Not enumerating all positive rational numbers

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Abstract: It is shown that the enumeration of rational numbers cannot be complete.

The impossibility of so-called supertaks [1] is generally accepted. By the equivalence of spatial and temporal axes it is clear already that bijections between different infinite sets are impossible too. But here we will give an independent proof.

Cantor's enumeration of the set $\mathbb{Q}_+$ positive rational numbers q is ordered by the ascending sum $(a+b)$ of numerator a and denominator b of $q = a/b$, and in case of equal sum, by ascending numerator a. Since all *fractions* will repeat themselves infinitely often, repetitions will be dropped when enumerating the *rational numbers*. This yields the sequence [2]

$$1/1, 1/2, 2/1, 1/3, 3/1, 1/4, 2/3, 3/2, 4/1, 1/5, 5/1, 1/6, 2/5, 3/4, 4/3, 5/2, 6/1, 1/7, ... \qquad [*]$$

It is easy to see that at least half of all fractions of this sequence belong to the first unit interval (0, 1]. Therefore, while every positive rational number q gets a natural index n in a finite step of this sequence, there remains always a set s_n of positive rational numbers less than n which have not got an index less than n

$$s_{n+1} = (s_n \cup \{q \mid n < q \leq n+1\}) \setminus \{q_{n+1}\} \quad \text{with} \quad s_1 = \{q \mid 0 < q \leq 1\} \setminus \{q_1\} \ .$$

Since all terms of the sequence (s_n) are infinite, $|s_n| = \infty$ for every $n \in \mathbb{N}$ and, according to real analysis, in the limit. But also the geometric measure of *connected unit intervals* below n without any indexed rational number, what we will call *undefiled* intervals, is increasing beyond every bound. This is shown by the following

<u>Theorem</u> For every $k \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$: $(n\text{-}k, n] \subset s_n$.

Proof: Let $a/1$ be the largest fraction indexed by n. Up to every such n at least half of the natural numbers are mapped on fractions of the first unit interval. $a(n)$ is increasing in steps without missing any natural number n, i.e., without gaps. Therefore n must be about twice as a, precisely:

$$n \geq 2a - 1$$

Examples are taken from Cantor's sequence [*] given above:

$a = 1, n = 1$
$a = 2, n = 3$
$a = 3, n = 5$
$a = 4, n = 9$
$a = 5, n = 11$
$a = 6, n = 17$
...

Therefore for any $n_0 \geq 6$ we can take $k = n_0/2$. Then the interval $(n_0/2, n_0] \subset s_{n_0}$.

This means, there are arbitrarily large sequences of undefiled unit intervals (containing no rational number with an index n or less) in the sets s_n. It is easy to find a completely undefiled interval of any length or every desired multiple in some set s_n.

To give a formal proof [3], let j, k, n denote natural numbers. Let $(k - 1, k]$ denote the kth positive unit interval. Further let $q_1, q_2, q_3, ...$ be any enumeration of all (cancelled) positive fractions.

Consider the sequence (S_n) of sets S_n of such unit intervals $s_k = (k - 1, k]$ which contain rational numbers not enumerated by $j \leq n$:

$$S_n = \{s_k \mid (k - 1, k] \wedge k \leq n \wedge \exists q(q \in \mathbb{Q}_+ \cap (k - 1, k] \wedge \neg\exists j \leq n: q = q_j)\}$$

This sequence of sets of unit intervals with the specific property of containing not enumerated fractions has the limit

$$\lim_{n\to\infty} S_n = \{s_k \mid k \in \mathbb{N}\}$$

i.e., in the limit, after having used up all natural numbers, there are all unit intervals containing together (and in fact each interval separately) infinitely many not enumerated fractions.

Remark: Sometimes it is claimed that, by some unknown power, "in the limit" all rational numbers get enumerated. But the above formalism has been applied specifically to intervals having not enumerated rational numbers. So the analytical limit must have this property too.

Remark: Why can't we specify a rational number that is not indexed? The reason is that every rational number that can be specified belongs to a tiny initial segment beyond which a potential infinity of rational numbers is following – and each of those has this very same property.

References

[1] W. Mückenheim: "Does Set Theory Cause Perceptual Problems?", viXra.org > Mind Science > viXra:1702.0280.
[2] G. Cantor, letter to R. Lipschitz, 19 Nov 1883.
[3] W.Mückenheim, Transfinity - A Source Book, 2017.