Fractal,

Polynomial,

π

by

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Abstract

This note presents formulas and fractals related with the polynomial:

$$p(x) = x^8 + 4x^7 - 10x^6 - 16x^5 + 19x^4 + 16x^3 - 10x^2 - 4x + 1$$

I. The Fractal

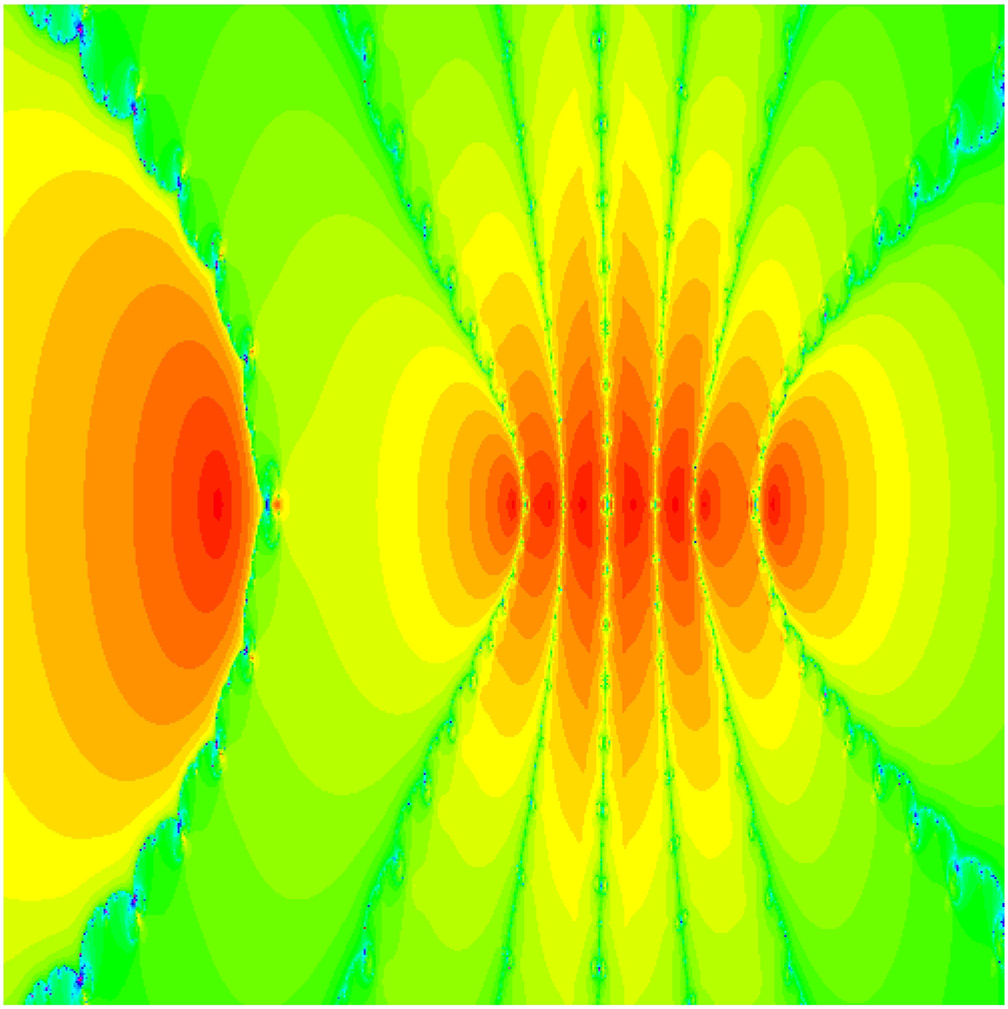

Figure 1. Newton-Julia set for:

$$p(x) = x^8 + 4x^7 - 10x^6 - 16x^5 + 19x^4 + 16x^3 - 10x^2 - 4x + 1$$

II. The Polynomial

$$p(x) = x^8 + 4x^7 - 10x^6 - 16x^5 + 19x^4 + 16x^3 - 10x^2 - 4x + 1 \tag{1}$$

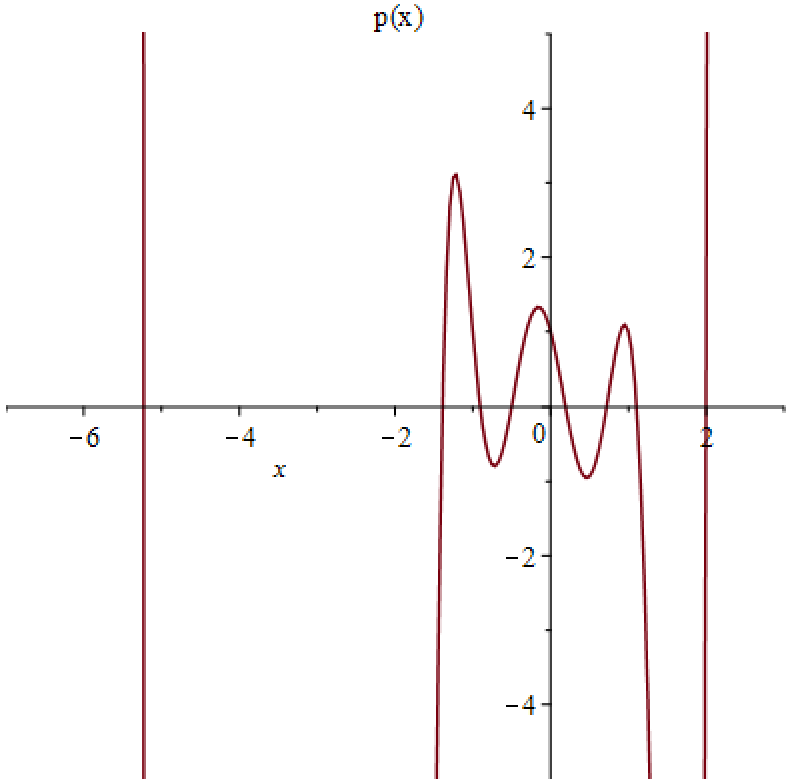

Figure 2. $p(x)$

$$p(x) = \left(x^2 + \left(1+\sqrt{2}+\sqrt{2}\sqrt{2+\sqrt{2}}\right)x - 1\right)\left(x^2 + \left(1+\sqrt{2}-\sqrt{2}\sqrt{2+\sqrt{2}}\right)x - 1\right)$$
$$\left(x^2 + \left(1-\sqrt{2}+\sqrt{2}\sqrt{2-\sqrt{2}}\right)x - 1\right)\left(x^2 + \left(1-\sqrt{2}-\sqrt{2}\sqrt{2-\sqrt{2}}\right)x - 1\right) \tag{2}$$

III. Roots

$$p\left(x_n\right)=0\,,\ n=1,2,3,4,5,6,7,8\,;\ x_n\in\mathbb{R} \tag{3}$$

$$x_1<x_2<x_3<x_4<x_5<x_6<x_7<x_8 \tag{4}$$

$$x_1=\frac{-\sqrt{11+4\sqrt{2}+4\sqrt{2+\sqrt{2}}+2\sqrt{2}\sqrt{2+\sqrt{2}}}-\sqrt{2}\sqrt{2+\sqrt{2}}-\sqrt{2}-1}{2} \tag{5}$$

$$x_2=\frac{-\sqrt{11-4\sqrt{2}-4\sqrt{2-\sqrt{2}}+2\sqrt{2}\sqrt{2-\sqrt{2}}}-\sqrt{2}\sqrt{2-\sqrt{2}}+\sqrt{2}-1}{2} \tag{6}$$

$$x_3=\frac{-\sqrt{11+4\sqrt{2}-4\sqrt{2+\sqrt{2}}-2\sqrt{2}\sqrt{2+\sqrt{2}}}+\sqrt{2}\sqrt{2+\sqrt{2}}-\sqrt{2}-1}{2} \tag{7}$$

$$x_4=\frac{-\sqrt{11-4\sqrt{2}+4\sqrt{2-\sqrt{2}}-2\sqrt{2}\sqrt{2-\sqrt{2}}}+\sqrt{2}\sqrt{2-\sqrt{2}}+\sqrt{2}-1}{2} \tag{8}$$

$$x_5=\frac{\sqrt{11+4\sqrt{2}+4\sqrt{2+\sqrt{2}}+2\sqrt{2}\sqrt{2+\sqrt{2}}}-\sqrt{2}\sqrt{2+\sqrt{2}}-\sqrt{2}-1}{2} \tag{9}$$

$$x_6=\frac{\sqrt{11-4\sqrt{2}-4\sqrt{2-\sqrt{2}}+2\sqrt{2}\sqrt{2-\sqrt{2}}}-\sqrt{2}\sqrt{2-\sqrt{2}}+\sqrt{2}-1}{2} \tag{10}$$

$$x_7=\frac{\sqrt{11+4\sqrt{2}-4\sqrt{2+\sqrt{2}}-2\sqrt{2}\sqrt{2+\sqrt{2}}}+\sqrt{2}\sqrt{2+\sqrt{2}}-\sqrt{2}-1}{2} \tag{11}$$

$$x_8=\frac{\sqrt{11-4\sqrt{2}+4\sqrt{2-\sqrt{2}}-2\sqrt{2}\sqrt{2-\sqrt{2}}}+\sqrt{2}\sqrt{2-\sqrt{2}}+\sqrt{2}-1}{2} \tag{12}$$

$$x_1x_5=x_2x_6=x_3x_7=x_4x_8=-1 \tag{13}$$

IV. Arctangents Relations

$$\pi = -\frac{16}{15}\left(\tan^{-1}(x_1) + \tan^{-1}(x_1^3)\right) \tag{14}$$

$$\pi = -\frac{16}{11}\left(\tan^{-1}(x_2) + \tan^{-1}(x_2^3)\right) \tag{15}$$

$$\pi = -\frac{16}{7}\left(\tan^{-1}(x_3) + \tan^{-1}(x_3^3)\right) \tag{16}$$

$$\pi = -\frac{16}{3}\left(\tan^{-1}(x_4) + \tan^{-1}(x_4^3)\right) \tag{17}$$

$$\pi = \frac{16}{1}\left(\tan^{-1}(x_5) + \tan^{-1}(x_5^3)\right) \tag{18}$$

$$\pi = \frac{16}{5}\left(\tan^{-1}(x_6) + \tan^{-1}(x_6^3)\right) \tag{19}$$

$$\pi = \frac{16}{9}\left(\tan^{-1}(x_7) + \tan^{-1}(x_7^3)\right) \tag{20}$$

$$\pi = \frac{16}{13}\left(\tan^{-1}(x_8) + \tan^{-1}(x_8^3)\right) \tag{21}$$

V. Series

$$\pi = -\frac{16}{7}\sum_{n=0}^{\infty}(-1)^n x_3^{6n+1}\left(\frac{1}{6n+1} + \frac{2x_3^2}{6n+3} + \frac{x_3^4}{6n+5}\right) \tag{22}$$

$$\pi = -\frac{16}{3}\sum_{n=0}^{\infty}(-1)^n x_4^{6n+1}\left(\frac{1}{6n+1} + \frac{2x_4^2}{6n+3} + \frac{x_4^4}{6n+5}\right) \tag{23}$$

$$\pi = \frac{16}{1}\sum_{n=0}^{\infty}(-1)^n x_5^{6n+1}\left(\frac{1}{6n+1} + \frac{2x_5^2}{6n+3} + \frac{x_5^4}{6n+5}\right) \tag{24}$$

$$\pi = \frac{16}{5}\sum_{n=0}^{\infty}(-1)^n x_6^{6n+1}\left(\frac{1}{6n+1} + \frac{2x_6^2}{6n+3} + \frac{x_6^4}{6n+5}\right) \tag{25}$$

VI. Trigonometric Relations

$$p(x)=\left(x^2-x\tan\left(\frac{\pi}{16}\right)-1\right)\left(x^2+x\tan\left(\frac{3\pi}{16}\right)-1\right)\left(x^2-x\tan\left(\frac{5\pi}{16}\right)-1\right)\left(x^2+x\tan\left(\frac{7\pi}{16}\right)-1\right) \tag{26}$$

$$x_1=\frac{1}{2}\left(-\tan\left(\frac{7\pi}{16}\right)-\sqrt{4+\left(\tan\left(\frac{7\pi}{16}\right)\right)^2}\right) \tag{27}$$

$$x_2=\frac{1}{2}\left(-\tan\left(\frac{3\pi}{16}\right)-\sqrt{4+\left(\tan\left(\frac{3\pi}{16}\right)\right)^2}\right) \tag{28}$$

$$x_3=\frac{1}{2}\left(\tan\left(\frac{\pi}{16}\right)-\sqrt{4+\left(\tan\left(\frac{\pi}{16}\right)\right)^2}\right) \tag{29}$$

$$x_4=\frac{1}{2}\left(\tan\left(\frac{5\pi}{16}\right)-\sqrt{4+\left(\tan\left(\frac{5\pi}{16}\right)\right)^2}\right) \tag{30}$$

$$x_5=\frac{1}{2}\left(-\tan\left(\frac{7\pi}{16}\right)+\sqrt{4+\left(\tan\left(\frac{7\pi}{16}\right)\right)^2}\right) \tag{31}$$

$$x_6=\frac{1}{2}\left(-\tan\left(\frac{3\pi}{16}\right)+\sqrt{4+\left(\tan\left(\frac{3\pi}{16}\right)\right)^2}\right) \tag{32}$$

$$x_7=\frac{1}{2}\left(\tan\left(\frac{\pi}{16}\right)+\sqrt{4+\left(\tan\left(\frac{\pi}{16}\right)\right)^2}\right) \tag{33}$$

$$x_8=\frac{1}{2}\left(\tan\left(\frac{5\pi}{16}\right)+\sqrt{4+\left(\tan\left(\frac{5\pi}{16}\right)\right)^2}\right) \tag{34}$$

$$x_5 = \cfrac{1}{\tan\left(\frac{7\pi}{16}\right) + \cfrac{1}{\tan\left(\frac{7\pi}{16}\right) + \cfrac{1}{\tan\left(\frac{7\pi}{16}\right) + \dots}}} \tag{35}$$

$$\frac{1}{x_5} = \sqrt{1 + \tan\left(\frac{7\pi}{16}\right)\sqrt{1 + \tan\left(\frac{7\pi}{16}\right)\sqrt{1 + \dots}}} \tag{36}$$

$$x_6 = \cfrac{1}{\tan\left(\frac{3\pi}{16}\right) + \cfrac{1}{\tan\left(\frac{3\pi}{16}\right) + \cfrac{1}{\tan\left(\frac{3\pi}{16}\right) + \dots}}} \tag{37}$$

$$\frac{1}{x_6} = \sqrt{1 + \tan\left(\frac{3\pi}{16}\right)\sqrt{1 + \tan\left(\frac{3\pi}{16}\right)\sqrt{1 + \dots}}} \tag{38}$$

$$x_7 = \tan\left(\frac{\pi}{16}\right) + \cfrac{1}{\tan\left(\frac{\pi}{16}\right) + \cfrac{1}{\tan\left(\frac{\pi}{16}\right) + \dots}} \tag{39}$$

$$x_7 = \sqrt{1 + \tan\left(\frac{\pi}{16}\right)\sqrt{1 + \tan\left(\frac{\pi}{16}\right)\sqrt{1 + \dots}}} \tag{40}$$

$$x_8 = \tan\left(\frac{5\pi}{16}\right) + \cfrac{1}{\tan\left(\frac{5\pi}{16}\right) + \cfrac{1}{\tan\left(\frac{5\pi}{16}\right) + \dots}} \tag{41}$$

$$x_8 = \sqrt{1 + \tan\left(\frac{5\pi}{16}\right)\sqrt{1 + \tan\left(\frac{5\pi}{16}\right)\sqrt{1 + \dots}}} \tag{42}$$

VII. Roots: Iterative Method

$$y_{n+1} = \frac{7y_n^8 + 24y_n^7 - 50y_n^6 - 64y_n^5 + 57y_n^4 + 32y_n^3 - 10y_n^2 - 1}{8y_n^7 + 28y_n^6 - 60y_n^5 - 80y_n^4 + 76y_n^3 + 48y_n^2 - 20y_n - 4} \quad , n \in \mathbb{N} \tag{43}$$

$$y_1 = -5 \Rightarrow y_n \to x_1 \tag{44}$$

$$y_1 = -3/2 \Rightarrow y_n \to x_2 \tag{45}$$

$$y_1 = -1 \Rightarrow y_n \to x_3 \tag{46}$$

$$y_1 = -1/2 \Rightarrow y_n \to x_4 \tag{47}$$

$$y_1 = 0 \Rightarrow y_n \to x_5 \tag{48}$$

$$y_1 = 3/4 \Rightarrow y_n \to x_6 \tag{49}$$

$$y_1 = 1 \Rightarrow y_n \to x_7 \tag{50}$$

$$y_1 = 2 \Rightarrow y_n \to x_8 \tag{51}$$

VIII. Trigonometric Identities

$$\tan\left(\frac{7\pi}{16}\right) - \tan\left(\frac{5\pi}{16}\right) + \tan\left(\frac{3\pi}{16}\right) - \tan\left(\frac{\pi}{16}\right) = 4 \tag{52}$$

$$\begin{aligned} &\tan\left(\frac{\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right) - \tan\left(\frac{\pi}{16}\right)\tan\left(\frac{5\pi}{16}\right) + \tan\left(\frac{\pi}{16}\right)\tan\left(\frac{7\pi}{16}\right) + \\ &\quad \tan\left(\frac{3\pi}{16}\right)\tan\left(\frac{5\pi}{16}\right) - \tan\left(\frac{3\pi}{16}\right)\tan\left(\frac{7\pi}{16}\right) + \tan\left(\frac{5\pi}{16}\right)\tan\left(\frac{7\pi}{16}\right) = 6 \end{aligned} \tag{53}$$

$$\begin{aligned} &-\tan\left(\frac{\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right)\tan\left(\frac{5\pi}{16}\right) + \tan\left(\frac{\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right)\tan\left(\frac{7\pi}{16}\right) - \\ &\quad \tan\left(\frac{\pi}{16}\right)\tan\left(\frac{5\pi}{16}\right)\tan\left(\frac{7\pi}{16}\right) + \tan\left(\frac{3\pi}{16}\right)\tan\left(\frac{5\pi}{16}\right)\tan\left(\frac{7\pi}{16}\right) = 4 \end{aligned} \tag{54}$$

$$\tan\left(\frac{\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right)\tan\left(\frac{5\pi}{16}\right)\tan\left(\frac{7\pi}{16}\right) = 1 \tag{55}$$

IX. Fractal: Details

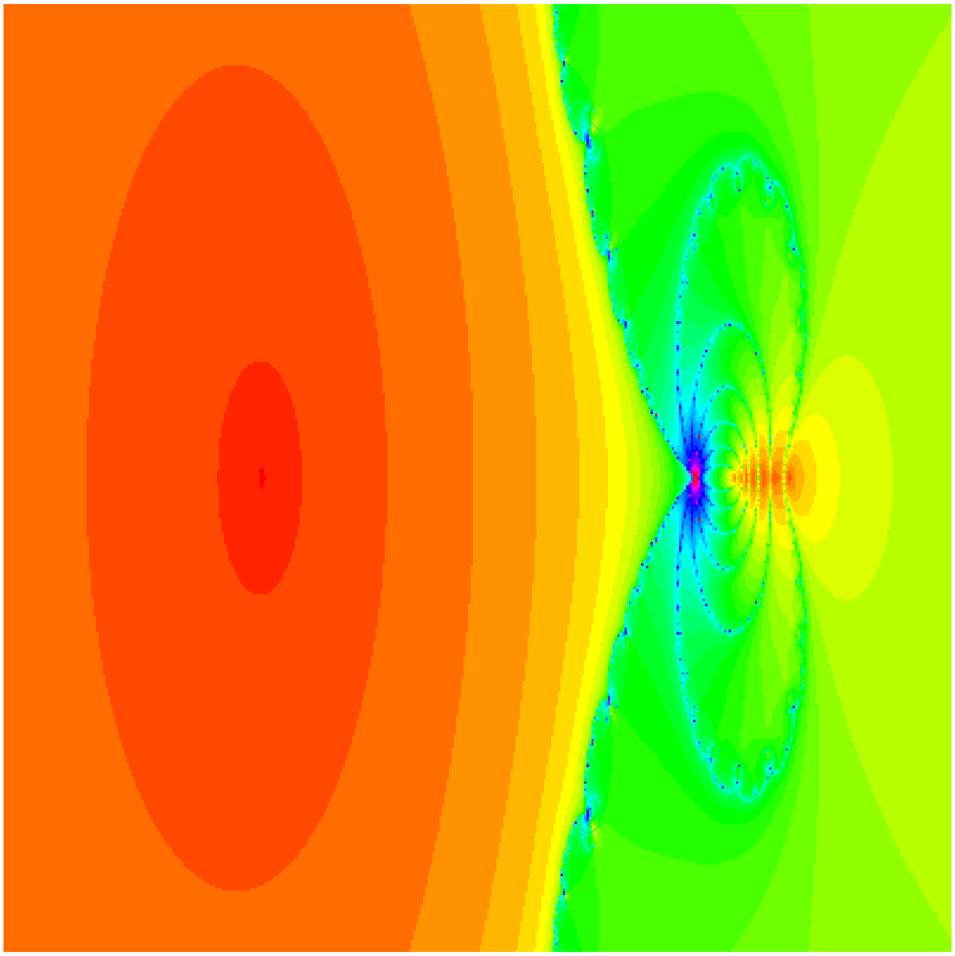

Figure 3. Newton-Julia set for $p(z)$.

Region: $(bl, ur) = (-5.6 - 0.25i, -4.2 + 0.25i)$

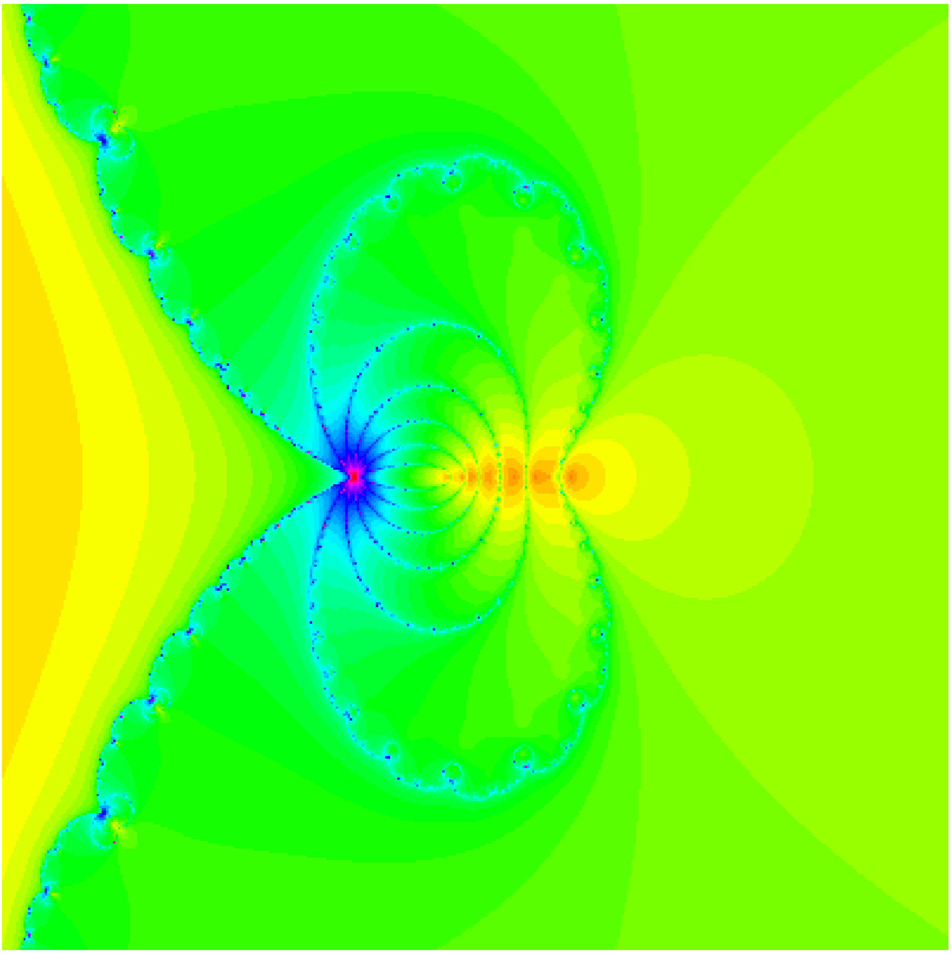

Figure 4. Newton-Julia set for $p(z)$.

Region: $(bl, ur) = (-4.8 - 0.25i, -4.2 + 0.25i)$

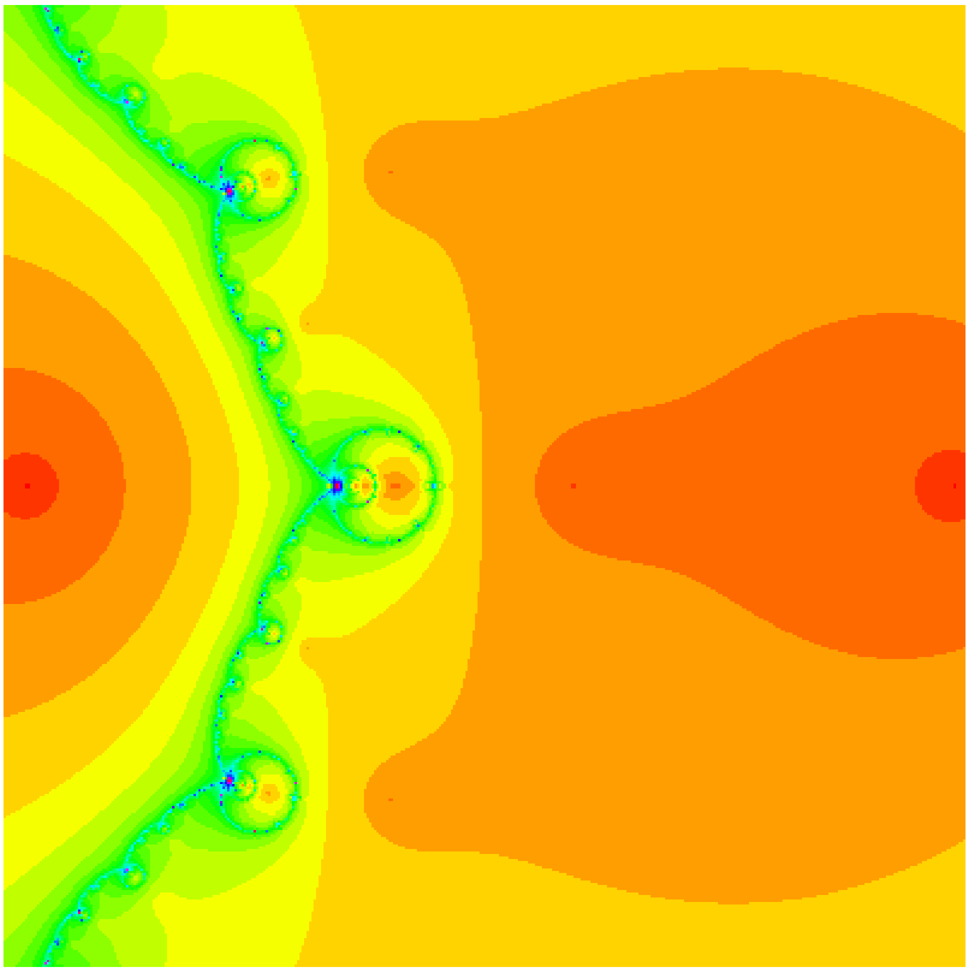

Figure 5. Newton-Julia set for $p(z)$

Region: $(bl,ur)=(-1.4-0.25i,-0.9+0.25i)$

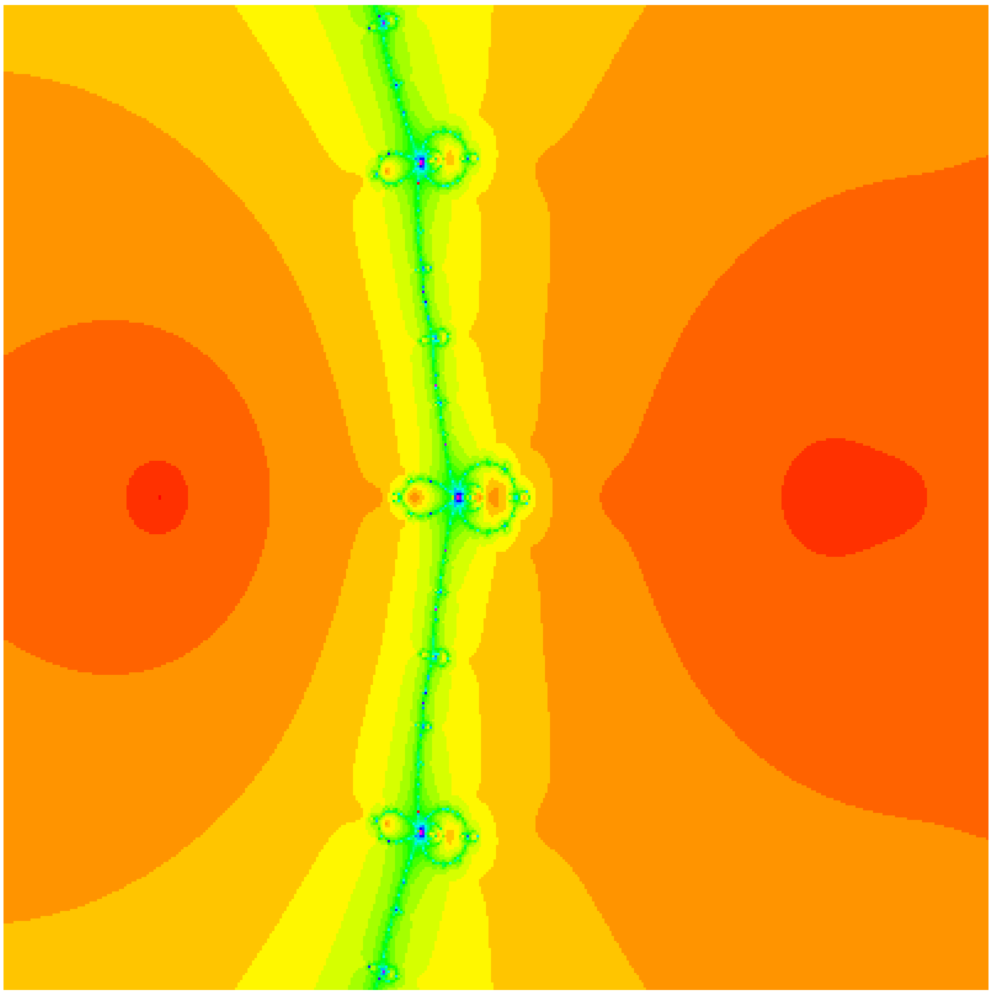

Figure 6. Newton-Julia set for $p(z)$.

Region: $(bl, ur) = (-1-0.25i, -0.4+0.25i)$

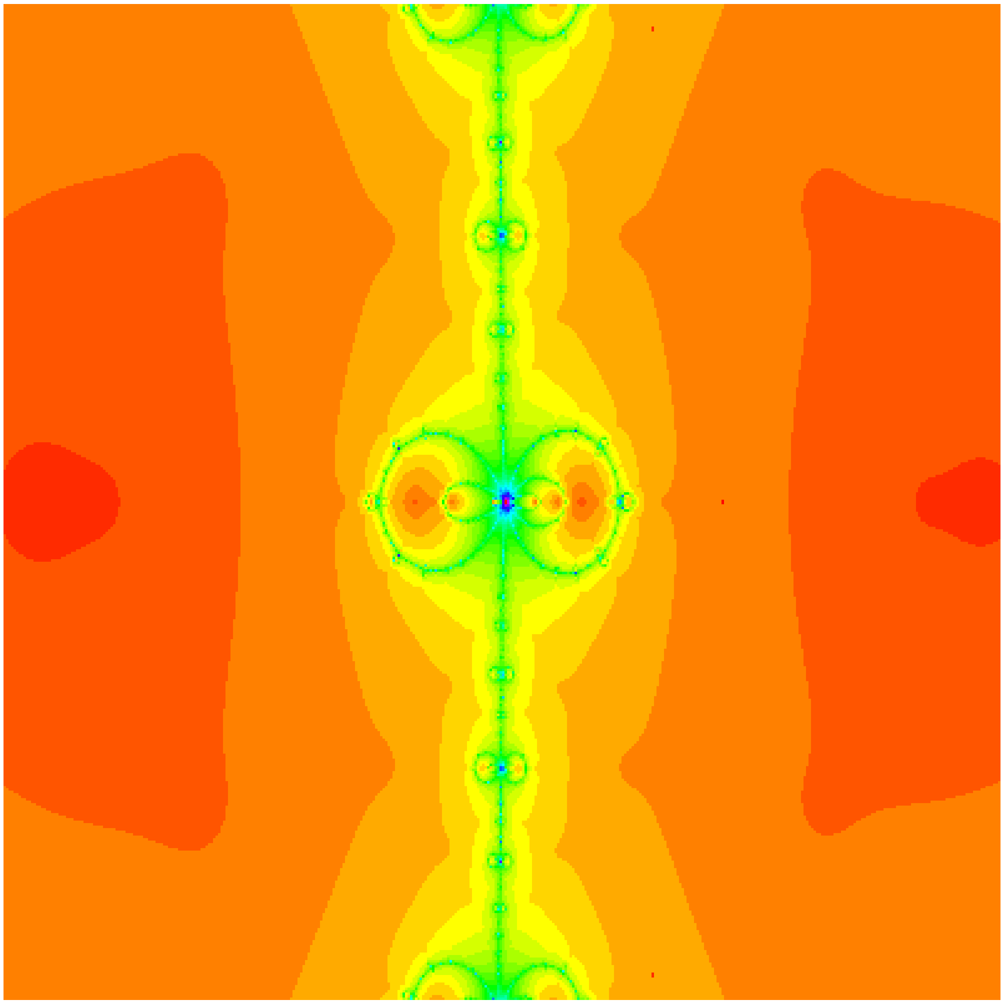

Figure 7. Newton-Julia set for $p(z)$.

Region: $(bl, ur) = (-0.52 - 0.25i, 0.2 + 0.25i)$

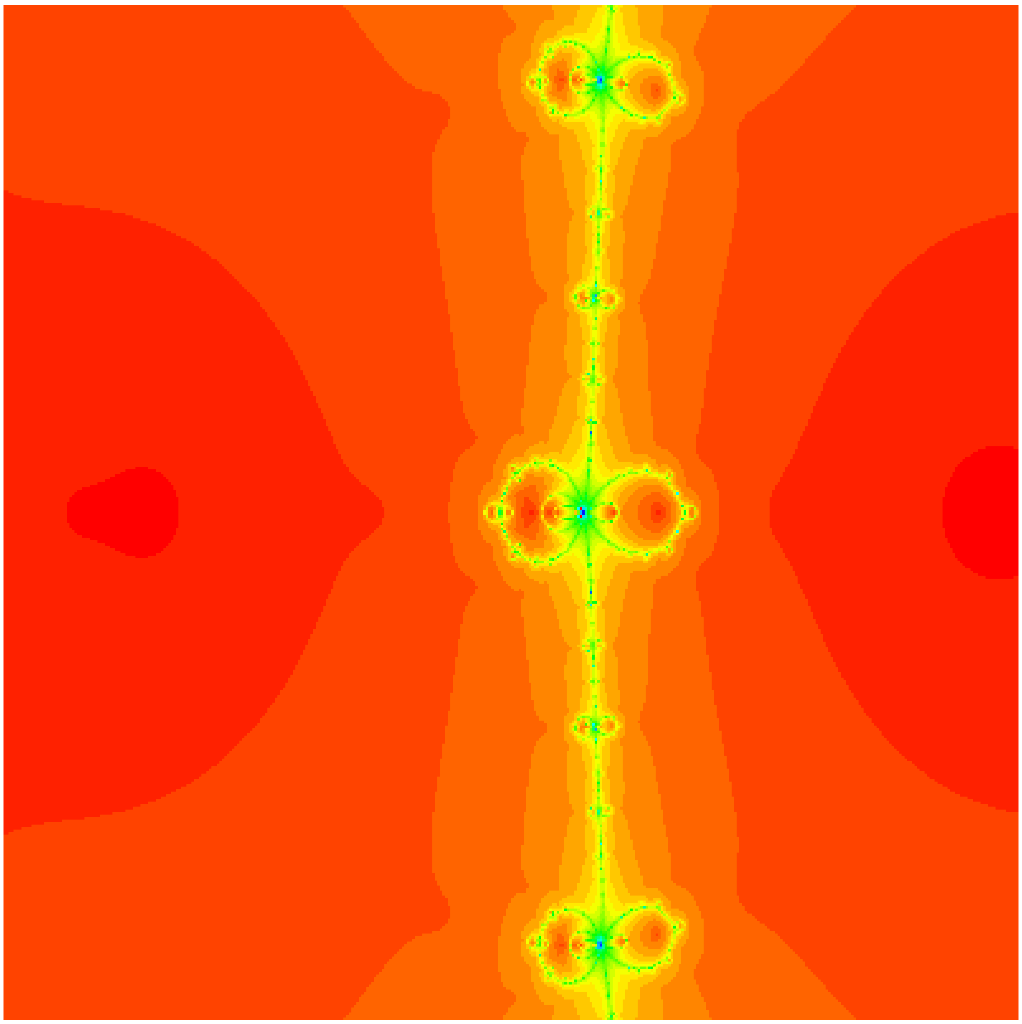

Figure 8. Newton-Julia set for $p(z)$.

Region: $(bl, ur) = (0.1 - 0.25i, 0.74 + 0.25i)$

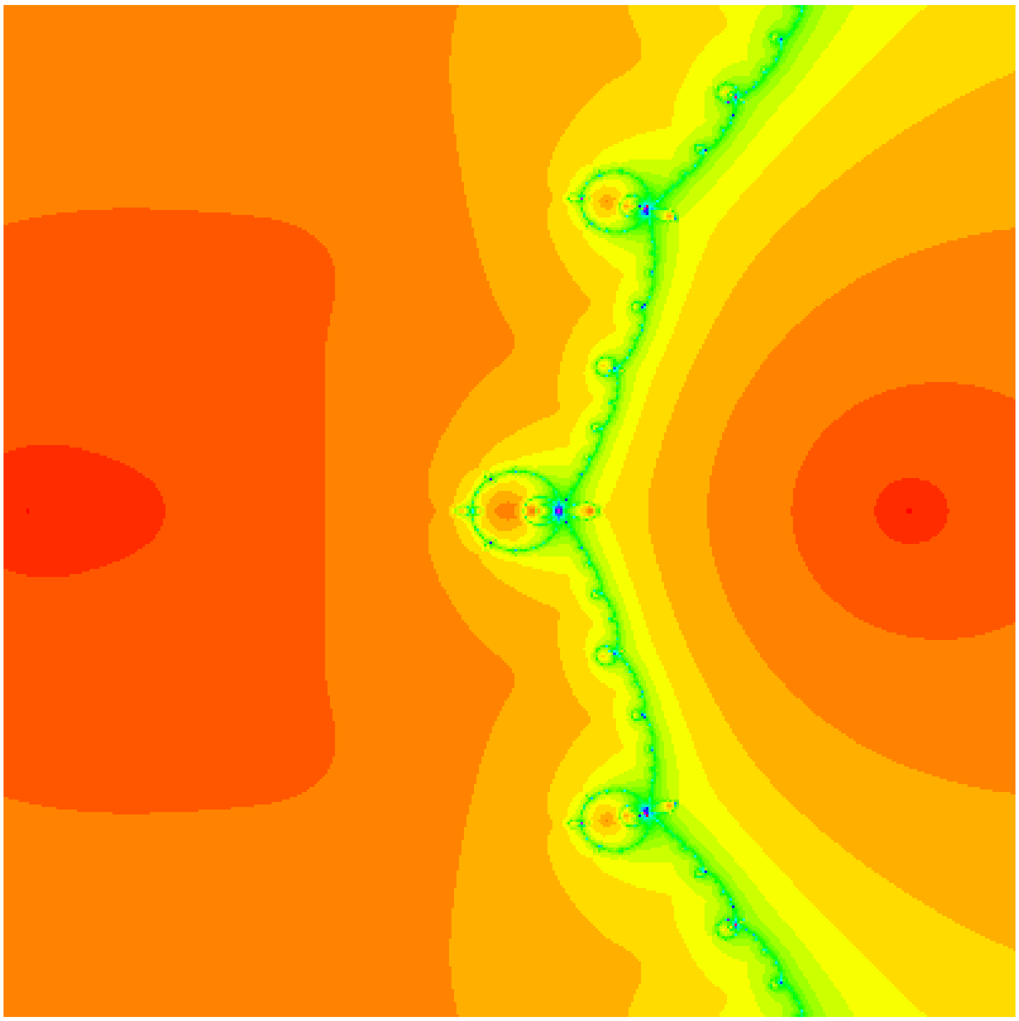

Figure 9. Newton-Julia set for $p(z)$.

Region: $(bl, ur) = (0.71 - 0.25i, 1.15 + 0.25i)$

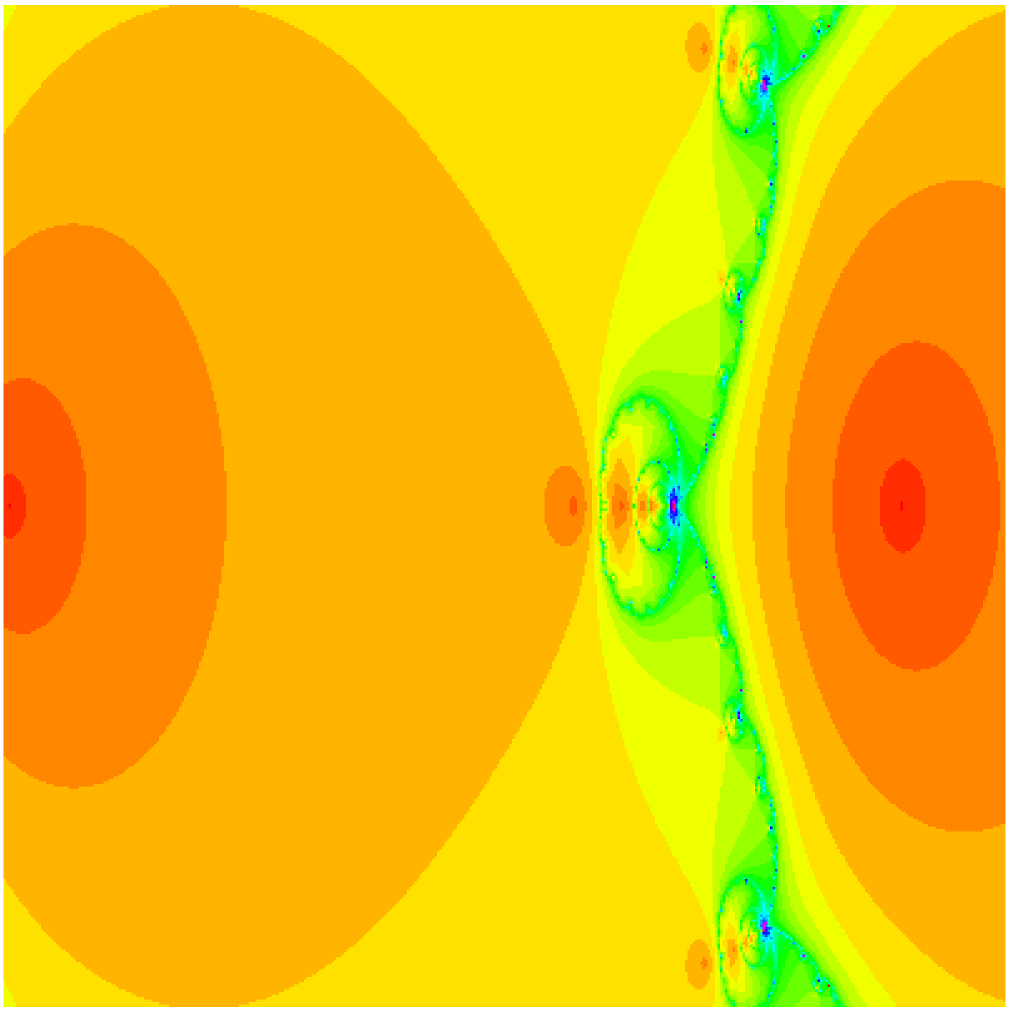

Figure 10. Newton-Julia set for $p(z)$.

Region: $(bl,ur)=(1.1-0.25i,2.1+0.25i)$

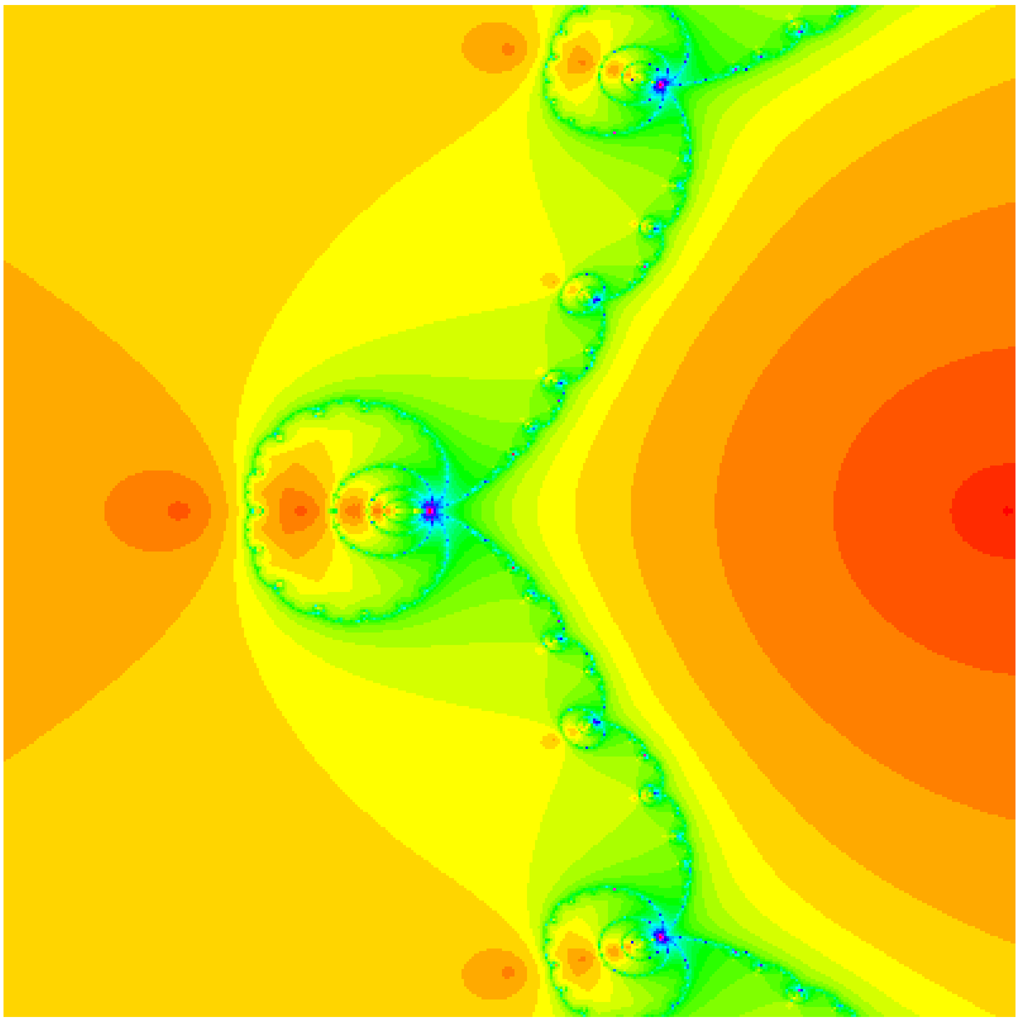

Figure 11. Newton-Julia set for $p(z)$.

Region: $(bl, ur) = (1.6 - 0.25i, 2 + 0.25i)$

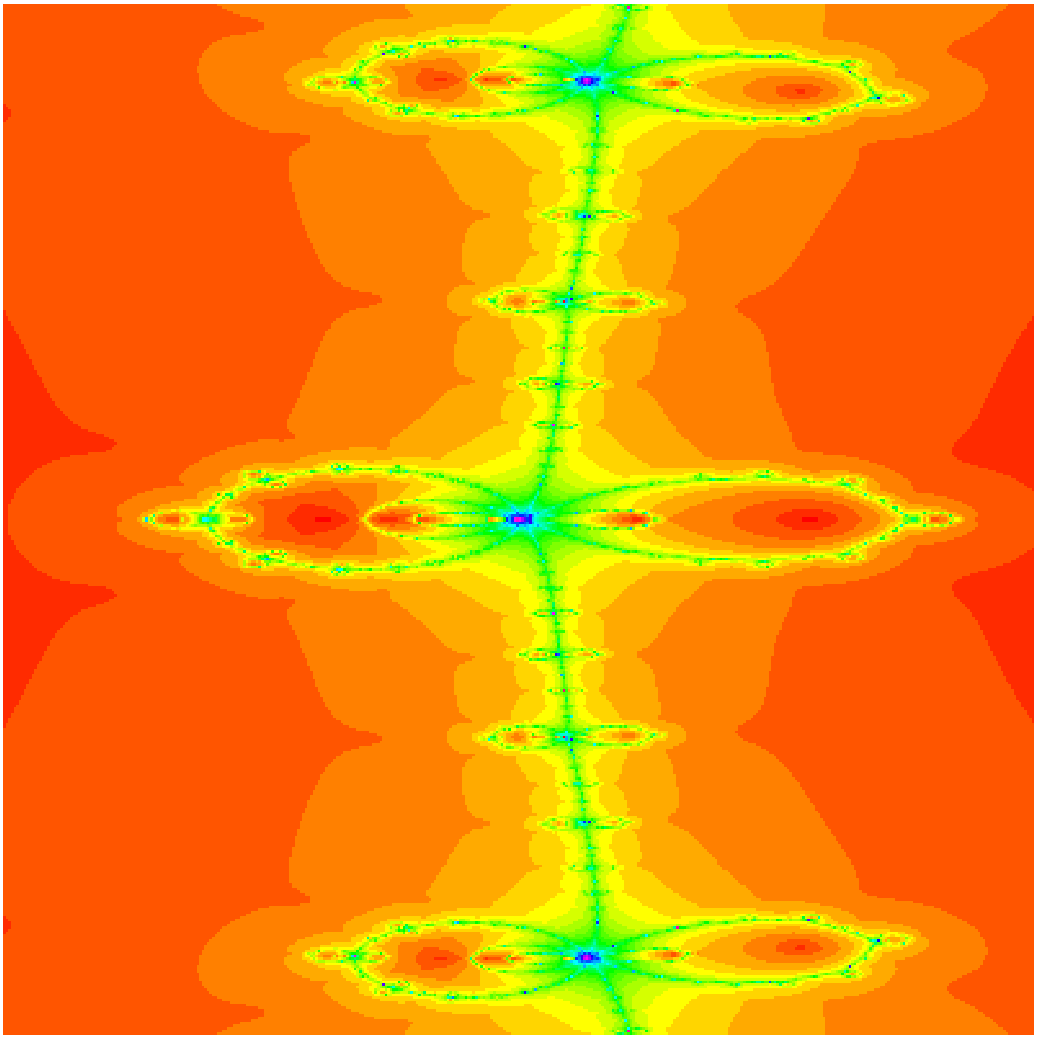

Figure 12. Newton-Julia set for $p(z)$.

Region: $(bl, ur) = (0.38 - 0.25i, 0.55 + 0.25i)$

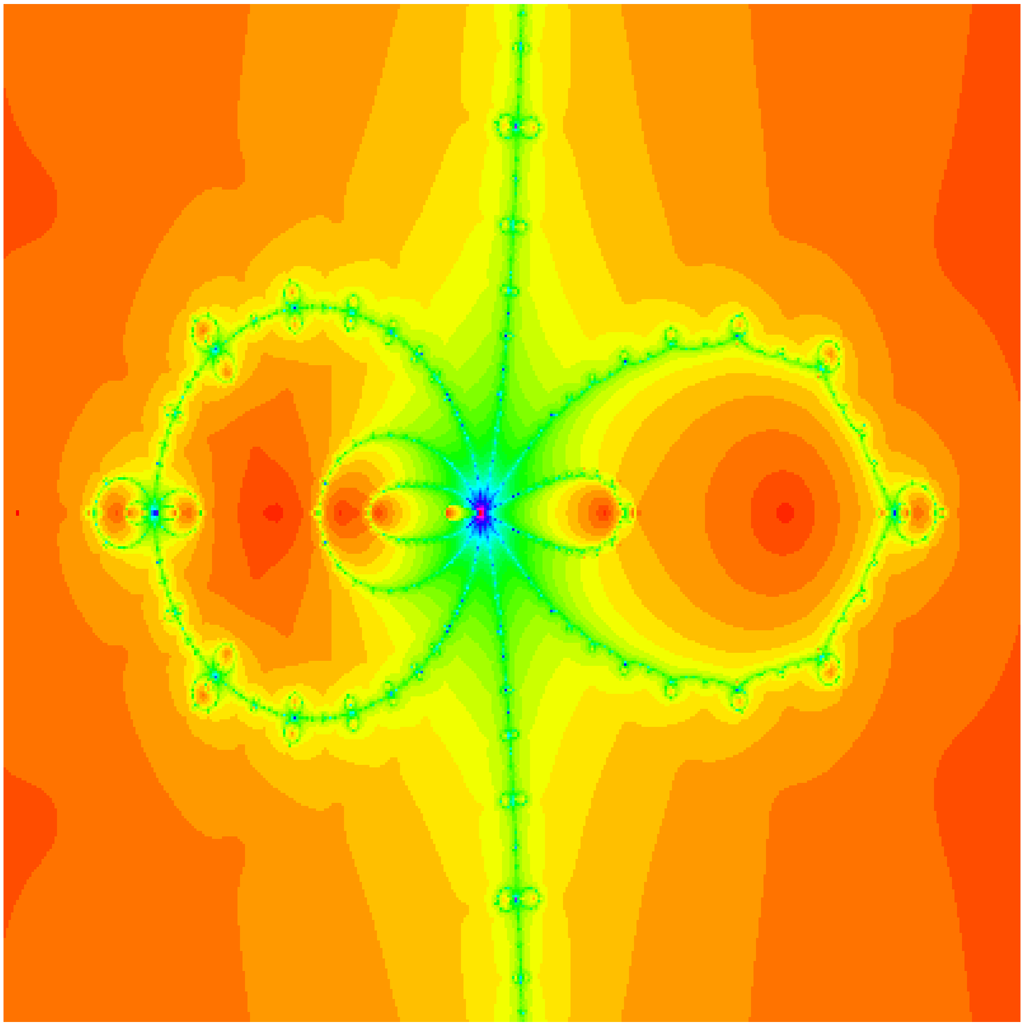

Figure 13. Newton-Julia set for $p(z)$.

Region: $(bl, ur) = (0.39 - 0.25i, 0.55 + 0.25i)$

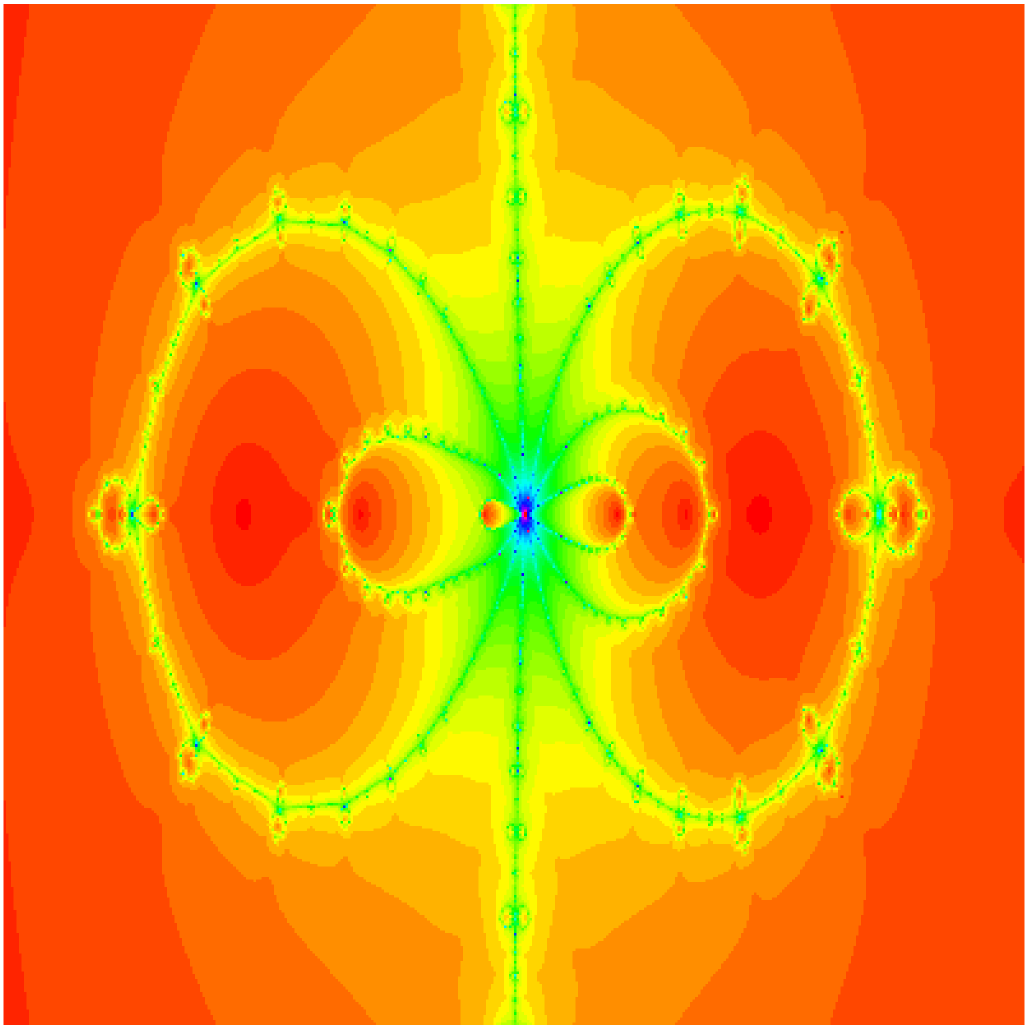

Figure 14. Newton-Julia set for $p(z)$.

Region: $(bl, ur) = (-0.28 - 0.06i, -0.04 + 0.06i)$

X. Associate Polynomial

$$q(y) = y^4 + 4y^3 - 6y^2 - 4y + 1 \tag{56}$$

$$q(y) = \left(y - \tan\left(\frac{\pi}{16}\right)\right)\left(y + \tan\left(\frac{3\pi}{16}\right)\right)\left(y - \tan\left(\frac{5\pi}{16}\right)\right)\left(y + \tan\left(\frac{7\pi}{16}\right)\right) \tag{57}$$

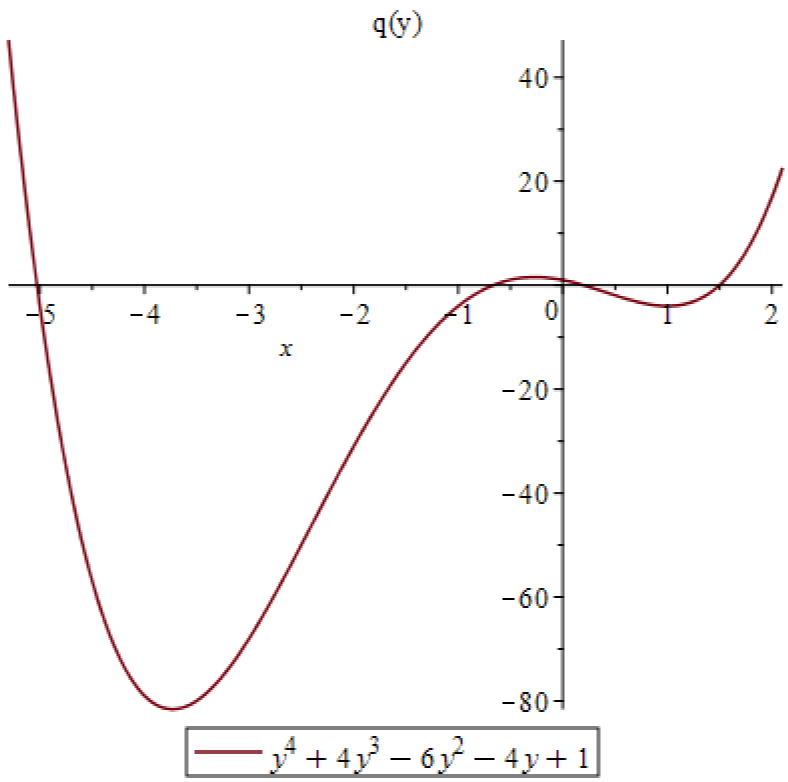

Figure 15.

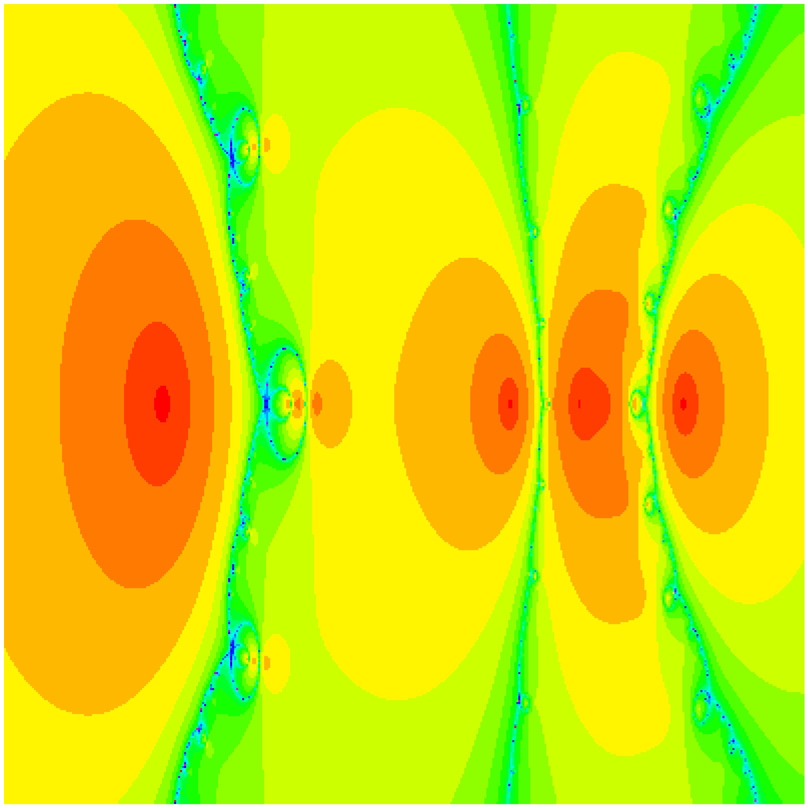

Figure 16. Newton-Julia set for $q(z)$.

XI. Formula

$$\pi^2 = 16x_5^2\left(1-x_5^2\right)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{\left(\left(2n-2k+1\right)^2-\left(1-x_5^2\right)^2\right)\left(\left(2k+1\right)^2-x_5^4\right)} \tag{58}$$

Referencias

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