product of two angular vectors can be developed using crux of vector product i.e. combination of perpendicular component of vector acting on magnitude of other. If a vector ***a*** acts on another vector ***b***, then by geometric definition of cross product we take magnitude of component of ***a*** that is transverse to ***b*** and multiply it by magnitude of ***b***. Formulae for spherical trigonometry in [5] i.e. like (14) assists the derivation. Then we have the magnitude of cross product as

$|\boldsymbol{a}\mathrm{x}\boldsymbol{b}| = a''b$ where $a''$ magnitude of component of ***a*** that is transverse to ***b***.

By using equations for spherical triangles, we get

$$|\boldsymbol{a}\ \mathrm{x}\ \boldsymbol{b}| = \arccos\left(\frac{\cos a}{\cos\left[\arcsin\left(\sin a.\sin\left(\frac{\pi}{2}-\theta\right)\right)\right]}\right).b \tag{19}$$

Crux behind the vector product clearly implies that the vector product of two vectors is perpendicular to both of them. This is possible only if the product has direction linearly independent to that of both. In the example, ***a*** & ***b*** are expressed in terms of basis ***m*** & ***n***. Hence the vector product should have direction linearly independent to ***m*** & ***n***. Let's denote the unit vector in the new direction by ***l***; thus the vector product (19) has direction ***l***. That is,

$$\boldsymbol{a}\ \mathrm{x}\ \boldsymbol{b} = \left[\arccos\left(\frac{\cos a}{\cos\left\{\arcsin\left(\sin a.\sin\left(\frac{\pi}{2}-\theta\right)\right)\right\}}\right).b\right]\boldsymbol{l} \tag{20}$$

Using (20) we obtain the essential properties of angular vector product as

$\boldsymbol{m}\ \mathrm{x}\ \boldsymbol{m} = 0$ and $\boldsymbol{n}\ \mathrm{x}\ \boldsymbol{n} = 0$ and $|\ \boldsymbol{m}\ \mathrm{x}\ \boldsymbol{n}\ | = |\ \boldsymbol{n}\ \mathrm{x}\ \boldsymbol{m}\ | = 1$

Also $\boldsymbol{m}\ \mathrm{x}\ \boldsymbol{n} = \boldsymbol{l}$ and $\boldsymbol{n}\ \mathrm{x}\ \boldsymbol{m} = -\boldsymbol{l}$

Using these properties, in terms of basis we obtain (equivalent to general expression)

$$\boldsymbol{a}\ \mathrm{x}\ \boldsymbol{b} = (\mathrm{MN'} - \mathrm{NM'})\boldsymbol{l} = -(\boldsymbol{b}\ \mathrm{x}\ \boldsymbol{a}) \tag{21}$$

In this subsection we have revealed basic details about 2-dimensional vectors or angular vectors which are elements of $V_2$ & have directions along $\mathcal{D}_2$. The formulary is consistent with that of $V_1$. Thus one may generalize the scalar & vector products for higher dimensional vectors in terms of basis. The algebraic properties of all types of vectors (at least of rectilinear & angular) are identical; and one can't distinguish between their algebras. If angular vectors are identified to be rectilinear vectors by appropriate morphism, then algebraically one can't reveal the fact. Different typed vectors are algebraically identical but have geometrically different.

During evolution of physics, we encountered many examples of angular vectors such as angular velocity, angular momentum, torque etc. We assumed them to be rectilinear vectors by assigning right hand thumb rule as the morphism. Further, these vectors are always cross products of other rectilinear vectors. As algebraic properties of angular vectors and rectilinear vectors are same, their algebras are indistinguishable and no trouble occurred in the analysis. But when their geometry is concerned, the difference explicitly arises. At first glance everyone feels that these vectors are fundamentally different from other rectilinear vectors, but abandons this fact as the analysis plays fine. Most effective sensation of geometry, next to physical realization is symmetry. These vectors indicate their difference when studied under symmetries. The scientific community compensated this matter by making two classes of vectors as pure vector (or polar vector) and pseudovector (or axial vector). The pseudovectors don't obey laws of symmetry e.g. reflection. Pseudovector is always associated with the cross product of two pure vectors [6]; and the cross product implies a vector acting on another vector. Mathematically the pseudovectors should be angular vectors (according to the theorems in last section, quantities having direction $\mathcal{D}_2$ are essentially angular vectors belonging to $V_2$). We have to accept the fact that scheme of pseudovectors is misleading (it is misinterpretation) and they are actually angular vectors. If we consider the angular vectors instead of pseudovectors, all physical systems are invariant under all trivial symmetry operations including reflection. Vector is just a tool to analyze physical system; the system must be invariant if frame of such tool is changed- this is possible only if we consider angular vectors instead of pseudovectors.

*3.2 Sangular Vectors*

As discussed earlier, in 4-dimensional space the vector space $V_3$ having elements as 3-dimensional vectors can be configured. Such vectors will exist on the 4-balls (or 3-spheres) existing in U; and have directions along $\mathcal{D}_3$ and norm in terms of $R_3$ i.e. solid angle. This norm will induce metric in terms of solid angle for the sangular vectors.

***Definition 9*****:** Elements of $V_3$ having direction along $\mathcal{D}_3$ are defined as sangular vectors.

U being 4-dimensional, can be configured as sangularly 2-dimensional vector space. Continuous random change in positions of a point object on surface of $S_2$ leads to manifestation of sangular vector.

For configuration of vectors, appropriate frame is needed. For rectilinear vectors it could be any of the conventional; for angular vectors in the frame a fixed point on the $S_1$ is needed from which angles can be measured. Analogously, for sangular vectors in the frame two points on $S_2$ are needed referring to which area traced by a point on $S_2$ (i.e. E in the conjecture) can be measured. Both ends of a diameter can be considered as the origin in the frame, these two points and the object point form triangle on the sphere. Area of such triangle divided by square of radius of the sphere yields the solid angle i.e. norm of the sangular vector of the object point in the frame. For quantification of area on the spheres, any two reference points would work, but we concluded end points of a diameter because this makes symmetry for choice of frames on the spheres. Further the end points of a diameter mean $S_0$, this would help for generalization for higher dimensional vectors.

Area of the spherical triangle formed by two reference points & an object point characterizes norm of the sangular vector of the object point. Area A of plane triangle is half of the product of base & height (b.h/2); and area *A* of spherical triangle having same base b & height h has different but comparable area due to spherical excess. We can write $A = g\, b.h/2$ where, g is the deviation due to spherical area. We don't need to explore g here.

Further, for any sangular vector, the base concerned is constant as out of the three points, two are always reference points (or end point of a diameter). The spherical distance between ends of a diameter is $\pi r$ i.e. $b = \pi r$. Using this substitution, we get area of the spherical triangle formed by point x as $A_x = g\, \pi r.h_x/2$, where location of x characterizes $h_x$. Using this in the conjecture we get

$$R_3(x) = \frac{g\pi h_x}{2r} \tag{22}$$

*3.3 Vectors in the universe*

In the universe U, rectilinear, angular & sangular types of vectors should exist. An angular vector spans over two dimensions as $S_1$ exists in 2-dimensional space. Similarly a sangular vector spans over three dimensions. Let in a frame, the four rectilinear basis dimensions of U are $x_1$, $x_2$, $x_3$ & $x_4$; let $x_1$ be time dimension. Let the unit angular vectors in planes $x_1x_2$, $x_1x_3$ & $x_1x_4$ be basis for $V_2$ in same frame. Note that any combination $x_ix_j$ with i being same & j varying over three others forms basis for angular vector space, and all basis sets are equivalent related by linear transformations.

In 4-dimensional space only two linearly independent sangular vectors can exist and a sangular vector spans over three rectilinear (Euclidean) dimensions.

In the universe U, there exist three types of vectors viz. rectilinear, angular and sangular. According to theorem 6, any quantity like displacement, momentum etc. should come in these four versions. The formalism on a type (or for general vector) is to be followed for all the types of vectors. This means if rate of change (w.r.t. a quantity) of a vector quantity **v** is defined as **u**, then it holds for any type of vector. Therefore if a quantity is conserved, then it should be conserved in all typed vectors.

According to Theorem 6, a vector quantity should exist in all the types of vectors. If it changes in U, then it must change locally i.e. the change must be manifested in spatially 3-imensional space with time evolution. Vector has magnitude & direction, if change happens in magnitude, then it is explicitly manifested as change in the path length along the ordered direction. But if a vector of fixed magnitude exists and can change via variation in direction only, then local geometry on U is important. If two linearly independent vectors of a type are manifested locally, then change in the typed vector via change in direction is manifested due to there are many vector directions possible. Local space of manifestation is spatial 3-dimensional portion of U with time evolution. As in 3-dimensional space at least two linearly independent vectors of rectilinear & angular type can exist, change in them due to direction can be manifested. This isn't the case with sangular vectors as only one such vector spans whole 3-dimensional (spatial) space.

According to theory of relativity, U is globally 4-dimensional continuum while locally is 3+1-dimensional having Minkowskian geometry. Thus if there exists 4-ball in U, then locally it is manifested as 3-ball with a dimension being evolution parameter. Two linearly independent sangular vectors can exist on 4-ball, but only one such on 3-ball. The 3-ball is projection of 4-ball aligned with local spatial space of manifestation $U_s$. If change in a sangular vector direction happens, then the change must be perpendicular to $U_s$. If a vector changes direction (or rotates) perpendicular to a subspace, then its projection on (or component in) the subspace should change. If a path along $\mathcal{D}_m$ having specific path length is changed (rotated) perpendicular to the accommodating $I_m$, then path length along the projection of the path in the $I_m$ will be changed depending on the amount of change (rotation). Thus even the path length is generally constant, for the projection in the subspace- it changes. Thus in effect, in local portion of U, change in sangular vector is manifested as change in its magnitude on the 3-ball (even if its magnitude on 4-ball is constant).

*3.4 Comparison of Magnitudes of different typed vectors*

Three types of vectors exist in the universe. For fruitful analysis, comparison between magnitudes of different typed vectors is must. All the m-dimensional vectors such that m is greater than 1 exist on the respective spheres or balls. Rectilinear vectors are fundamental vectors quantified in terms of $R_1$. The universe is infinitesimally piecewise rectilinear. All the comparison should be done with respect to magnitude of rectilinear vector.

Consider a rectilinear vector $\mathbf{v}_R$ of norm $|\mathbf{v}_R|$, it should exist along $\mathcal{D}_1$ i.e. straight line. But the same norm i.e. curved line segment of the length $|\mathbf{v}_R|$ can exist on spheres. Initially, let's find comparison with angular vector existing on a sphere of radius r. Magnitude i.e. norm of an angular vector $\mathbf{v}_A$ is given by $R_2$ i.e. $|\mathbf{v}_A| = P/r$, P being difference between the $L_1$s of extremities of the vector. As $L_1$ is length, P is length on the $S_1$. Comparison can be obtained by substituting $|\mathbf{v}_R|$ for P meaning that same path length is used to construct both the vectors. Then we get

$$\left|\mathbf{v}_A\right| = \frac{\left|\mathbf{v}_R\right|}{r} \tag{23}$$

Relation (23) provides comparison of magnitudes of the angular & rectilinear vectors if same amount of geometric content (in terms of Lebesgue measure) is used to generate both the vector. This relation is similar to $\theta = l/r$ of arc length & angle.

Norm of sangular vector is given by difference in $R_3$s of its extremities. Thus norm $|\mathbf{v}_S|$ of sangular vector $\mathbf{v}_S$ is ratio of area due to $\mathbf{v}_S$ on the sphere to square of the radius. It is as given in (22). There $h_x$ is curved length which can be regarded as magnitude of the corresponding rectilinear vector for comparison. In other words, for comparison purpose $R_3(x)$ in (22) is magnitude of a sangular vector $\mathbf{v}_S$ while $h_x$ is magnitude of a corresponding rectilinear vector $\mathbf{v}_R$. It takes the form

$$\left|\mathbf{v}_S\right| = \frac{g\pi\left|\mathbf{v}_R\right|}{2r} \tag{24}$$

This equality provides abstract comparison of magnitudes. Here g is general function and we haven't explored it. The relative magnitudes of the three types of vectors are essential in the physical theory proposed in [1].

## 4 Conclusion

Vectors have ordered directions that not needed to be rectilinear always. The paper provides generalization of conventional interpretation of vectors. It concludes that a type of ordered direction exists for every number of Euclidean dimensions. Paths with the path lengths along such ordered directions satisfy axioms of the vectors, hence they can be considered as vectors. Thus every number of dimensions comes with a type of vector. Algebra of all the typed vectors is identical. Expressions in terms of basis or components for scalar product & vector product are identical. But different typed vectors differ in magnitude; an n-dimensional vector has magnitude in terms of $R_n$. Elements of arbitrary algebraic vector space may be interpreted as of any type in corresponding geometrical (configuration) space. All types of the vector form Banach spaces and have metric induced topologies.

In 4-dimensional Euclidean space, three types of vectors exist viz. rectilinear, angular & sangular. A gross comparison of their magnitudes is obtained as (23) & (24). The types of the vectors retain their directions infinitesimally i.e. it is meaningless to say that an angular (or sangular) direction is infinitesimally rectilinear. This makes the generalized vectors different from that through the differential geometry.

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