Primality Criterion for Safe Primes

Predrag Terzić

Podgorica , Montenegro
e-mail: pedja.terzic@hotmail.com

February 12 , 2017

Abstract: Polynomial time primality test for safe primes is introduced .

Keywords: Primality test , Polynomial time , Prime numbers .
AMS Classification: 11A51 .

1 Introduction

In 1750 Euler stated following theorem

Theorem 1.1. *Let* $p \equiv 3 \pmod 4$ *be prime ,*
then $2p+1$ *is prime iff* $2p+1 \mid 2^p - 1$.

In 1775 Lagrange gave a proof of the theorem . In this note we provide a proof to the theorem that is similar to the Euler-Lagrange theorem .

2 The Main Result

Theorem 2.1. *Let* $p \equiv 5 \pmod 6$ *be prime ,*
then $2p+1$ *is prime iff* $2p+1 \mid 3^p - 1$.

Proof. Suppose $q = 2p+1$ is prime. $q \equiv 11 \pmod{12}$ so 3 is quadratic residue module q and it follows that there is an integer n such that $n^2 \equiv 3 \pmod q$. This shows $3^p = 3^{(q-1)/2} \equiv n^{q-1} \equiv 1 \pmod q$ showing $2p+1$ divides $3^p - 1$.

Conversely, let $2p+1$ be factor of $3^p - 1$. Suppose that $2p+1$ is composite and let q be its least prime factor. Then $3^p \equiv 1 \pmod q$ and so we have $p = k \cdot \mathrm{ord_q}(3)$ for some integer k . Since p is prime there are two possibilities $\mathrm{ord_q}(3) = 1$ or $\mathrm{ord_q}(3) = p$. The first possibility cannot be true because q is an odd prime number so $\mathrm{ord_q}(3) = p$. On the other hand $\mathrm{ord_q}(3) \mid q-1$, hence p divides $q-1$. This shows $q > p$ and it follows $2p+1 > q^2 > p^2$ which is contradiction since $p > 3$, hence $2p+1$ is prime .

Q.E.D.