Geometric (Clifford) Algebra Calculation of the Trajectory of a Gas Molecule Desorbed from the Earth's Surface

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Abstract

As a step toward understanding why the Earth's atmosphere "rotates" with the Earth, we use using Geometric (Clifford) Algebra to investigate the trajectory of a single molecule that desorbs vertically upward from the Equator, then falls back to Earth without colliding with any other molecules. Sample calculations are presented for a molecule whose vertical velocity is equal to the surface velocity of the Earth at the Equator (463 m/s) and for one with a vertical velocity three times as high. The latter velocity is sufficient for the molecule to reach the Kármán Line (100,000 m). We find that both molecules fall to Earth behind the point from which they desorbed: by 0.25 degrees of latitude for the higher vertical velocity, but by only 0.001 degrees for the lower.

1 Introduction

A question sometimes asked in on-line science forums is,

> The atmosphere must 'spin' with the rotating Earth, because if the atmosphere didn't, then people living at the Equator would experience winds with a velocity of approximately 1600 km/hr. But how can the atmosphere 'spin' ?"

An important part of the answer to that question is that what we refer to as "the atmosphere" is a vast number of individual molecules. Those molecules interact weakly with each other, forming what is essentially an ideal gas. However, each is attracted gravitationally to the Earth.

Why, then, don't those molecules fall to the Earth, then bind to each other and to molecules that make up rocks and soil, remaining there forever? On a very cold planet, that is exactly what would happen. But

the surface temperature of the Earth is high enough to give the molecules sufficient energy to desorb back into the atmosphere occasionally.

As an example, think of the molecules in a drop of water. Each has a certain thermal energy, which causes it to vibrate back and forth within the drop. However, not all have the same energy: the molecules' energy is Maxwell-Boltzman distributed. At any time, some of the molecules vibrate vigorously enough to break free of the drop, thereby becoming part of what we call the "atmosphere".

What happens after a molecule desorbs into the atmosphere? Most likely, it will collide with other air molecules within 60 nm of the spot from which it desorbed [1] . Of course, each of those other molecules is also attracted gravitationally to the Earth, and would soon have fallen to the surface were it not for the collisions. Each may do so anyway. Thus the details of molecules' movements within the atmosphere become complicated.

But do we need to know those details in order to know how the atmosphere "spins"? As a step toward answering that question, this document identifies and studies the trajectory of a single molecule that desorbs from the surface of the Earth at the Equator. To identify that trajectory, we will use Geometric (Clifford) Algebra, following the method presented by Hestenes in *New Foundations for Classical Mechanics* ([2]).

2 Our Model

A molecule of mass m desorbs from the Earth's surface at the Equator, with a velocity whose tangential component is equal (in direction as well as magnitude) to the that of the surface, and whose radial component is some multiple γ of the tangential component (Fig. 1). Note that the molecule —to an observer standing alongside the point from which it desorbed —would appear to fly "straight up".

3 Background

The trajectory of an object launched from the surface of the Earth is closely approximated by a parabola, but is actually an ellipse ([3], [4]). In this section, we'll discuss how to use Geometric Algebra to identify that trajectory.

3.1 Orbital Mechanics

The key concept in orbital mechanics is that during any given orbit, two quantities are constant (i.e., they do not vary with time):

- The angular momentum ($\mathbf{L}$). As formulated by [2] in Equation (1.2) on p. 196,

$$\mathbf{L} = m\mathbf{r} \wedge \mathbf{v}, \tag{1}$$

  where m is the satellite's mass; $\mathbf{r}$ is its position with respect to a fixed point referred to as the *center of force*, and $\mathbf{v}$ is its velocity with respect to the center of force. When the mass of the satellite is negligible compared to that of the primary (as is the case in a

A satellite's *primary* is the body around which the satellite revolves.

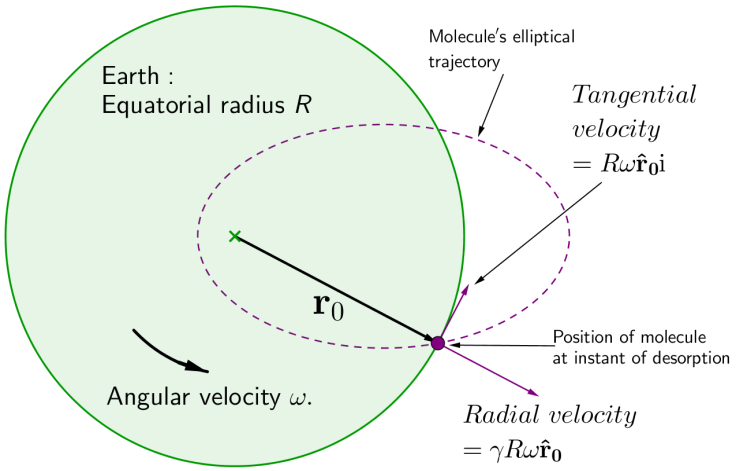

Figure 1: Our model: A molecule of mass m desorbs from the Equator with a tangential velocity equal to that of the Earth's surface at the point of desorption, and with a radial component equal to some scalar multiple γ of the tangential velocity. (See text for additional details.)

molecule orbiting the Earth), the center of force can be taken as the primary's barycenter. Note that $\mathbf{L}$ is a bivector.

- The eccentricity vector ($\boldsymbol{\epsilon}$). The eccentricity vector is dimensionless, and is related to other quantities by [2]'s Equation (3.3), page 205:

$$\mathbf{L}\mathbf{v} = k\left(\hat{\mathbf{r}} + \boldsymbol{\epsilon}\right). \tag{2}$$

  where $k = GmM$, with G being the gravitational constant ($6.67408 \times 10^{-11} m^3 kg^{-1} s^{-2} = 6.67408 \times 10^{-11} N m^2 kg^{-2}$); m being the satellite's mass; and M being the primary's mass.

We should take a moment to ensure that we understand the significance of the foregoing. Firstly, if m does not vary with time, then neither does the product $m\mathbf{r} \wedge \mathbf{v}$ (which is $\mathbf{L}$) . Therefore, if we know the value of $m\mathbf{r} \wedge \mathbf{v}$ for any point in an object's trajectory, then we know it for all times in that trajectory. Similarly, if we know the value of $\frac{\mathbf{L}\mathbf{V}}{k} - \hat{\mathbf{r}}$ $(= \boldsymbol{\epsilon})$ at any point in the orbit, then we know it for all points in the orbit. Therefore, in the situation that we are modeling we can calculate $\mathbf{L}$ and $\boldsymbol{\epsilon}$ from the molecule's initial position and velocity.

From Equations (1) and (2), we can deduce that the trajectory of the orbiting body is a conic curve, one of whose foci is the primary's barycenter. The eccentricity (ϵ) of that curve is a scalar constant, and is related to other quantities involved in the orbit by the following non-numbered equation presented on p. 206 of [2]:

$$k^2\left(\epsilon^2 - 1\right) = L^2\left(v^2 - \frac{2k}{mr}\right). \tag{3}$$

Here, again, we have a result whose significance we should be sure to understand before proceeding: we will be able to calculate ϵ knowing $\|\mathbf{v}\|$ and $\|\mathbf{r}\|$ at any instant in the orbit, including at the instant of our molecule's desorption.

In the situation that we are modeling, we will be interested only in cases where the desorbed molecule falls back to Earth rather than escaping from the atmosphere entirely. For such cases, the value of ϵ is less than 1, and the trajectory is an ellipse. Therefore, our next subject will be the analytical geometry of that ellipse.

3.2 Analytical Geometry of the Elliptical Orbit

Much can be said about this beautiful subject, but the facts of immediate interest to us are (1) the lengths of the ellipse's major and minor axes; and (2) the location of the ellipse's center with respect to the center of the Earth. We'll start from Fig. 2 and the following non-numbered equation that is found at the bottom of page 206 of [2]:

$$k\left[\boldsymbol{\epsilon}\cdot\mathbf{r}+r\right]=\frac{L^2}{m}. \tag{4}$$

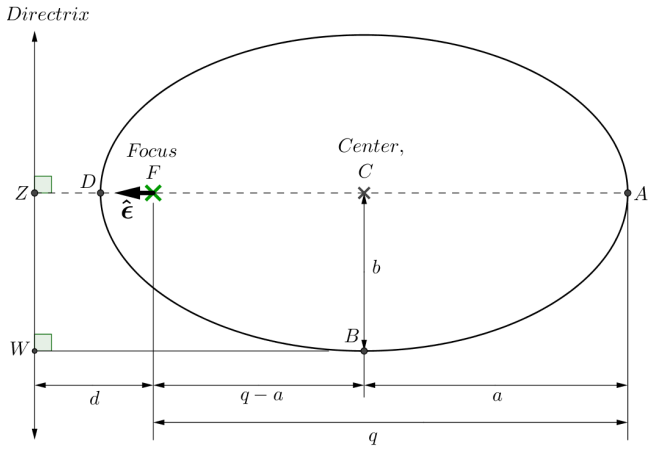

Figure 2: Our model: An elliptical orbit with the Earth's center at focus F, showing dimensions discussed and derived in the text.

To find the length of the semi-major axis (a), we consider points A and D in the orbit. When the satellite is at A, $\mathbf{r}=\bar{}q\hat{\boldsymbol{\epsilon}}$. Using that value in Eq. (4),

$$\begin{aligned}
k\left[\boldsymbol{\epsilon}\cdot(\bar{}q\hat{\boldsymbol{\epsilon}})+q\right]&=\frac{L^2}{m};\\
k\left[(\epsilon\hat{\boldsymbol{\epsilon}})\cdot(\bar{}q\hat{\boldsymbol{\epsilon}})+q\right]&=\frac{L^2}{m};\\
k\left[\bar{}\epsilon\hat{\boldsymbol{\epsilon}}+q\right]&=\frac{L^2}{m};\\
\therefore\quad q&=\frac{L^2}{km\left(1-\epsilon\right)}.
\end{aligned} \tag{5}$$

When the satellite is at D, $\mathbf{r}=(2a-q)\,\hat{\boldsymbol{\epsilon}}$. Using that value in Eq. (4), and the expression for q that we identified in Eq. (5), we arrive at

$$a=\frac{L^2}{km\left(1-\epsilon\right)}. \tag{6}$$

We could have obtained Eq. (6) from Eq. (5) more directly by using the fact that the distance between the center of an ellipse and either of of its foci is ϵa. Thus, $q = 1 + \epsilon a$, etc.

Similarly, we could now use known properties of ellipses to find b directly from a. Instead, we'll take this opportunity to review the fundamental properties of a conic curve, as expressed via GA in pp. 90-90 of [2]. We'll begin by identifying the distance d, in Fig. 2.

According to the definition of a conic curve, $AF = \epsilon AZ = \epsilon\,(q + d)$. As we've already shown, $AF = q = \dfrac{L^2}{km\,(1 - \epsilon)}$. Putting these ideas together,

$$d = \frac{L^2}{\epsilon km}. \tag{7}$$

The definition of a conic curve also leads us to $BF = \epsilon BW = \epsilon BZ$. Therefore,

$$\sqrt{b^2 + CF^2} = \epsilon\,(DF + d) \text{ and}$$
$$\sqrt{b^2 + (q - a)^2} = \epsilon\,(q - a + d)\,.$$

Substituting the expressions we've obtained for q and d, and solving for b, we find that

$$b = \frac{L^2}{km\sqrt{1 - \epsilon^2}}. \tag{8}$$

This result, and (6), are consistent with the fact that in an ellipse, $\epsilon = \sqrt{1 - (b/a)^2}$.

We've now identified the ellipse's dimensions and $\hat{\boldsymbol{\epsilon}}$, the direction of its major axis. Next, we need to find the satellite's position as a function of time. That will be our next subject.

3.3 Kepler's Solution for Elliptical Orbits

We'll follow Hestenes's presentation ([2], pp. 216-219) with reference to Fig. 3. Said figure is a reproduction, given an orientation more suitable for our task, of [2]'s Fig. 4.1, found on p. 216 of that work.

As noted by [2] in connection with his Eq. 4.3, found on p. 216, Kepler parameterized the elliptical circle in terms of an angle ϕ that is defined with the help of an auxiliary circle that circumscribes the ellipse:

$$\mathbf{r} = \mathbf{a}\,(\cos\phi - \epsilon) + \mathbf{b}\sin\phi, \tag{9}$$

where $\mathbf{a} = a\hat{\boldsymbol{\epsilon}}$ and $\mathbf{b} = b\hat{\boldsymbol{\epsilon}}\mathbf{i}$, with a and b as given by Eqs. (6) and (8) respectively. To determine how $\mathbf{r}$ varies with time, we need, first, to calculate T, the orbit's period. According to [2]'s Eq. (2.9), on p. 200,

$$T = \frac{2\pi mab}{L}. \tag{10}$$

Using that result, we can relate the angle ϕ to time via what is known as *Kepler's equation* for planetary motion:

$$\left[\frac{2\pi}{T}\right] t = \phi - \epsilon\sin\phi. \tag{11}$$

In that equation, the zero of time is associated with the pericenter.

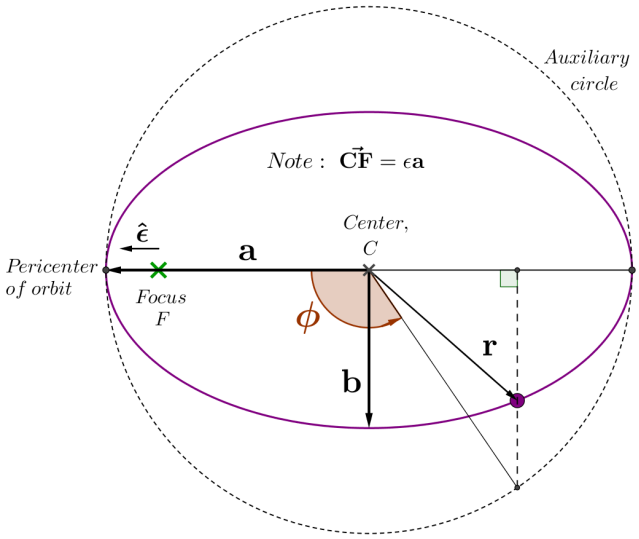

Figure 3: Kepler's parameterization of an elliptical orbit in terms of the angle ϕ.

Having arrived at Eq. (11), Hestenes presents a convenient way of approximating its solution with good accuracy when ϵ is small; *i.e.*, when the orbit is reasonably circular. However, that solution will not work for the highly elliptical orbits like that of a desorbed molecule ([4]).

Dynamic-geometry programs like GeoGebra are also useful for checking the correctness of the calculated ellipse. For example, by checking whether the line that's tangent to the ellipse at the desorption point is parallel to the initial-velocity vector.

How, then, might we solve Eq. (11)? If we were making a dynamic-geometry construction of the orbit with a program like GeoGebra, we could find the intersection point of the functions $y = \left[\frac{2\pi}{T}\right] t$ (which is constant for given t) and $y = x - \epsilon \sin x$. Another solution method—available to us because the value of ϕ will be close to π throughout the trajectory—is to approximate $\sin\phi$ with a Taylor expansion about $\phi = \pi$ (Fig. 4) . In that way, we would obtain a cubic equation to solve for ϕ :

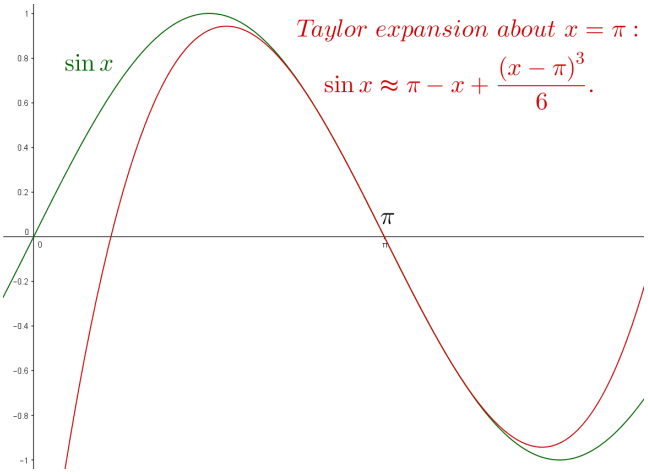

Figure 4: Comparison of $\sin x$ to its Taylor expansion about $x = \pi$.

$$\left[\frac{2\pi}{T}\right] t = \phi - \epsilon\left[\pi - \phi + \frac{(\phi-\pi)^3}{6}\right].$$

4 Solution: Trajectory of the Desorbed Molecule

Our solution will use Section 3's equations in roughly the same order as that in which they were presented therein.

4.1 Identify Dimensions and Orientation of the Elliptical Orbit

4.1.1 Calculate the Angular Momentum (L)

In our model, the values of $\mathbf{r}$ and $\mathbf{v}$ at the instant of desorption are $R\hat{\mathbf{r}}_0$ and $\gamma R\omega\hat{\mathbf{r}}_0 + R\omega\hat{\mathbf{r}}_0\mathbf{i}$, respectively. Therefore, from Eq. (1),

$$\begin{aligned}\mathbf{L} &= m\left(R\hat{\mathbf{r}}_0\right) \wedge \left(\gamma R\omega\hat{\mathbf{r}}_0 + R\omega\hat{\mathbf{r}}_0\mathbf{i}\right)\\ &= mR^2\omega\hat{\mathbf{r}}_0 \wedge \left(\hat{\mathbf{r}}_0\mathbf{i}\right)\\ &= mR^2\omega\mathbf{i}. \end{aligned} \tag{12}$$

Notice that only the tangential component of the molecule's initial velocity affects $\mathbf{L}$.

4.1.2 The Eccentricity Vector ($\boldsymbol{\epsilon}$)

From Eq. (2),

$$\mathbf{Lv} = k\left(\hat{\mathbf{r}} + \boldsymbol{\epsilon}\right).$$

Therefore, using the values of $\mathbf{v}$ and $\mathbf{r}$ at the instant of desorption,

$$\begin{aligned}\boldsymbol{\epsilon} &= \frac{\mathbf{Lv_0}}{k} - \hat{\mathbf{r}}_0\\ &= \frac{mR^2\omega\mathbf{i}}{GmM}\left(\gamma R\omega\hat{\mathbf{r}}_0 + R\omega\hat{\mathbf{r}}_0\mathbf{i}\right) - \hat{\mathbf{r}}_0\\ &= \frac{\omega}{GM/R^2}\left(\gamma R\omega\mathbf{i}\hat{\mathbf{r}}_0 + R\omega\mathbf{i}\hat{\mathbf{r}}_0\mathbf{i}\right) - \hat{\mathbf{r}}_0\\ &= \frac{R\omega^2}{g}\left(\hat{\mathbf{r}}_0 - \gamma\hat{\mathbf{r}}_0\mathbf{i}\right) - \hat{\mathbf{r}}_0\\ &= \left[\frac{R\omega^2}{g} - 1\right]\hat{\mathbf{r}}_0 - \left[\frac{R\omega^2\gamma}{g}\right]\hat{\mathbf{r}}_0\mathbf{i},\end{aligned}$$

where $g\ (= GM/R^2)$ is the gravitational acceleration at the Earth's surface.

4.1.3 The Eccentricity (ϵ)

According to Eq. (3),

$$k^2\left(\epsilon^2 - 1\right) = L^2\left(v^2 - \frac{2k}{mr}\right).$$

Solving for ϵ, and using the values of $\mathbf{r}$ and $\mathbf{v}$ at the instant of desorption,

$$\begin{aligned}
\epsilon &= \sqrt{\frac{L^2}{k^2}\left[v^2 - \frac{2k}{mr}\right] + 1} \\
&= \sqrt{\left(\frac{\|L\|}{GmM}\right)^2 \left[(\gamma R\omega\hat{\mathbf{r}}_0 + R\omega\hat{\mathbf{r}}_0\mathbf{i})^2 - \frac{2GmM}{mR}\right] + 1} \\
&= \sqrt{\left(\frac{mR^2\omega}{GmM}\right)^2 \left[(\gamma R\omega\hat{\mathbf{r}}_0 + R\omega\hat{\mathbf{r}}_0\mathbf{i})^2 - \frac{2GmM}{mR}\right] + 1} \\
&= \sqrt{\left(\frac{\omega}{g}\right)^2 \left[(\gamma R\omega\hat{\mathbf{r}}_0 + R\omega\hat{\mathbf{r}}_0\mathbf{i})^2 - 2gR\right] + 1} \\
&= \sqrt{\left(\frac{\omega}{g}\right)^2 \left[(R\omega)^2 (1+\gamma^2) - 2gR\right] + 1} \\
&= \sqrt{\left(\frac{R\omega^2}{g}\right)^2 \left[1 + \gamma^2 - \frac{2g}{R\omega^2}\right] + 1}. \qquad (13)
\end{aligned}$$

4.1.4 The Vectors a and b

Eq. (6) showed us that the length of the semi-major axis is

$$a = \frac{L^2}{km\,(1-\epsilon^2)}.$$

We'll write $\dfrac{L^2}{km}$ as

$$\begin{aligned}
\frac{L^2}{km} &= \left(\frac{\|\mathbf{L}\|}{k}\right)\left(\frac{\|\mathbf{L}\|}{m}\right) \\
&= \left(\frac{mR^2\omega}{GmM}\right)\left(\frac{mR^2\omega}{m}\right) \\
&= \frac{R^2\omega^2}{g},
\end{aligned}$$

from which

$$\|\mathbf{a}\| = \frac{R^2\omega^2}{g\,(1-\epsilon^2)}.$$

If we now substitute the expression for ϵ from Eq. (3), we find that

$$\|\mathbf{a}\| = \frac{g}{\omega^2\left[\dfrac{2g}{R\omega^2} - (1+\gamma^2)\right]} \qquad (14)$$

Similarly, from Eq. (8),

$$b = \frac{L^2}{km\sqrt{1-\epsilon^2}}.$$

Therefore,

$$\begin{aligned}
\|\mathbf{b}\| &= \frac{R^2\omega^2}{g\sqrt{1-\epsilon^2}} \\
&= \frac{R}{\sqrt{\left[\dfrac{2g}{R\omega^2} - (1+\gamma^2)\right]}}. \qquad (15)
\end{aligned}$$

Now, we use $\mathbf{a} = \|\mathbf{a}\|\hat{\boldsymbol{\epsilon}}$ and $\mathbf{b} = \|\mathbf{b}\|\hat{\boldsymbol{\epsilon}}\mathbf{i}$.

4.2 The Satellite's Position as a Function of Time

4.2.1 The orbital Period (T)

We begin this subject by determining T, which according to Eq. (10) is

$$T = \frac{2\pi mab}{L}.$$

Using our expressions for $\|\mathbf{a}\|$, $\|\mathbf{b}\|$, and $\mathbf{L}$ (Eqs. (4.1.4), (16), and (12)),

$$T = \frac{2\pi m\left[\dfrac{g}{\omega^2\left[\dfrac{2g}{R\omega^2} - (1+\gamma^2)\right]}\right]\left[\dfrac{R}{\sqrt{\left[\dfrac{2g}{R\omega^2} - (1+\gamma^2)\right]}}\right]}{mR^2\omega}$$

$$= \frac{2\pi g}{R\omega^3\left[\dfrac{2g}{R\omega^2} - (1+\gamma^2)\right]^{3/2}}. \tag{16}$$

We will also find the following quantity useful:

$$\frac{2\pi}{T} = \frac{R\omega^3}{g}\left[\frac{2g}{R\omega^2} - \left(1+\gamma^2\right)\right]^{3/2}. \tag{17}$$

4.2.2 Value of ϕ at the Instant of Desorption

As mentioned in connection with Kepler's equation for planetary motion (Eq. (11)), the zero in time for an orbit is associated with the pericenter. Our molecule does not desorb from that point, so we need to determine the value of the time (according to the Kepler equation) associated with the point (we'll denote it by $\mathbf{r}_d$) from which the molecule does desorb.

One way to think of the time variable in this problem is to pretend that the molecule started from the pericenter at time $t = 0$, and desorbed as soon as it reached the Earth's surface.

To do so, we must first determine the value of ϕ for that point. We'll call that value ϕ_d (Fig. 5), and identify it by finding the values of $\cos\phi_d$ and $\sin\phi_d$. From Eq. (9),

$$\mathbf{r}_d = \mathbf{a}\left(\cos\phi_d - \epsilon\right) + \mathbf{b}\sin\phi_d.$$

Because $\mathbf{a}$ and $\mathbf{b}$ are perpendicular,

$$\begin{aligned}\mathbf{r}_d \cdot \mathbf{a} &= \|\mathbf{a}\|^2\left(\cos\phi_d - \epsilon\right), \text{ and}\\ \mathbf{r}_d \cdot \mathbf{b} &= \|\mathbf{b}\|^2\sin\phi_d,\end{aligned}$$

from which

$$\begin{aligned}\cos\phi_d &= \frac{\mathbf{r}\cdot\mathbf{a}}{\|\mathbf{a}\|^2} + \epsilon, \text{ and}\\ \sin\phi_d &= \frac{\mathbf{r}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}.\end{aligned} \tag{18}$$

In the case of a desorbing molecule, we know that ϕ_d must be between 0 and 2π radians, so we would not need to know the value of $\sin\phi_d$: the value and algebraic sign of $\cos\phi_d$ suffice to identify ϕ_d unambiguously.

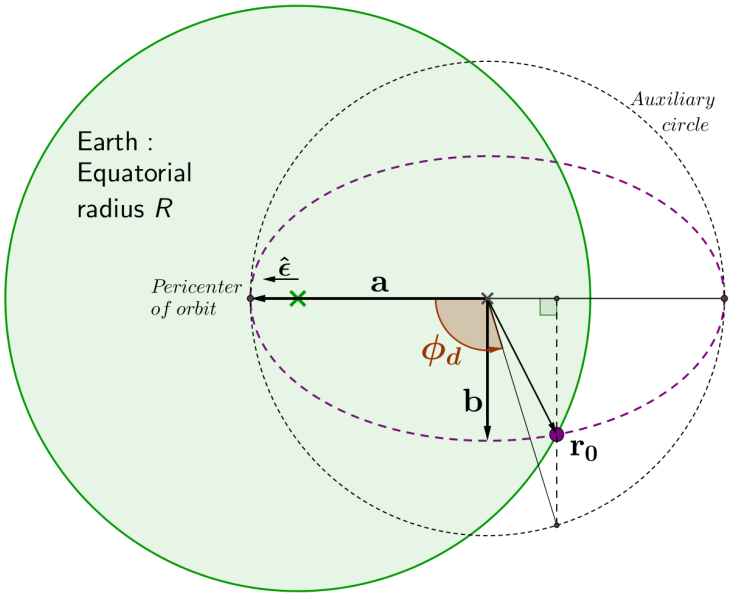

Figure 5: The elliptical orbit for our model, showing the angle (ϕ_d) associated with the instant of the molecule's desorption.

4.2.3 The Position of the Molecule in Terms of Time Elapsed Since Desorption

Although the variable t in the Kepler equation for planetary motion represents "time since perigee", a more-convenient variable, for our purposes, is "time elapsed since desorption". The two measures of time are related by

$$\begin{aligned} \textit{time from pericenter to } \mathbf{r} = &(\textit{time from pericenter to desorption}) \\ &+ (\textit{time elapsed since desorption})\,. \end{aligned}$$

Let's rewrite that equation, using Kepler's variable t, and two new variables, t_d and t_e:

$$\begin{aligned} \overbrace{\textit{time from pericenter to } \mathbf{r}}^{\textit{Kepler's "t"}} = &\overbrace{(\textit{time from pericenter to desorption})}^{t_d} \\ &+ \underbrace{(\textit{time elapsed since desorption})}_{t_e}\,. \end{aligned} \tag{19}$$

To calculate t_d, we solve Kepler's equation for planetary motion, for the angle $\phi = \phi_d$:

$$t_d = \left[\frac{T}{2\pi}\right][\phi_d - \epsilon \sin\phi_d]\,. \tag{20}$$

Thus, Kepler's equation

$$\frac{2\pi t}{T} = \phi - \epsilon \sin\phi$$

becomes

$$\frac{2\pi}{T}(t_d + t_e) = \phi(t_e) - \epsilon \sin\phi(t_e)\,,$$

and

$$\left(\frac{2\pi}{T}\right) t_e + \phi_d - \epsilon \sin \phi_d = \phi\left(t_e\right) - \epsilon \sin \phi\left(t_e\right). \tag{21}$$

5 Sample Calculations

5.1 What do We want to Calculate, and Why?

The purpose of this investigation is to help us understand why the atmosphere can "rotate" with the Earth. With that goal in mind, what calculations might be useful to us? As one example, we might wish to know where the desorbed molecule lands with respect to its starting point. Finding that answer will require us to calculate

1. The angle through which the Earth rotates from the time the molecule desorbs, until it strikes the ground. To calculate that angle, we will need to know how much time elapses between desorption and "landing".
2. The central angle (analogous to β in Fig. 6) that is subtended by the arc between the molecule's points of desorption and landing.

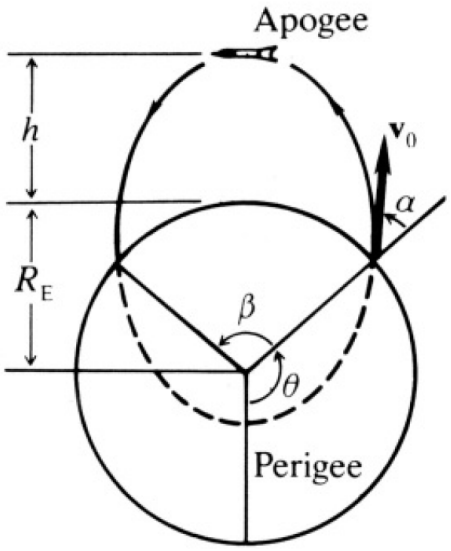

Figure 6: Reproduction of Fig. 3.5 from [2], p. 215.

We also need to identify the relevant range of γ values. Bearing in mind, again, that we wish to understand how the atmosphere can rotate, we probably needn't consider values of γ that would be sufficient for molecules to ascend beyond the Kármán line (100 km, [5]). To estimate the value of γ that would be needed to reach that altitude, we'll use the familiar equation ${v_f}^2 - {v_0}^2 = 2as$. We want to know the value of v_0 for which v_f is zero at 100,000 m. Our value of a is ^-g ($= {}^-9.8 m/s^2$), giving $v_0 = 1400 m/s$.

In comparison, the tangential velocity of the Earth's surface at the Equator is $R\omega$. Taking R as 6370 km, and ω as 7.27×10^{-5} radians/s, the

Calculation of ω: The Earth rotates 2π radians in 24 hours, therefore $\omega = 7.27 \times 10^{-5}$ radians/s.

tangential velocity is 463 m/s. Therefore, the estimated value of γ needed to reach the Kármán line is $1400m/s \div 463m/s = 3$.

Curiously, the tangential velocity at the Equator is very nearly equal to the mean thermal velocity of air (464 m/s) at 20°C ([6]). Therefore, we'll use $\gamma = 1$ as our other value of γ.

5.2 How to Calculate the Molecule's Time of Flight and Angular Displacement

To calculate how much time elapses between desorption and landing (*i.e.*, the molecule's "time of flight"), we consider the complete orbit that the molecule would make if the Earth weren't in the way. The time elapsed between desorption and landing is

$$\boxed{\text{Time between desorption and landing}} = T - \left\{ \boxed{\text{Time from pericenter to desorption point}} + \boxed{\text{Time from landing point to pericenter}} \right\}$$

Because of symmetry, the two terms on the right-hand side of that equation are equal. The "time from pericenter to desorption point" is the quantity that we called t_d in Eq. (20). Therefore,

$$\boxed{\text{Time between desorption and landing}} = T \left[1 - \frac{\phi_d - \epsilon \sin \phi_d}{\pi} \right]. \tag{22}$$

From that result, we can now write

$$\boxed{\text{Angle through which Earth rotates during molecule´s flight}} = \omega T \left[1 - \frac{\phi_d - \epsilon \sin \phi_d}{\pi} \right]. \tag{23}$$

We want to compare that angle of rotation to the angle β (Fig. 6), which we can calculate using one of GA's most-useful capabilities: rotations of vectors. By comparing Figs. 5 and 5, we can see that rotating the vector $\hat{\mathbf{r}}_0$ through the angle $\beta/2$ aligns it with $\bar{\hat{\boldsymbol{\epsilon}}}$. Using GA, we can write that fact as

$$\hat{\mathbf{r}}_0 \exp^{\mathbf{i}\frac{\beta}{2}} = \bar{\hat{\boldsymbol{\epsilon}}}.$$

Multiplying both sides by $\hat{\mathbf{r}}_0$,

See [7] for more information about rotations, and how to use them in solving geometry problems.

$$\begin{aligned} \hat{\mathbf{r}}_0 \hat{\mathbf{r}}_0 \exp^{\mathbf{i}\frac{\beta}{2}} &= \bar{\hat{\mathbf{r}}}_0 \hat{\boldsymbol{\epsilon}}; \\ \exp^{\mathbf{i}\frac{\beta}{2}} &= \bar{\hat{\mathbf{r}}}_0 \hat{\boldsymbol{\epsilon}}; \\ \cos\frac{\beta}{2} + \mathbf{i}\sin\frac{\beta}{2} &= \bar{\frac{1}{\epsilon}} \left\{ \hat{\mathbf{r}}_0 \cdot \boldsymbol{\epsilon} + \mathbf{r}_0 \wedge \boldsymbol{\epsilon} \right\}. \end{aligned}$$

Now, using the expression for $\boldsymbol{\epsilon}$ given in Eq. (13),

$$\begin{aligned}\cos\frac{\beta}{2}+\mathbf{i}\sin\frac{\beta}{2} &= \text{-}\frac{1}{\epsilon}\left\{\hat{\mathbf{r}}_0\cdot\text{-}\left[\left(\frac{R\omega^2}{g}-1\right)\hat{\mathbf{r}}_0-\left(\frac{R\omega^2\gamma}{g}\right)\hat{\mathbf{r}}_0\mathbf{i}\right]\right.\\ &\quad\left.+\mathbf{r}_0\wedge\text{-}\left[\left(\frac{R\omega^2}{g}-1\right)\hat{\mathbf{r}}_0-\left(\frac{R\omega^2\gamma}{g}\right)\hat{\mathbf{r}}_0\mathbf{i}\right]\right\}\\ &= \left(1-\frac{R\omega^2}{g}\right)+\left(\frac{R\omega^2\gamma}{g}\right)\mathbf{i}.\end{aligned}$$

$$\begin{aligned}\hat{\mathbf{r}}_0\wedge(\hat{\mathbf{r}}_0\mathbf{i}) &= \langle\hat{\mathbf{r}}_0\,(\hat{\mathbf{r}}_0\mathbf{i})\rangle_2\\ &= \langle(\hat{\mathbf{r}}_0\hat{\mathbf{r}}_0)\,\mathbf{i}\rangle_2\\ &= \mathbf{i}.\end{aligned}$$

Equating the scalar and bivector parts of the two sides of that equation, $\cos\frac{\beta}{2}=1-\frac{R\omega^2}{g}$, and $\sin\frac{\beta}{2}=\frac{R\omega^2\gamma}{g}$, from which

$$\tan\frac{\beta}{2}=\frac{\gamma}{\frac{g}{R\omega^2}-1}. \tag{24}$$

5.3 The Molecule's Time of Flight and Angular Displacement for $\gamma = 1$ and $\gamma = 3$

In Table 1, α is the angle through which the Earth rotates while the molecule is in flight (*i.e.*, between the molecule's desorption and landing): $\alpha = R\omega(Time\ of\ flight)$.

A study of Table 1 shows that the molecule does reach the height of the Kármán Line (100,000 m) for $\gamma = 3$. We can also deduce that the molecule will fall to Earth behind its starting point , because α is larger than β (even if only slightly, as in the case of $\gamma = 1$). We should note that at temperatures near 20°C, very few molecules in air have velocities as high as $\gamma = 3$ (1400 m/s) [8].

Characteristic	$\gamma = 1$	$\gamma = 3$
Initial tangential velocity (m/s)	463	463
Initial radial velocity (m/s)	463	1400
Eccentricity (ϵ)	0.996568	0.996616
$\|\mathbf{a}\|$ (m)	3,195,986	3,240,670
$\|\mathbf{b}\|$ (m)	264,542	266,386
Maximum altitude (m)	11,000	100,400
ϕ_d	175.24°	165.67°
t_d (s)	852.6	773.8
Time of flight (s)	95.1	290.6
Angle α	0.396°	1.211°
Angle β	0.395°	1.186°
$\alpha - \beta$	0.001°	0.025°

Table 1: Characteristics of trajectories for molecules with $\gamma = 1$ and $\gamma = 3$.

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