# Elementary Proof of the Goldbach Conjecture

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**13 February 2017**


## Abstract

Christian Goldbach (March 18, 1690 – November 20, 1764) was a German mathematician. He is remembered today for Goldbach's conjecture.

Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states: Every even integer greater than 2 can be expressed as the sum of two primes.

On 7 June 1742, the German mathematician Christian Goldbach wrote a letter to Leonhard Euler (letter XLIII) in which he proposed the following conjecture:
Every even integer which can be written as the sum of two primes (the strong conjecture)
He then proposed a second conjecture in the margin of his letter:

Every odd integer greater than 7 can be written as the sum of three primes (the weak conjecture).

A Goldbach number is a positive even integer that can be expressed as the sum of two odd primes. Since four is the only even number greater than two that requires the even prime 2 in order to be written as the sum of two primes, another form of the statement of Goldbach's conjecture is that all even integers greater than 4 are Goldbach numbers.

The "strong" conjecture has been shown to hold up through $4 \times 10^{18}$, but remains unproven for almost 300 years despite considerable effort by many mathematicians throughout history.

In number theory, Goldbach's weak conjecture, also known as the odd Goldbach conjecture, the ternary Goldbach problem, or the 3-primes problem, states that
Every odd number greater than 5 can be expressed as the sum of three primes. (A prime may be used more than once in the same sum). In 2013, Harald Helfgott proved Goldbach's weak conjecture.

The author would like to give many thanks to Helfgott's proof of the weak conjecture, because this proof of the strong conjecture is completely dependent on Helfgott's proof. Without Helfgott's proof, this elementary proof would not be possible.

## Proof

Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states: Every even integer greater than 2 can be expressed as the sum of two primes.

The Goldbach Conjecture states that for every even integer N, and N > 2, then N = $P_1$ + $P_2$, where $P_1$, and $P_2$, are prime numbers. The first two Goldbach numbers are as follows:

When N = 4, then 4 = 2 + 2, and since 2 is prime then the Goldbach Conjecture is satisfied. When N = 6, then 6 = 3 + 3, and since 3 is prime then the Goldbach Conjecture is satisfied again.

All prime numbers greater than 2 are odd numbers, however since the sum of two odd numbers is always even, then it is possible that Goldbach's conjecture is true. Further more, the set of prime numbers is a subset of the odd integers, therefore, it is plausible for Goldbach's conjecture to be true.

From the above, it follows that all even numbers greater than 4, then the even number is the sum of two odd numbers. Now we must prove that there always exists at least one set of two prime numbers that satisfy the two odd numbers, although there could be additional non-prime odd numbers whose sum equals the even number.

The strong Goldbach conjecture implies the conjecture that **all odd numbers greater than 5 are the sum of three odd primes**, which is known today variously as the **"weak" Goldbach conjecture**, the "odd" Goldbach conjecture, or the "ternary" Goldbach conjecture. While the weak Goldbach conjecture has finally been proved, by Helfgott, in 2013, [1][2] however the strong conjecture has remained unsolved. In this paper we shall use Helfgott's prove of the "odd" Goldbach conjecture to prove the strong conjecture of even numbers.

Every even number is an odd number -1, therefore since the weak Goldbach conjecture has been proven, as discussed above, then every even number is the sum of 3 primes – 1. Mathematically, for every integer k > 1:

$$\textbf{Every even number} = 2k = p_1 + p_2 + p_3 - 1$$

$$2k - p_3 + 1 = p_1 + p_2$$

Now all we need to do is prove that $\mathbf{2k - p_3 + 1}$ represents every even number > 2. We already know that **2k** represents every even number > 2, so we need to prove that:

$\mathbf{2k - p_3 + 1}$ also represents every even number > 1

First it is a straight forward observation that $\mathbf{2k - p_3 + 1}$ will always be even since $\mathbf{2k - p_3}$ is odd since an even number – an odd number is always odd, but since 1 is added to $\mathbf{2k - p_3}$, then $\mathbf{2k - p_3 + 1}$ is always even. Rearranging, shows

$$\mathbf{2k - p_3 + 1 = 2k - (p_3 - 1)}$$

Therefore, $\mathbf{2k - (p_3 - 1)}$ is always an even number that is $\mathbf{(p_3 - 1)}$ less than **2k.** Since **2k** represents every even number > 2, then $\mathbf{2k - (p_3 - 1)}$ represents every even number $\mathbf{(p_3 - 1)}$ less than **2k.** Therefore, we only need to prove that **2k**, for every integer k > 1, is the sum of two primes.

Since there are an infinite number of even numbers, therefore, **2k**, for K > 1, is infinite since it includes all of the even integers > 2. Since **2k** is infinite we will never reach the final even number **2k.** Additionally, since **2k** is infinite, $\mathbf{2k - (p_3 - 1)}$ is also infinite. As **2k** approaches infinity, specifically:

$$\mathbf{2k} \Rightarrow \infty$$

There is always an even number **2k** greater than the previous even number **2k.** Additionally each of the infinite even numbers **2k** can be represented as $\mathbf{2k - (p_3 - 1)}$**.** Therefore every even number for infinity can be represented as $\mathbf{2k - (p_3 - 1)}$**.** Additionally, we have shown above that:

$$\mathbf{2k - (p_3 - 1) = p_1 + p_2}$$

Therefore, we have proved that every even number is the sum of two prime numbers. Thus the strong Goldbach Conjecture is proven by this elementary proof.

**References:**


1) Helfgott, H.A. (2013). "Major arcs for Goldbach's theorem". arXiv:1305.2897 Freely accessible [math.NT].
2) Helfgott, H.A. (2012). "Minor arcs for Goldbach's problem". arXiv:1205.5252 Freely accessible [math.NT].