# ELEMENTARY EQUALITIES BETWEEN RADICALS

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ABSTRACT:
In this note we briefly examine some elementary radical identities found in Ramanujan's work

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## 1. Introducción

TEOREMA 1. (Ramanujan)

$$(a)\qquad \left(\sqrt[3]{2}-1\right)^{1/3}=\left(4\sqrt[3]{\frac{2}{3}}-5\sqrt[3]{\frac{1}{3}}\right)^{1/8}=\sqrt[3]{\frac{1}{9}}-\sqrt[3]{\frac{2}{9}}+\sqrt[3]{\frac{4}{9}}$$

$$(b)\qquad \left(7\sqrt[3]{20}-19\right)^{1/6}=\sqrt[3]{\frac{5}{3}}-\sqrt[3]{\frac{2}{3}}$$

$$(c)\qquad \left(\frac{3+2\sqrt[4]{5}}{3-2\sqrt[4]{5}}\right)^{1/4}=\frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}$$

$$(d)\qquad \left(\sqrt[5]{\frac{32}{5}}-\sqrt[5]{\frac{27}{5}}\right)^{1/3}=\sqrt[5]{\frac{1}{25}}+\sqrt[5]{\frac{3}{25}}-\sqrt[5]{\frac{9}{25}}$$

$$(e)\qquad \left(\sqrt[5]{\frac{1}{5}}+\sqrt[5]{\frac{4}{5}}\right)^{1/2}=\sqrt[5]{\frac{16}{125}}+\sqrt[5]{\frac{8}{125}}+\sqrt[5]{\frac{2}{125}}-\sqrt[5]{\frac{1}{125}}=\left(1+\sqrt[5]{2}+\sqrt[5]{8}\right)^{1/5}$$

$$(f)\qquad \left(\sqrt[3]{28}-3\right)^{1/2}=\frac{\sqrt[3]{98}-\sqrt[3]{28}-1}{3}$$

$$(g)\qquad \left(\sqrt[3]{5}-\sqrt[3]{4}\right)^{1/2}=\frac{\sqrt[3]{2}+\sqrt[3]{20}-\sqrt[3]{25}}{3}$$

$$(h)\qquad \sqrt[3]{\frac{1}{3}}+\sqrt[3]{\frac{5}{3}}=\sqrt[6]{3}\sqrt{\frac{\sqrt[3]{5}-1}{2-\sqrt[3]{5}}}=\sqrt[3]{\frac{3+\sqrt[3]{5}}{\sqrt[3]{5}-1}}=\sqrt[5]{\frac{3\sqrt[3]{3}+\sqrt[3]{15}}{2-\sqrt[3]{5}}}$$

En esta nota mostramos una colección de fórmulas relacionadas con los radicales de Ramanujan.

## 2. Fórmulas

Teorema 2.

$(a)\qquad \left(\sqrt[3]{2}-1\right)^{-1/3}=\left(4\sqrt[3]{\frac{2}{3}}-5\sqrt[3]{\frac{1}{3}}\right)^{-1/8}=\left(\sqrt[3]{\frac{1}{9}}-\sqrt[3]{\frac{2}{9}}+\sqrt[3]{\frac{4}{9}}\right)^{-1}=\sqrt[3]{\frac{1}{3}}+\sqrt[3]{\frac{2}{3}}$

$(b)\qquad \left(7\sqrt[3]{20}-19\right)^{-1/6}=\left(\sqrt[3]{\frac{5}{3}}-\sqrt[3]{\frac{2}{3}}\right)^{-1}=\sqrt[3]{\frac{4}{9}}+\sqrt[3]{\frac{10}{9}}+\sqrt[3]{\frac{25}{9}}$

$(c)\qquad \left(\frac{3+2\sqrt[4]{5}}{3-2\sqrt[4]{5}}\right)^{-1/4}=\left(\frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}\right)^{-1}=\frac{3-\sqrt[4]{5}+\sqrt{5}-\sqrt[4]{125}}{2}$

$(d)\qquad \left(\sqrt[5]{\frac{32}{5}}-\sqrt[5]{\frac{27}{5}}\right)^{-1/3}=\left(\sqrt[5]{\frac{1}{25}}+\sqrt[5]{\frac{3}{25}}-\sqrt[5]{\frac{9}{25}}\right)^{-1}$

$=2\sqrt[5]{\frac{1}{125}}+\sqrt[5]{\frac{3}{125}}+\sqrt[5]{\frac{9}{125}}+\sqrt[5]{\frac{81}{125}}$

$(e)\qquad \left(\sqrt[5]{\frac{1}{5}}+\sqrt[5]{\frac{4}{5}}\right)^{-1/2}=\left(\sqrt[5]{\frac{16}{125}}+\sqrt[5]{\frac{8}{125}}+\sqrt[5]{\frac{2}{125}}-\sqrt[5]{\frac{1}{125}}\right)^{-1}=\left(1+\sqrt[5]{2}+\sqrt[5]{8}\right)^{-\frac{1}{5}}$

$=\sqrt[5]{\frac{4}{25}}+\sqrt[5]{\frac{2}{25}}-\sqrt[5]{\frac{1}{25}}$

$(f)\qquad \left(\sqrt[3]{28}-3\right)^{-1/2}=\left(\frac{\sqrt[3]{98}-\sqrt[3]{28}-1}{3}\right)^{-1}=\frac{5+2\sqrt[3]{28}+\sqrt[3]{98}}{3}$

$(g)\qquad \left(\sqrt[3]{5}-\sqrt[3]{4}\right)^{-1/2}=\left(\frac{\sqrt[3]{2}+\sqrt[3]{20}-\sqrt[3]{25}}{3}\right)^{-1}=\frac{2\sqrt[3]{4}+\sqrt[3]{5}+\sqrt[3]{50}}{3}$

Teorema 3.

$(a)\qquad \left(\sqrt[3]{2}-1\right)^{-1}=1+\sqrt[3]{2}+\sqrt[3]{4}$

$(b)\qquad \left(4\sqrt[3]{\frac{2}{3}}-5\sqrt[3]{\frac{1}{3}}\right)^{-1}=25\sqrt[3]{\frac{1}{9}}+20\sqrt[3]{\frac{2}{9}}+16\sqrt[3]{\frac{4}{9}}$

$(c)\qquad \left(7\sqrt[3]{20}-19\right)^{-1}=361+133\sqrt[3]{20}+98\sqrt[3]{50}$

$(d)\qquad \left(\frac{3+2\sqrt[4]{5}}{3-2\sqrt[4]{5}}\right)^{-1}=161-108\sqrt[4]{5}+72\sqrt{5}-48\sqrt[4]{125}$

$(e)$ $$\left(\sqrt[5]{\frac{32}{5}}-\sqrt[5]{\frac{27}{5}}\right)^{-1}=\frac{16+12\sqrt[5]{3}+9\sqrt[5]{9}+8\sqrt[5]{27}+6\sqrt[5]{81}}{\sqrt[5]{5^4}}$$

$(f)$ $$\left(\sqrt[5]{\frac{1}{5}}+\sqrt[5]{\frac{4}{5}}\right)^{-1}=\frac{\left(-1+\sqrt[5]{2}+\sqrt[5]{4}\right)^2}{\sqrt[5]{5^4}}$$

$(g)$ $$\left(\sqrt[3]{28}-3\right)^{-1}=9+3\sqrt[3]{28}+2\sqrt[3]{98}$$

$(h)$ $$\left(\sqrt[3]{5}-\sqrt[3]{4}\right)^{-1}=2\sqrt[3]{2}+\sqrt[3]{20}+\sqrt[3]{25}$$

Teorema 4.

$(a)$ $$\pi=4tan^{-1}\left(\sqrt[5]{\frac{16}{125}}+\sqrt[5]{\frac{8}{125}}+\sqrt[5]{\frac{2}{125}}-\sqrt[5]{\frac{1}{125}}-1\right)+4tan^{-1}\left(2\sqrt[5]{\frac{4}{25}}+2\sqrt[5]{\frac{2}{25}}-2\sqrt[5]{\frac{1}{25}}-1\right)$$

$(b)$ $$\pi=4tan^{-1}\left(\sqrt[3]{\frac{1}{3}}+\sqrt[3]{\frac{2}{3}}-1\right)+4tan^{-1}\left(2\sqrt[3]{\frac{1}{9}}-2\sqrt[3]{\frac{2}{9}}+2\sqrt[3]{\frac{4}{9}}-1\right)$$

$(c)$ $$\pi=4tan^{-1}\left(2\sqrt[3]{\frac{4}{9}}+2\sqrt[3]{\frac{10}{9}}+2\sqrt[3]{\frac{25}{9}}-1\right)-4tan^{-1}\left(1-\sqrt[3]{\frac{5}{3}}+\sqrt[3]{\frac{2}{3}}\right)$$

$(d)$ $$\pi=4tan^{-1}\left(4\sqrt[5]{\frac{1}{125}}+2\sqrt[5]{\frac{3}{125}}+2\sqrt[5]{\frac{9}{125}}+2\sqrt[5]{\frac{81}{125}}-1\right)-4tan^{-1}\left(1-\sqrt[5]{\frac{1}{25}}-\sqrt[5]{\frac{3}{25}}+\sqrt[5]{\frac{9}{25}}\right)$$

Teorema 5.

$(a)$ $$\left(3\sqrt[3]{15}-3\sqrt[3]{12}\right)^{1/2}=\sqrt[3]{\frac{2}{3}}+\sqrt[3]{\frac{20}{3}}-\sqrt[3]{\frac{25}{3}}$$

$(b)$ $\quad \left(63\sqrt[3]{20}-171\right)^{1/6}=\sqrt[3]{5}-\sqrt[3]{2}$

$(c)$ $\quad \left(10-5\sqrt[5]{27}\right)^{1/3}=1+\sqrt[5]{3}-\sqrt[5]{9}$

$(d)$ $\quad \left(\frac{\sqrt[3]{5}-1}{2-\sqrt[3]{5}}\right)^{1/2}=\sqrt[6]{\frac{1}{27}}+\sqrt[6]{\frac{25}{27}}$

$(e)$ $\quad \left(7\sqrt[3]{20}-19\right)^{1/2}=1+\sqrt[3]{20}-\sqrt[3]{50}$

$(f)$ $\quad \sqrt[5]{1+4\sqrt[5]{2}+\sqrt[5]{4}+2\sqrt[5]{8}+2\sqrt[5]{16}}=\sqrt[5]{\frac{1}{5}}+\sqrt[5]{\frac{4}{5}}$

$(g)$ $\quad \left(3\sqrt[3]{84}-9\sqrt[3]{3}\right)^{1/2}=\sqrt[3]{\frac{98}{3}}-\sqrt[3]{\frac{28}{3}}-\sqrt[3]{\frac{1}{3}}$

Teorema 6.

$(a)$ $\quad \sqrt[3]{5}-\sqrt[3]{4}=\sqrt{2\sqrt[3]{2}-2\sqrt[3]{20}+\sqrt[3]{25}}$

$(b)$ $\quad \sqrt[3]{5}-\sqrt[3]{4}=\sqrt[3]{1+6\sqrt[3]{10}-3\sqrt[3]{100}}$

$(c)$ $\quad \sqrt[3]{5}-\sqrt[3]{4}=\sqrt[4]{-16\sqrt[3]{4}-11\sqrt[3]{5}+12\sqrt[3]{50}}$

Teorema 7.

Para $n\in\mathbb{N}=\{1,2,3,\ldots\}$ se tiene:

$$\sqrt[3]{\frac{1}{9}}-\sqrt[3]{\frac{2}{9}}+\sqrt[3]{\frac{4}{9}}=\left(a_n+b_n\sqrt[3]{2}+c_n\sqrt[3]{4}\right)^{1/3n}$$

donde

$$a_{n+1}=-a_n+2c_n$$

$$b_{n+1}=a_n-b_n$$

$$c_{n+1}=b_n-c_n$$

$$a_1=-1\,,b_1=1\,,c_1=0$$

Teorema 8.

Para $n \in \mathbb{N} = \{1,2,3, \dots\}$ se tiene:

$$\sqrt[3]{5} - \sqrt[3]{4} = \left(a_n \sqrt[3]{2} + b_n \sqrt[3]{20} + c_n \sqrt[3]{25}\right)^{1/(3n+2)}$$

donde

$$a_{n+1} = a_n - 30b_n + 30c_n$$

$$b_{n+1} = 6a_n + b_n - 15c_n$$

$$c_{n+1} = -6a_n + 12b_n + c_n$$

$$a_1 = 2\,, b_1 = -2\,, c_1 = 1$$

Teorema 9.

Para $n \in \mathbb{N} = \{1,2,3, \dots\}$ se tiene:

$$\sqrt[3]{5} - \sqrt[3]{4} = \left(a_n \sqrt[3]{4} + b_n \sqrt[3]{5} + c_n \sqrt[3]{50}\right)^{1/(3n+1)}$$

donde

$$a_{n+1} = a_n - 15b_n + 30c_n$$

$$b_{n+1} = 12a_n + b_n - 10c_n$$

$$c_{n+1} = -6a_n + 6b_n + c_n$$

$$a_1 = -16\,, b_1 = -11\,, c_1 = 12$$

Teorema 10.

Para $n \in \mathbb{N} = \{1,2,3, \dots\}$ se tiene:

$$\sqrt[3]{5} - \sqrt[3]{4} = \left(a_n + b_n \sqrt[3]{10} + c_n \sqrt[3]{100}\right)^{1/(3n)}$$

donde

$$a_{n+1} = a_n - 30b_n + 60c_n$$

$$b_{n+1} = 6a_n + b_n - 30c_n$$

$$c_{n+1} = -3a_n + 6b_n + c_n$$

$$a_1 = 1\,, b_1 = 6\,, c_1 = -3$$

Teorema 11.

Para $n \in \mathbb{N} = \{1,2,3,\ldots\}$ se tiene:

$$\left(\sqrt[3]{28} - 3\right)^{n/2} = \frac{1}{3^n}\left(a_n + b_n\sqrt[3]{28} + c_n\sqrt[3]{98}\right)$$

donde

$$a_{n+1} = -a_n + 14b_n - 14c_n$$

$$b_{n+1} = -a_n - b_n + c_n$$

$$c_{n+1} = a_n - 2b_n - c_n$$

$$a_1 = -1\,, b_1 = -1\,, c_1 = 1$$

Teorema 12.

Para $n \in \mathbb{N} = \{1,2,3,\ldots\}$ se tiene:

$$\left(\sqrt[5]{\frac{1}{5}} + \sqrt[5]{\frac{4}{5}}\right)^{n/2} = \left(a_n + b_n\sqrt[5]{2} + c_n\sqrt[5]{4} + d_n\sqrt[5]{8} + e_n\sqrt[5]{16}\right)^{1/5}$$

donde

$$a_{n+1} = a_n + 2c_n + 2e_n$$

$$b_{n+1} = a_n + b_n + 2d_n$$

$$c_{n+1} = b_n + c_n + 2e_n$$

$$d_{n+1} = a_n + c_n + d_n$$

$$e_{n+1} = b_n + d_n + e_n$$

$$a_1 = 1\,, b_1 = 1\,, c_1 = 0\,, d_1 = 1\,, e_1 = 0$$

Teorema 13.

$(a)$ $$\sqrt[3]{\frac{1}{3}}+\sqrt[3]{\frac{5}{3}}=\frac{\sqrt[3]{9}+\sqrt[3]{45}}{3}=\sqrt[3]{2+\sqrt[3]{5}+\sqrt[3]{25}}$$

$(b)$ $$\left(\sqrt[3]{\frac{1}{3}}+\sqrt[3]{\frac{5}{3}}\right)^{-1}=\frac{1}{2}\sqrt[3]{-1+3\sqrt[3]{5}-\sqrt[3]{25}}=\frac{\sqrt[3]{3}-\sqrt[3]{15}+\sqrt[3]{75}}{3}$$
$$=\frac{1}{2}\left(\sqrt[3]{\frac{1}{9}}-\sqrt[3]{\frac{5}{9}}+\sqrt[3]{\frac{25}{9}}\right)$$

$(c)$ $$\sqrt[3]{\frac{1}{3}}+\sqrt[3]{\frac{1}{15}}=\sqrt[6]{\frac{3}{5}}\sqrt{\frac{\sqrt[3]{5}-1}{2\sqrt[3]{5}-\sqrt[3]{25}}}=\sqrt[3]{\frac{3+\sqrt[3]{5}}{5\sqrt[3]{5}-5}}=\sqrt[5]{\frac{3\sqrt[3]{3}+\sqrt[3]{15}}{10\sqrt[3]{25}-25}}=\sqrt[3]{\frac{2+\sqrt[3]{5}+\sqrt[3]{25}}{5}}$$

Teorema 14.

$(a)$ $$\pi=4tan^{-1}\left(\sqrt[3]{\frac{1}{3}}+\sqrt[3]{\frac{5}{3}}\right)-4tan^{-1}\left(\frac{1+\sqrt[3]{3}-\sqrt[3]{5}-\sqrt[3]{9}+\sqrt[3]{25}+\sqrt[3]{45}-\sqrt[3]{75}}{3}\right)$$

$(b)$ $$\pi=4tan^{-1}\left(\sqrt[3]{\frac{1}{3}}+\sqrt[3]{\frac{5}{3}}-1\right)+4tan^{-1}\left(\frac{\sqrt[3]{3}-\sqrt[3]{15}+\sqrt[3]{75}-3}{3}\right)$$

Teorema 15.

Para $n\in\mathbb{N}_0=\{0,1,2,3,\ldots\}$ se tiene:

$$\left(\sqrt[3]{\frac{4}{3}}-\sqrt[3]{\frac{1}{3}}\right)^{3n}=\frac{1}{3^n}\left(a_{3n}+b_{3n}\sqrt[3]{2}+c_{3n}\sqrt[3]{4}\right)$$

$$\left(\sqrt[3]{\frac{4}{3}}-\sqrt[3]{\frac{1}{3}}\right)^{3n+1}=\frac{1}{3^n}\left(a_{3n+1}\sqrt[3]{\frac{1}{3}}+b_{3n+1}\sqrt[3]{\frac{2}{3}}+c_{3n+1}\sqrt[3]{\frac{4}{3}}\right)$$

$$\left(\sqrt[3]{\frac{4}{3}}-\sqrt[3]{\frac{1}{3}}\right)^{3n+2}=\frac{1}{3^n}\left(a_{3n+2}\sqrt[3]{\frac{1}{9}}+b_{3n+2}\sqrt[3]{\frac{2}{9}}+c_{3n+2}\sqrt[3]{\frac{4}{9}}\right)$$

donde

$$a_{n+1}=-a_n+2b_n$$

$$b_{n+1} = -b_n + 2c_n$$

$$c_{n+1} = a_n - c_n$$

$$a_0 = 1\,, b_0 = 0\,, c_0 = 0$$

Teorema 16.

$$\sqrt[3]{\frac{4}{3}} - \sqrt[3]{\frac{1}{3}} = \left(\sqrt[3]{2} - 1\right)^{2/3}$$

Teorema 17.

$(a)$ $\quad g^4 = 5 \;\Rightarrow\; \pi = 4tan^{-1}(2 + g + g^2 + g^3) - 4tan^{-1}\left(\sqrt[5]{\frac{4g-4}{3+2g}}\right)$

$(b)$ $\quad g^5 = 2 \;\Rightarrow\; \pi = 4tan^{-1}\left(\frac{1}{g + g^2}\right) + 4tan^{-1}\left(\sqrt{\frac{5g-5}{3+g}}\right)$

$(c)$ $\quad g^5 = 2 \;\Rightarrow\; \pi = 4tan^{-1}\left(\frac{5}{(1 + g^2 + g^3 + g^9)^2}\right) + 4tan^{-1}\left(\sqrt{\frac{4g-3}{1+g^2}}\right)$

$(d)$ $\quad g^5 = 3 \;\Rightarrow\; \pi = 4tan^{-1}\left(\frac{1}{1 + g^2 + g^3 + g^4}\right) + 4tan^{-1}\left(\sqrt{\frac{5g-5}{1+g^2}}\right)$

Teorema 18.

$(a)$ $\quad g^2 = \sqrt[3]{5} - \sqrt[3]{4} \;\Rightarrow\; \pi = 4tan^{-1}(g) + 4tan^{-1}\left(\frac{8 - 17g + 17g^2 + 6g^3 - 6g^4 + 6g^5 + g^6 - g^7 + g^8}{9}\right)$

$(b)$ $\quad g^3 = \sqrt[3]{2} - 1 \;\Rightarrow\; \pi = 4tan^{-1}(g) + 4tan^{-1}(g^8 - g^7 + g^6 + 2g^5 - 2g^4 + 2g^3 + g^2 - g)$

$(c)$ $\quad g^2 = \sqrt[3]{28} - 3 \;\Rightarrow\; \pi = 4tan^{-1}(g) + 4tan^{-1}\left(\frac{2 + g^2}{3}\right)$

$$(d) \qquad g^6 = 7\sqrt[3]{20} - 19 \Rightarrow \pi = 4tan^{-1}(g) + 4tan^{-1}\left(\frac{18 - 37g + 37g^2 - 4g^3 + 4g^4 - 4g^5 + g^6 - g^7 + g^8}{19}\right)$$

$$(e) \qquad g = \frac{\sqrt[4]{5} - 1}{\sqrt[4]{5} + 1} \Rightarrow \pi = 4tan^{-1}(g) + 4tan^{-1}\left(\frac{9 - 13g + 7g^2 - g^3}{10}\right)$$

## Referencias


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