Proof of the Polignac Prime Conjecture and other Conjectures

Stephen Marshall

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Abstract

The Polignac prime conjecture, was made by Alphonse de Polignac in 1849. Alphonse de Polignac (1826 – 1863) was a French mathematician whose father, Jules de Polignac (1780-1847) was prime minister of Charles X until the Bourbon dynasty was overthrown in1830. Polignac attended the École Polytechnique (commonly known as Polytechnique) a French public institution of higher education and research, located in Palaiseau near Paris. In 1849, the year Alphonse de Polignac was admitted to Polytechnique, he made what's known as Polignac's conjecture:

For every positive integer k, there are infinitely many prime gaps of size 2k.

Alphonse de Polignac made other significant contributions to number theory, including the de Polignac's formula, which gives the prime factorization of n!, the factorial of n, where $n \geq 1$ is a positive integer.

This paper presents a complete and exhaustive proof of the Polignac Prime Conjecture. The approach to this proof uses same logic that Euclid used to prove there are an infinite number of prime numbers.

Proof of Polignac's Conjecture

In number theory, Polignac's conjecture states:

For any positive integer k, then for any positive even number 2k, there are infinitely many prime gaps of size 2k. In other words, there are infinitely many cases of two consecutive prime numbers with difference 2k. Mathematically stated:

There exist infinitely many cases where, both p and p + 2k are prime.

First we shall assume that the set of Polignac Primes are finite and then we shall prove that this is false, which shall prove that are Polignac Primes are infinite

The divergence of the harmonic series was independently proved by Johann Bernoulli in 1689 in a counter-intuitive manner (reference 1). His proof is worthy of deep study, as it shows the counter-intuitive nature of infinity. We will use Bernoulli's proof and apply it toward proving the Polignac prime numbers are infinite.

Let the finite set of, p, Polignac primes be listed in reverse order from the largest to smallest Polignac primes as follows:

$n_1 = p_1 + 2k$ = largest Polignac prime

$n_2 = p_1$ = smaller largest Polignac prime

$n_3 = p_3 + 2k$ = second largest Polignac prime

$n_4 = p_4$ = smaller second largest Polignac prime

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$n_p = p_p$ = smallest Polignac prime number

This reverse ordering of the finite set of Polignac prime numbers is key to our proof. We assume that the following Polignac prime reciprocal series have a finite sum, which we call S.

$$\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}+\cdots+\frac{1}{n_p}>\frac{1}{2n_1}+\frac{1}{3n_2}+\frac{1}{4n_3}+\cdots+\frac{1}{mn_p}=S$$

Where, m is the denominator factor for the smallest Polignac prime number that exists in our finite set.

We now proceed to derive a contradiction in the following manner. First we rewrite each term occurring in S thus:

$$\frac{1}{3n_2}=\frac{2}{6n_2}=\frac{1}{6n_2}+\frac{1}{6n_2},\ \frac{1}{4n_3}=\frac{3}{12n_3}=\frac{1}{12n_3}+\frac{1}{12n_3}+\frac{1}{12n_3},\ \ldots,$$

Next we write the resulting fractions in an array as shown below:

$$\begin{array}{ccccccc}
\frac{1}{2n_1} & \frac{1}{6n_2} & \frac{1}{12n_3} & \frac{1}{20n_4} & \frac{1}{30n_5} & \frac{1}{42n_6} & \frac{1}{56n_7}\cdots \\
 & \frac{1}{6n_2} & \frac{1}{12n_3} & \frac{1}{20n_4} & \frac{1}{30n_5} & \frac{1}{42n_6} & \frac{1}{56n_7}\cdots \\
 & & \frac{1}{12n_3} & \frac{1}{20n_4} & \frac{1}{30n_5} & \frac{1}{42n_6} & \frac{1}{56n_7}\cdots \\
 & & & \frac{1}{20n_4} & \frac{1}{30n_5} & \frac{1}{42n_6} & \frac{1}{56n_7}\cdots \\
 & & & & \frac{1}{30n_5} & \frac{1}{42n_6} & \frac{1}{56n_7}\cdots \\
 & & & & & \frac{1}{42n_6} & \frac{1}{56n_7}\cdots \\
 & & & & & & \frac{1}{56n_7}\cdots
\end{array}$$

Note that the column sums are just the fractions of the Polignac primes; thus S is the sum of all the fractions occurring in the array. As Bernoulli did, we now sums the rows using the telescoping technique. Next we assign symbols to the row sums as shown below,

$$A = \frac{1}{2n_1} + \frac{1}{6n_2} + \frac{1}{12n_3} + \frac{1}{20n_4} + \frac{1}{30n_5} + \frac{1}{42n_6} + \frac{1}{56n_7} + \dots,$$

$$B = \frac{1}{6n_2} + \frac{1}{12n_3} + \frac{1}{20n_4} + \frac{1}{30n_5} + \frac{1}{42n_6} + \frac{1}{56n_7} + \dots,$$

$$C = \frac{1}{12n_3} + \frac{1}{20n_4} + \frac{1}{30n_5} + \frac{1}{42n_6} + \frac{1}{56n_7} + \dots,$$

$$D = \frac{1}{20n_4} + \frac{1}{30n_5} + \frac{1}{42n_6} + \frac{1}{56n_7} + \dots,$$

We now rearrange as follows:

$$A = (\frac{1}{n_1} - \frac{1}{2n_1}) + (\frac{1}{2n_2} - \frac{1}{3n_2}) + (\frac{1}{3n_3} - \frac{1}{4n_3}) + (\frac{1}{4n_4} - \frac{1}{5n_4}) + \dots$$

Since, $n_1 > n_2 > n_3 > n_4$

$$A = \frac{1}{n_1} + (\frac{1}{2n_2} - \frac{1}{2n_1}) + (\frac{1}{3n_3} - \frac{1}{3n_2}) + (\frac{1}{4n_4} - \frac{1}{4n_3}) + (\frac{1}{5n_5} - \frac{1}{5n_4}) + \dots$$

Since, $\left(\frac{1}{2n_2} - \frac{1}{2n_1}\right) > 0$, $\left(\frac{1}{3n_3} - \frac{1}{3n_2}\right) > 0$, $(\frac{1}{4n_4} - \frac{1}{4n_3}) > 0$, $(\frac{1}{5n_5} - \frac{1}{5n_4}) > 0$

Then, $A > \frac{1}{n_1}$

$$B = \left(\frac{1}{2n_2} - \frac{1}{3n_2}\right) + \left(\frac{1}{3n_3} - \frac{1}{4n_3}\right) + \left(\frac{1}{4n_4} - \frac{1}{5n_4}\right) + \left(\frac{1}{5n_5} - \frac{1}{6n_5}\right)\ldots$$

Since, $n_1 > n_2 > n_3 > n_4$, the same rearranging that we did with *A* can be done with *B*.

Then, $B > \frac{1}{2n_2}$

$$C = \left(\frac{1}{3n_3} - \frac{1}{4n_3}\right) + \left(\frac{1}{4n_4} - \frac{1}{5n_4}\right) + \left(\frac{1}{5n_5} - \frac{1}{6n_5}\right) + \left(\frac{1}{6n_5} - \frac{1}{7n_5}\right) \ldots$$

Since, $n_1 > n_2 > n_3 > n_4$, the same rearranging that we did with *A* can be done with *C*.

Then, $C > \frac{1}{3n_3}$

$$D = \left(\frac{1}{4n_4} - \frac{1}{5n_4}\right) + \left(\frac{1}{5n_5} - \frac{1}{6n_5}\right) + \left(\frac{1}{6n_5} - \frac{1}{7n_5}\right) + \left(\frac{1}{7n_6} - \frac{1}{8n_6}\right) \ldots$$

Since, $n_1 > n_2 > n_3 > n_4$, the same rearranging that we did with *A* can be done with *D*.

Then, $D > \frac{1}{4n_4}$

and so on. Thus the sum *S,* which we had written in the form $A + B + C + D + \ldots$, turns out to be greater than

$$S > \frac{1}{n_1} + \frac{1}{2n_2} + \frac{1}{3n_3} + \frac{1}{4n_4} + \cdots$$

At the start we had defined *S* to be the following finite series,

$$S = \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{kn_\text{p}}$$

And we defined that, $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} + \cdots > S = \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots$

However, we just proved that $S > \frac{1}{n_1} + \frac{1}{2n_2} + \frac{1}{3n_3} + \frac{1}{4n_4} + \cdots > S = \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{kn_\text{p}}$

However, this is a contradiction, since in the finite realm S can't be equal to and greater than $\frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{kn_\text{p}}$ at the same time. Therefore, S must be infinite.

Now we can rewrite the S, the Polignac prime series as,

$$S > \frac{1}{n_1} + \frac{1}{2n_2} + \frac{1}{3n_3} + \frac{1}{4n_4} + \cdots > \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{kn_\text{p}} = S$$

This implies that $S > S$

However, no finite number can satisfy such an equation. Therefore, we have a contradiction and must conclude that $S = \infty$**.** Remember our definition of S from the above series:

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \cdots + \frac{1}{n_\text{p}} > S = \infty$$

Therefore, $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \cdots + \frac{1}{n_\text{p}} > \infty$

Therefore, we have proven that the reciprocal Polignac prime series diverges to infinity. Obviously, this cannot possibly happen if there are only finitely many Polignac prime reciprocals, therefore the Polignac prime reciprocals are infinite in number. Since the Polignac prime reciprocals are infinite in number, the Polignac prime numbers must be infinite as well.

This proof shows the shows the counter-intuitive nature of infinity, and why it has taken so long to prove the Polignac primes are infinite.

Our proof of infinite Polignac primes also proves several other Conjectures since Polignac primes are generalized form of other conjectures. For k = 1, the Polignac primes become the **Twin Prime Conjecture** and proves there are an infinite number of twin primes. For k = 2, the proof of the Polignac Conjecture proves there are infinitely many **Cousin Primes** (p, $p + 4$). For k = 3, proof of the Polignac Conjecture proves there are infinitely many **Sexy Primes** (p, $p + 6$).

The author expresses many thanks to the work of Johann Bernoulli in 1689, without his work this proof would not have been possible. It was solely through the study of Johann Bernoulli's work that the author was inspired to see this divergent proof. The author would also like to express many thanks to Shailesh Shirali's work in which he documented Johann Bernoulli's work in the most fascinating and interesting way.

References:

1. On the Infinitude of the Prime Numbers, Shailesh A Shirali, Kishi Valley School (Krishnamurti Foundation of India), Kishi Valley, Anclhra Pradesh, India

2. Guy, Richard K., Unsolved problems in number theory, (3rd edition). Springer-Verlag (2004).

3. From Euclid to Present: A Collection of Proofs regarding the Infinitude of Primes, Lindsey Harrison, December 14, 2013.

4. (Dun) W. Dunham,"Journey Through Genius: The Great Theorems of Mathematics," Penguin Books, John Wiley and Sons, 1990.