The union is not the limit.

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Abstract: Contrary to the assumptions of transfinite set theory, limit and union of infinite sequences of sets differ. We will show this for the set ℕ of natural numbers by the newly devised powerful tool of arithmogeometry. The basic theorem of set theory $\forall n \in \mathbb{N}: n < \aleph_0$ precludes the identity $|\mathbb{N}| =_{def} \aleph_0$. Further differences of union and limit and contradictions with analysis will be shown.

Arithmogeometry

The sequence of all finite initial segments $F(n) = \{1, 2, 3, ..., n\}$ of the set ℕ of natural numbers n is represented in the following figure which we call an arithmogeometric figure because it includes arithmetic and geometry:

1
1, 2 (1)
1, 2, 3
... .

It has the advantage that its contents can be interpreted as the set F of all finite initial segments of ℕ where only the braces have been replaced by carriage returns

$$\{\{1\}, \{1, 2\}, \{1, 2, 3\}, ...\} = F$$

as well as the union $\cup F$, i.e., as the set ℕ of all natural numbers

$$\{1, 1, 2, 1, 2, 3, ...\} \equiv \{1, 2, 3, ...\} = \mathbb{N} .$$

According to set theory

$$|\mathbb{N}| = |\cup F| = \aleph_0 \tag{2}$$

where, according to a very basic theorem [1]

$$\forall n \in \mathbb{N}: n < \aleph_0 . \tag{3}$$

Now, the union $\cup F$ can be accomplished by the following procedure: Every row of (1) is shifted into the first row, maintaining the column of each element. The shifted element replaces the element in the first row (i.e., the same natural number or a blank). When all rows have been shifted accordingly, then all natural numbers are in the first row. But obviously the first row has not $\aleph_0$ elements, if this means more elements than every row, because only these rows have contributed. By the practice of shifting rows the basic theorem (3) of set theory precludes (2).

The contrary claim would be tantamount to the claim that the figure has more elements than all rows of the figure, or that shifting of rows increases their finite cardinal numbers. Then however translation invariance of mathematical expressions like finite strings would be violated, namely the fact (among others necessary for mathematical discourse) that the place of writing a string must not change the meaning or cardinality of the string.

So we can conclude *that if* (3) *holds*,[1] then $|\mathbb{N}|$ is *not* a fixed quantity $\aleph_0$ greater than all natural numbers. Since $|\mathbb{N}| < \aleph_0$ cannot be equal to a natural number either, the only alternative is potential infinity: $|\mathbb{N}|$ is of always growing, never finished size. It is suitable to denote it by ∞ and to reserve $\aleph_0$ for the cardinal number of the least upper bound ω of the sequence (n) of natural numbers n such that $|\omega| = \aleph_0$.

The familiar equality

$$\lim_{n\to\infty} F(n) = \bigcup_{n\in\mathbb{N}} F(n) \tag{4}$$

turns out to be mistaken. Note that also in analysis a strictly monotonously increasing sequence does never assume its limit. Since this limit (which is not the set-theoretic limit, see below) is

$$\lim_{n\to\infty} n = \infty \tag{5}$$

or, as Cantor later wrote, ω, and since all natural numbers n are contained in the finite initial segments $F(n)$ as last elements, we get

$$\lim_{n\to\infty} F(n) = \lim_{n\to\infty} \{1, 2, 3, ..., n\} = \{1, 2, 3, ..., \omega\} .$$

The union of the rows of (1) however does not contain any transfinite number

$$\bigcup_{n\in\mathbb{N}} F(n) = \{1, 2, 3, ...\} = \mathbb{N} .$$

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[1] If on the other hand $|\mathbb{N}| = \aleph_0$ is taken as a definition, then (3) must fail.

Recursion of finite initial segments

The difference between limit and union can also be obtained from the fact that

$$\forall n \in \mathbb{N}: F(n) = F(n\text{-}1) \cup \{n\} \quad \text{with} \quad F(0) = \{\ \}\ .$$

There are as many unions as finite initial segments $F(n)$. If there are infinitely many (ω) finite initial segments, then there are infinitely many (ω) unions. None of them yields an infinite set, since every $F(n)$ is finite. It is impossible that another union, the union number $\omega + 1$, collecting only what has been collected before already and not adding anything more, could increase the former result by an infinity, namely by the difference between the limit ω and each $F(n)$.

It is strange that many mathematicians are so captivated by (4) that they can't comprehend this simple fact: If infinitely many unions with always increasing the number of elements do not yield the limit, then another union without any addition cannot, by simplest logic, increase the result to reach the limit. The limit is, as every strictly increasing sequence shows, not reached by finite steps – and every increase by a number n is a finite step.

As a result we find that $\aleph_0 = |\omega|$ is not the cardinality of the set $\mathbb{N}$. This set $\mathbb{N}$ is infinite, but it is only potentially infinite, having not a cardinal number because a finite cardinality cannot be attached to it and the first infinite cardinal number $\aleph_0$ has been excluded.

Failure of set theoretic limit

Finally, in order to see that for sequences of sets in general the set theoretic limit, here denoted by Lim, differs from the analytical limit, denoted by lim

$$\text{Lim } S(n) \neq \lim S(n)$$

we consider the sequence of closed real intervals $[1/n, 1]$.

In set theory a sequence (M_n) of sets M_n has the limit Lim M_n [2] if and only if

$$\text{Lim } M_n = \text{LimSup } M_n = \text{LimInf } M_n \qquad (6)$$

where

$$\text{LimSup } M_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} M_k$$

and

$$\text{LimInf } M_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} M_k\ .$$

According to this formalism the union of closed real intervals $[1/n, 1]$ is their union, namely the half-open real interval $(0, 1]$. In analysis however, we find from

$$\lim_{n\to\infty}\frac{1}{n}=0$$

the analytical and only correct limit

$$\lim_{n\to\infty}[\frac{1}{n},1]=[\lim_{n\to\infty}\frac{1}{n},1]=[0,1]$$

because the limit of the sequence of fractions $1/n$ cannot depend on the question whether they represent singular points or edges of intervals extending to the right-hand side which is opposite to the limit-point 0 on the left-hand side. Again one finds this clear logical conclusion often to be in ruins, covered by set theoretical rubble.

To show the general inappropriateness of the set theoretical limit and its being in contradiction with analytical results we will discuss three further simple examples.

• First consider the sequence of real intervals $[\frac{(-1)^n}{n},1]$ which has the analytical limit

$$\lim_{n\to\infty}[\frac{(-1)^n}{n},1] = [\lim_{n\to\infty}\frac{(-1)^n}{n},1] = [0,1] .$$

In set theory this sequence has no limit because every supremum

$$\text{Sup}\, M_n = \bigcup_{k=n}^{\infty} M_k = [0,1]$$

contains 0 and so does the intersection of all suprema, but no infinimum

$$\text{Inf}\ M_n = \bigcap_{k=n}^{\infty} M_k = (0,1]$$

contains 0, so that 0 is also lacking in the union of all infima. According to (6) there is no limit because $\text{LimSup}\, M_n \neq \text{LimInf}\ M_n$.

• As second example we consider the limit of the sequence of finite cardinal $\mathbb{N}_0$ numbers. This sequence can be represented in many different ways using Arabic digits, circles as unary symbols, Zermelo's system, or von Neumann's system

{0},	{1},	{2},	{3},	... → { }
{ },	{o},	{oo},	{ooo},	... → {ooo...}
{ },	{0},	{1},	{2},	... → { }
{ },	{0},	{0, 1},	{0, 1, 2},	... → {0, 1, 2, ... } = $\mathbb{N}_0$.

All symbols in the same column, looking very different though, denote one and the same cardinal number *n*. The limits however have not only different shape but denote completely different objects.

• The last example is unsurpassable in showing the uselessness of the set theoretic limit.

Scrooge McDuck every day earns 10 $ and spends 1 $. Since a comic character lives forever his wealth grows immeasurably – at least when applying the analytical limit of the sequence $(9n)_{n\in\mathbb{N}}$. However if he issues always the dollars received first and if he applies (6), then he will go bankrupt since the sequence of sets {*n*, *n*+1, *n*+2, ..., 10*n*} has an emty limit set. Not necessary to mention that this result is inacceptable, in particular because it depends on the indices of the spent dollars. Such a dependence cannot be accepted in any scientific theory.

It has to be mentioned however that this very limit, transformed from the time axis to the spatial axis of real numbers and with the ratio *infinite* instead of 10, is necessary to enumerate all rational numbers by natural numbers such that the set of not enumerated rational numbers like the set of dollars kept by McDuck is empty. [3]

References

[1] Karel Hrbacek, Thomas Jech: "Introduction to Set Theory", 2nd ed., Marcel Dekker, New York (1984) p. 83
[2] S.I. Resnick: "A probability path", Birkhäuser, Boston (1998) p. 6
[3] W. Mückenheim: "Does Set Theory Cause Perceptual Problems?", viXra.org, Mind Science, 1702.0280